Fusion Plasma Theory II. 2019

Week 4

Ch. 18. Low-frequency Waves in Magnetized Plasmas.

18.1. Dielectric Tensor

* For low frequency waves, ion dynamics has to be considered.
* We will also consider finite pressure and arbitrary angle of propagation
* Take $\vec{B}_0 = B_0 \hat{z}$, $\vec{k} = k_x \hat{x} + k_z \hat{z}$, $= k(\sin\theta \hat{x} + \cos\theta \hat{z})$

(*) For each species, we have

$$m n_0 \frac{\partial}{\partial t} \vec{u}_1 = q n_0 (\vec{E}_1 + \vec{u}_1 \times \vec{B}_0) - \gamma T \vec{\nabla} n_1 \qquad (18.1)$$

and

$$\vec{\nabla} \cdot (n_0 \vec{u}_1) = -\frac{\partial}{\partial t} n_1 \qquad (18,5)$$

* $$\vec{J}_1 = \sum_{sp} n_0 q \, \vec{u}_1 \equiv \overleftrightarrow{\sigma} \cdot \vec{E}_1 \qquad (18,9)$$

⊛ $\overleftrightarrow{\sigma}$: a complex frequency-dependent tensor electrical conductivity.

* Wave Eqn :

$$k^2 \vec{E}_1 - \vec{k}(\vec{k} \cdot \vec{E}_1) = \left(\frac{\omega}{c}\right)^2 \left(\vec{E}_1 + i\vec{j}_1/\epsilon_0\omega\right)$$

$$\Downarrow$$

$$\left(\frac{\omega}{c}\right)^2 \mu_0 \, \overleftrightarrow{\epsilon} \cdot \vec{E}_1$$

⊛ Di-electric tensor :

$$\overleftrightarrow{\epsilon} = \epsilon_0 \left(\overleftrightarrow{I} + i\overleftrightarrow{\sigma}/\epsilon_0\omega \right) \qquad (18,13)$$

By defining $\overleftrightarrow{X} \equiv \overleftrightarrow{I} - \vec{k}\vec{k}/k^2$, the wave equation can be written in a compact form,

$$\boxed{\left(\omega^2 \mu_0 \overleftrightarrow{\epsilon} - k^2 \overleftrightarrow{X}\right) \cdot \vec{E}_1 = 0} \qquad (18,15)$$

* Eq (18.15) leads to the "warm" plasma dispersion relation.

18.2. The Cold-Plasma Dispersion Relation,

* Take $T=0$ and define,

- $\tilde{n} \equiv ck/\omega = c/v_p$, $\rightarrow$ $c\vec{k}/\omega \equiv \tilde{n}(\sin\theta\,\hat{x} + \cos\theta\,\hat{z})$

- $R \equiv 1 - \dfrac{\omega_p^2}{\omega(\omega-\omega_c)} - \dfrac{\Omega_p^2}{\omega(\omega+\Omega_c)}$

- $L \equiv 1 - \dfrac{\omega_p^2}{\omega(\omega+\omega_c)} - \dfrac{\Omega_p^2}{\omega(\omega-\Omega_c)}$

- $S \equiv \dfrac{R+L}{2}$

- $D \equiv \dfrac{R-L}{2}$

- $P \equiv 1 - \dfrac{\omega_p^2}{\omega^2} - \dfrac{\Omega_p^2}{\omega^2}$

ω_p, Ω_p : electron and ion plasma freq's

ω_c, Ω_c : " cyclotron " .

(*) Then, the wave equation (18.15) can be written as

$$\begin{bmatrix} \hat{x}\hat{x}(S-\tilde{n}^2\cos^2\theta) & -\hat{x}\hat{y}\, iD & +\hat{x}\hat{z}\tilde{n}^2\sin\theta\cos\theta \\ +\hat{y}\hat{x}\, iD & +\hat{y}\hat{y}(S-\tilde{n}^2) & +0 \\ +\hat{z}\hat{x}\tilde{n}^2\sin\theta\cos\theta & +0 & +\hat{z}\hat{z}(P-\tilde{n}^2\sin^2\theta) \end{bmatrix} \cdot \vec{E}_1 = 0 \qquad (18.16)$$

* $\boxed{\text{Homework: Problem 18.1 on page 288.}}$

Determinant of (18.16) $=0$

$$\Rightarrow (S^2P-D^2P) - \underline{\tilde{n}^2}(SP\cos^2\theta + SP + S^2\sin^2\theta - D^2\sin^2\theta)$$
$$+\underline{\tilde{n}^4}(P\cos^2\theta + S\sin^2\theta) = 0.$$

→ Two branches of Dispersion Relation

$$\Rightarrow \quad \boxed{\tan^2\theta = \frac{-P(\tilde{n}^2 - R)(\tilde{n}^2 - L)}{(S\tilde{n}^2 - RL)(\tilde{n}^2 - P)}} \qquad (18.24)$$

A useful form of cold-plasma D.R.

