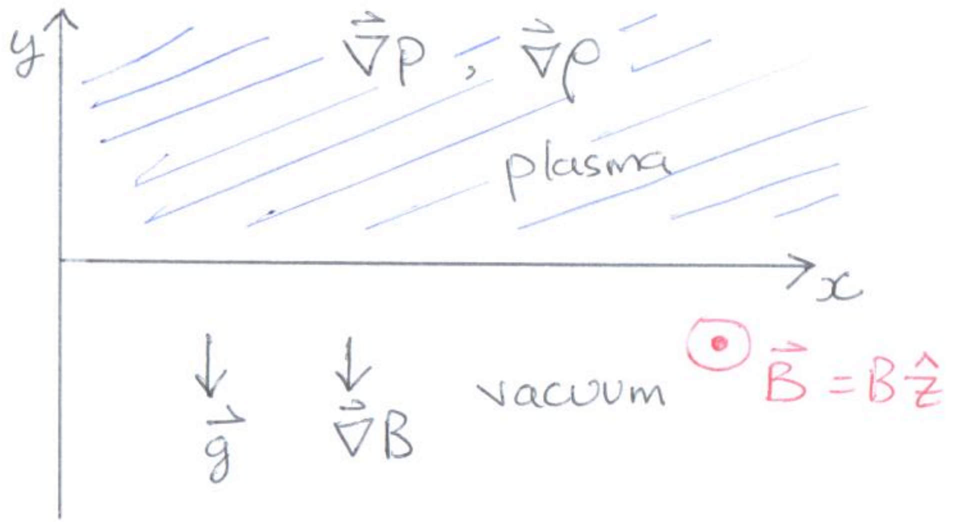
Fusion Plasma Theory II. 2019

Week 5

Ch. 19. Rayleigh-Taylor and flute instabilities

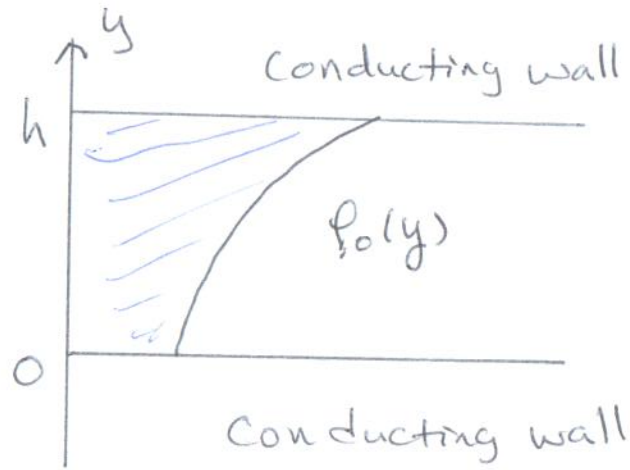
⊛ Gravitational Rayleigh-Taylor Instability



* Illustration of heavy fluid on top and lighter fluid at bottom.

(extreme case)

* Let's consider:



* MHD equilibrium:

$$\frac{\partial}{\partial y}\left(P_0 + \frac{B_0^2}{2\mu_0}\right) + \rho_0 g = 0 \qquad (19.2)$$

* Representation of Perturbed quantity:

$$\psi \propto \hat{\psi}(\underline{y}) \exp[ik_x x + ik_z z - i\omega t] \qquad (19.4)$$

direction in which equilibrium is inhomogeneous.

- Let's concentrate on "$k_z = 0$" perturbation for which the magnetic field lines remain straight (no cost of energy associated with field-line bending).

- One can show that $\frac{\partial B_{1y}}{\partial t} = 0$ and $\frac{\partial B_{1x}}{\partial t} = 0$ (19.6).