(*) For // propagation, $\theta = 0$ by definition.

We have two solutions, $\tilde{n}^2 = R$ and $\tilde{n}^2 = L$ which are the right- and left-circularly polarized waves,

(*) For $\perp$ propagation, $\tan^2\theta \to \infty$.

We have two solutions, $\tilde{n}^2 = P$ and $\tilde{n}^2 = \dfrac{RL}{S}$.

O-wave X-wave

Note that ion dynamics contributions from are included via the definitions of R, L, S and P.

(*) Resonances $\Rightarrow \tilde{n} \to \infty$ ($k \to \infty$, $\lambda \to 0$),

$\Rightarrow$ $\tan^2\theta = -P/S$, i.e., resonance freqs. vary with the angle of propagation.

* For $\theta = 0$, $P = 0$ corresponds to $\omega = \omega_p$ resonance.

$S \to \infty$ correspond to $\begin{cases} R \to \infty, & \omega = \omega_c \text{ resonance,} \\ \text{and} & \\ L \to \infty, & \omega = \Omega_c \text{ ''} \end{cases}$,

* For $\theta = \pi/2$,

$S \to 0$ $\to$ upper and lower-hybrid resonances, including ion dynamics.

(*) Cutoffs $\Rightarrow$ We must go back to Eq. (18.18)$_s$ to get

$PRL = 0$,

$P = 0 \to \omega_p$ cutoff of O-wave and Langmuir wave

$R = 0 \to \omega_R$ cutoffs with ion dynamics included,
$L = 0 \to \omega_L$

Ch 2 Ideal MHD Plasmas

- MHD equilibrium ($\partial/\partial t \to 0$)

$$\mathbf{j}_0 \times \mathbf{B}_o = \nabla p_0 + \rho_0(\mathbf{u}_0 \cdot \nabla)\mathbf{u}_0 \tag{34}$$

the sum of the forces acting on the plasma is zero
- what happens if the plasma is perturbed from this state?

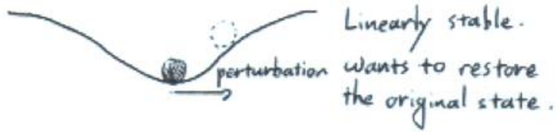

Figure 3

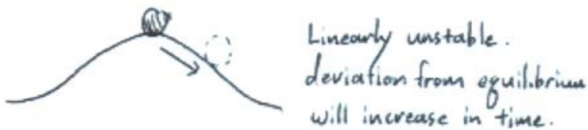

Figure 4

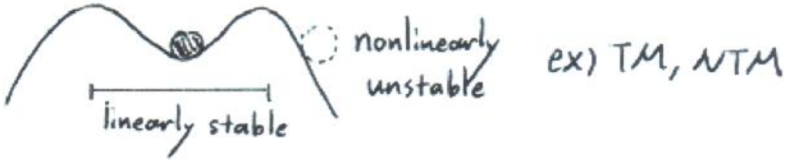

Figure 5

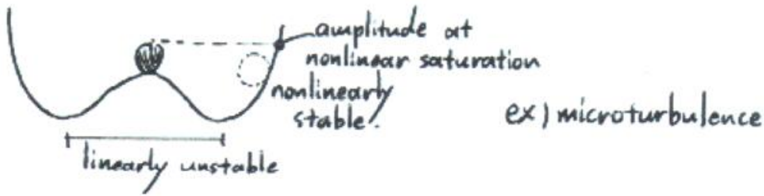

Figure 6

Figure 7

"Linear" $\equiv$ perturbation amplitude is infinitesimally small
$\Rightarrow$ ignore nonlinear terms (a product of more than two perturbed quantities)
In reality, we should consider a finite (not infinitesimal!) amplitude perturbation.

- How do we formulate the ideal MHD stability (or instability) ?

$\Rightarrow$ Linearization

$$\rho = \rho_0 + \delta\rho; \text{ where } \delta\rho(\mathbf{x}, t) = \delta\rho(\mathbf{x}) e^{-i\omega t}$$
$$p = p_o + \delta p; \text{ where } \delta p(\mathbf{x}, t) = \delta p(\mathbf{x}) e^{-i\omega t}$$
$$\mathbf{B} = \mathbf{B}_0 + \delta\mathbf{B}; \text{ where } \delta\mathbf{B}(\mathbf{x}, t) = \delta\mathbf{B}(\mathbf{x}) e^{-i\omega t}$$
$$\mathbf{u} = \underbrace{\cancel{\mathbf{u}_0}}_{=0 \text{ for simplicity}} + \delta\mathbf{u}; \text{ where } \delta\mathbf{u}(\mathbf{x}, t) = \delta\mathbf{u}(\mathbf{x}) e^{-i\omega t}$$

Linearized equations : every term has ONE perturbed quantity.

$\mathrm{Im}\,\omega > 0$: exponential instability

$e^{(\mathrm{Im}\,\omega)t} \nearrow$ as $t \nearrow$

$\gamma \equiv \mathrm{Im}\,\omega$: linear growth rate

$\mathrm{Im}\,\omega < 0$: exponential stability.

$\boxed{\text{Example}}$ MHD waves in <u>uniform</u> (i.e. both homogeneous and infinite) magnetic field and plasmas, then $\mathbf{B}_0 = B_0\hat{\mathbf{e}}_z, \mathbf{J}_0 = 0, \mathbf{u}_0 = 0, \nabla p_0 = \boldsymbol{\nabla}\rho_0 = 0$ Linearize the system of ideal MHD equations, assuming $\delta\mathbf{A}(\mathbf{x}, t) = \delta\mathbf{A}e^{-i\omega t + i\mathbf{k}\cdot\mathbf{x}}$ with $\mathbf{k} = k_\parallel \mathbf{e}_z + k_\perp \mathbf{e}_y$ without loss of generality.

2.1 Waves in MHD Plasmas

$$\frac{\partial}{\partial t}\rho + \boldsymbol{\nabla}\cdot(\rho\,\mathbf{u}) = 0 \tag{35}$$

$$\frac{d}{dt}\left(\frac{p}{\rho^\gamma}\right) = 0 \tag{36}$$

$$\frac{\partial}{\partial t}\mathbf{B} = -\boldsymbol{\nabla}\times\mathbf{E} = \boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B}) \tag{37}$$

$$\mu_0\mathbf{j} = \boldsymbol{\nabla}\times\mathbf{B} \tag{38}$$

$$\rho\frac{d\mathbf{u}}{dt} = \mathbf{j}\times\mathbf{B} - \boldsymbol{\nabla}p \tag{39}$$

$\nabla\delta(\cdots) \Rightarrow i\mathbf{k}\delta(\cdots)$, $\partial/\partial t \Rightarrow -i\omega$, $\delta(\rho\, u_0) = 0$ since we assumed $\mathbf{u}_0 = 0$

$$-i\omega\delta\rho + \rho_0(i\mathbf{k}\cdot\delta\mathbf{u}) = 0 \tag{40}$$
$$-i\omega\delta p + \gamma p_0(i\mathbf{k}\cdot\delta\mathbf{u}) = 0 \tag{41}$$
$$-i\omega\delta\mathbf{B} + i\mathbf{k}\times(\delta\mathbf{u}\times\mathbf{B}_0) = 0 \tag{42}$$
$$\mu_0\delta\mathbf{j} = i\mathbf{k}\times\delta\mathbf{B} \tag{43}$$
$$-i\omega\rho\delta\mathbf{u} = \delta\mathbf{j}\times\mathbf{B}_0 + \mathbf{j}_0\times\delta\mathbf{B} - i\mathbf{k}\delta p \tag{44}$$

Equations (40)-(44) can be expressed in terms of $\delta\mathbf{u}$:

$$(\omega^2 - k_\parallel^2 v_\mathrm{A}^2)\delta u_x = 0 \tag{45}$$
$$(\omega^2 - k_\perp^2 c_s^2 - k^2 v_\mathrm{A}^2)\delta u_y - k_\perp k_\parallel c_s^2\delta u_z = 0 \tag{46}$$
$$-k_\perp k_\parallel c_s^2\delta u_y + (\omega^2 - k_\parallel^2 c_s^2)\delta u_z = 0 \tag{47}$$

where
$v_\mathrm{A} = (B_0^2/\mu_0\rho_0)^{1/2}$: Alfvén speed
$c_s = (\gamma p_0/\rho_0)^{1/2}$: (adiabatic) sound speed.

$\boxed{\text{Homework}}$ Why equations (40)-(44) are expressed as 3×3 matrix in terms of $\delta\mathbf{u}$?