* Linearized (1st order) equation:

$$\rho_0 \frac{\partial}{\partial t} \vec{u}_1 = \rho_1 \vec{g} - \vec{\nabla}\left(p_1 + \frac{B_0 B_{z1}}{\mu_0}\right). \quad (19.7)$$

* We can ~~omit~~ eliminate the last term by taking " $\hat{z} \cdot \vec{\nabla} \times$ "

$$\rightarrow \quad -i\omega\left(ik\rho_0 u_y - \frac{\partial}{\partial y}(\rho_0 u_x)\right) = -ik\rho_1 g \quad (19.8)$$

- Assuming incompressibility:

$$ik u_x + \frac{\partial u_y}{\partial y} = 0 \quad (19.9)$$

- Continuity Eqn:

$$\frac{\partial}{\partial t}\rho_1 + \vec{u}_1 \cdot \vec{\nabla}\rho_0 = 0 \quad (19.10)$$

$$\rightarrow \quad \rho_1 = \frac{\rho_0 u_y}{i\omega S} \quad (19.11)$$

* Eliminating ρ_1 and u_x in favor of u_y, we get

$$\boxed{\frac{1}{\rho_0}\frac{\partial}{\partial y}\left(\rho_0 \frac{\partial u_y}{\partial y}\right) - k^2\left(1 + \frac{g}{s\omega^2}\right) u_y = 0} \quad (19.12)$$

* Eigenmode Equation (2nd order ordinary differential).

* We solve for eigenvalues "ω" and eigenfunction "u_y" with boundary conditions,

$$u_y = 0 \quad \text{at} \quad \underbrace{y = 0,\ h}_{\text{Conducting boundary}} \quad (19.13)$$

* Recall $\rho_0(y) \propto \exp(y/s)$,

* Eq. (19.12) can be transformed to a simpler (Schrödinger eqn-type) form, by introducing an integrating factor "$\cancel{\exp}(-y/2s)$"

* Let

$$u_y(y) = v_y(y)\exp(-y/2s)$$

$\Rightarrow$ Eq. (19.13) becomes

$$\frac{\partial^2}{\partial y^2} v_y + \underbrace{\left(\frac{1}{4s^2} + k^2\left(1+\frac{g}{s\omega^2}\right)\right)}_{\text{constant.}} v_y = 0,$$

$\Rightarrow$

$$v_y(y) = \sin\left(\frac{n\pi y}{h}\right), \quad n = 1, 2, \cdots$$

with eigenvalues satisfying

$$\boxed{k^2\left(1+\frac{g}{s\omega^2}\right) + \frac{1}{4s^2} = -\frac{n^2\pi^2}{h^2}} \qquad (19.15)$$