The eigenvalues of Equations (44)-(46) consist of :

$\omega^2 = k_{\parallel}^2 v_{\mathrm{A}}^2 \cdots$ (I)

$\omega^2 = \frac{1}{2}k^2(v_{\mathrm{A}}^2 + c_s^2)[1 \pm (1-\alpha^2)^{1/2}]$ where $\alpha^2 = 4(k_{\parallel}^2/k^2)\left|c_s^2 v_{\mathrm{A}}^2/(c_s^2+v_{\mathrm{A}}^2)\right|^2$

For $(2\mu_0 p_0/B_0^2 \equiv)\ \beta = c_s^2/v_{\mathrm{A}}^2 \ll 1$, the last two solutions simplify.

$\omega^2 \simeq k^2 v_{\mathrm{A}}^2 \cdots$ (II)

$\omega^2 \simeq k_{\parallel}^2 c_s^2 \cdots$ (III)

For uniform plasmas $L \to \infty$, $1/L < 1/\lambda$ thus

$$\left|\frac{\nabla\text{ (equilibrium quantity)}}{\text{(equilibrium quantity)}}\right| < \left|\frac{\nabla\text{ (perturbed quantity)}}{\text{(perturbed quantity)}}\right|$$

Note that for every solution, $\mathrm{Re}(\omega) \neq 0$ but $\mathrm{Im}(\omega) = 0$. There is *no* instability.

This is because for a ($\because$ uniform plasma, there is no free energy source)

(I) Shear Alfvén Wave

It causes *bending* of $\mathbf{B}$.
With $\mathbf{B}_0 = B_0\mathbf{e}_z$, $\delta\mathbf{u}$ follows $\delta\mathbf{B}$, both perturbations are in x-direction.
Relatively low frequency wave, and can become easily unstable.
Note that $W_m \propto |\mathbf{B}|^2 = |\mathbf{B}_0|^2 + |\delta\mathbf{B}|^2$ since $\mathbf{B}_0 \cdot \delta\mathbf{B} = 0$.
Therefore, the increase of magnetic field energy W_m due to the field line bending is only a second order in $\delta\mathbf{B}$ (i.e. it doesn't cost much magnetic energy).
Many (almost all) examples of MHD instability in low-β tokamak are versions of shear Alfvén wave (i.e. $\omega^2 = k_\parallel^2 v_\mathrm{A}^2 + \cdots$).
IBM, RBM, TM, NTM, Kink, TAE, $\cdots$

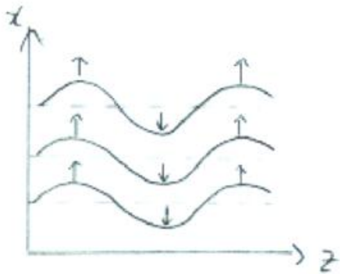

Figure 8 : Shear Alfvén wave

(II) Fast magnetosonic wave

In low-β limit plasmas, this becomes the compressional Alfvén wave.
Since typically $|k_\perp| \gg |k_\parallel|$ in tokamaks ($\because 2\pi R_0 \gg 2a$ typically),
$\omega^2_{\text{comp. Alfvén}} \gg \omega^2_{\text{shear Alfvén}}$
This wave compresses magnetic field!
$\delta\mathbf{B}_\parallel$ is non-zero, so the first order term in $|\mathbf{B}_0 + \delta\mathbf{B}|^2$ is non-zero.
Therefore, it's generally harder to excite.

(III) Slow magnetosonic wave

In low-β plasmas, this becomes a sound wave.

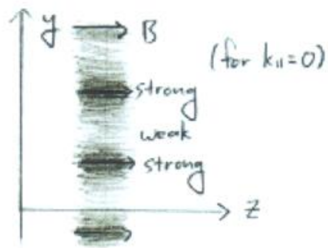

Figure 9 : Compressional Alfvén waves

This does *not* cause much of $\delta\mathbf{B}$. It exists even in the electrostatic limit. But this wave compresses plasmas. Many microinstabilities are versions of a sound wave.

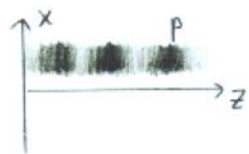

Figure 10 : Sound Wave

In low-β plasmas, typically, we have

$$\underset{\text{(Compressional Alfvén)}}{k^2 v_{\mathrm{A}}^2} \quad \gg \quad \underset{\text{(Shear Alfvén)}}{k_{\|}^2 v_{\mathrm{A}}^2} \quad \gg \quad \underset{\text{(Sound)}}{k_{\|}^2 c_s^2} \tag{48}$$