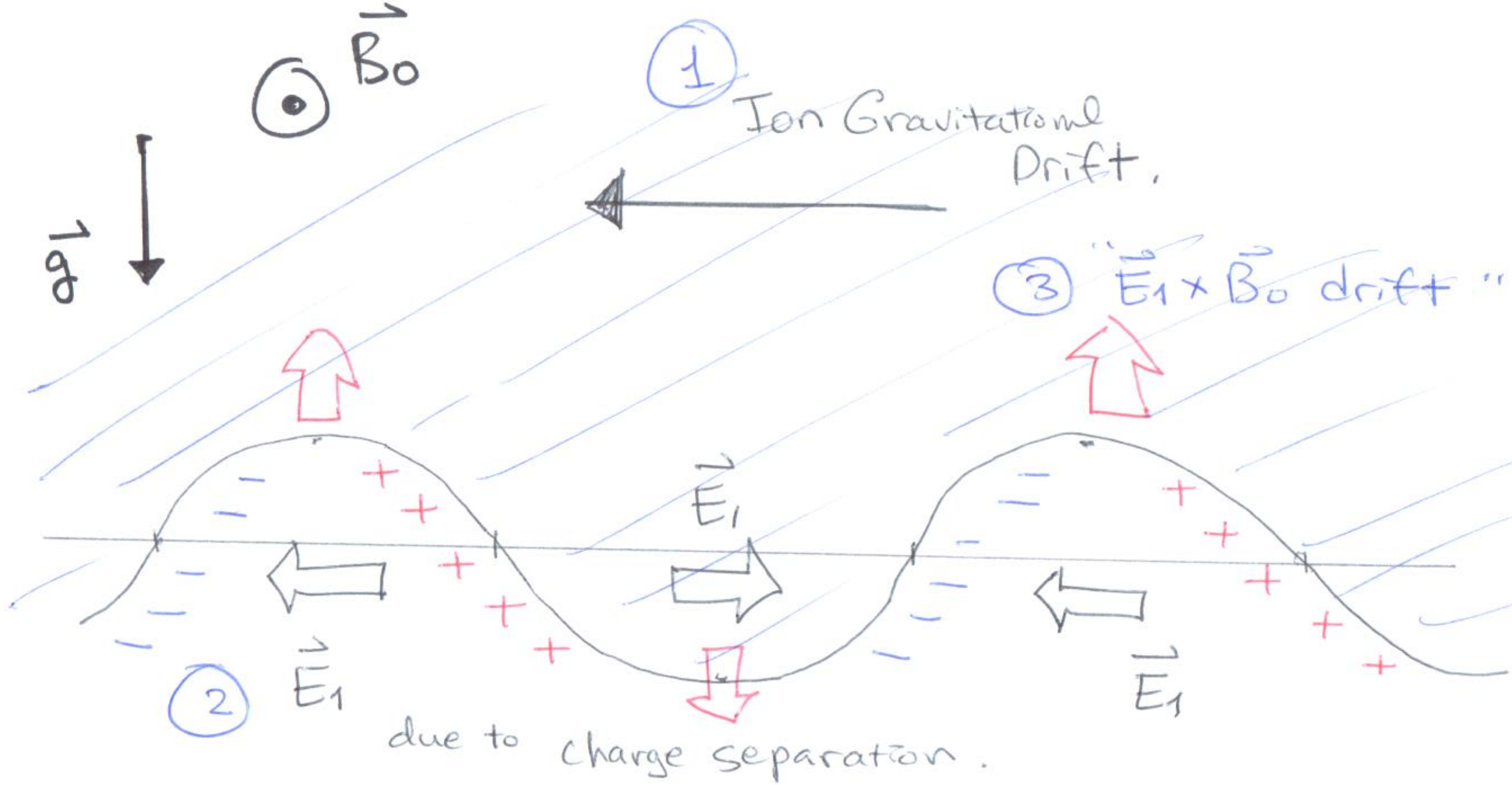
$$* \quad \omega = \pm i \left(\frac{g}{s}\right)^{1/2} \left(\frac{h^2 k^2}{n^2\pi^2 + h^2 k^2 + h^2/4s^2}\right)^{1/2} \qquad (19.26)$$

"+" sign corresponds to instability.

— Fastest growing mode corresponds to that with $n = 1$ and large k values.

— For $hk \gg \pi$ and $ks \gg 1$,

$\gamma \to (g/s)^{1/2}$, i.e, "free fall" due to acceleration by "g".



* Seed perturbation of plasma density will amplify in time based on this feedback loop.

Fig 19.4.

* One can show that the gravitational potential energy is lowered by the perturbation $(\propto \xi^2)$.

* Energy released goes into kinetic energy of plasma.

* Flute Instability due to $\vec{B}$ field Curvature *

* $\vec{B}$ field curvature can be considered an effective gravity:

$$\vec{V}_d = \frac{M}{e}\left(\frac{V_\perp^2}{2} + V_\parallel^2\right)\frac{\vec{R}_c \times \vec{B}}{R_c^2 B^2} \quad (19.30)$$

$\Longleftrightarrow$ gravitational Drift $\propto \vec{g} \times \vec{B}$ (19.29)

$$g_{eff} = \left(\frac{v_\perp^2}{2} + V_\parallel^2\right)\frac{1}{R_c},$$

taking thermal average,

$$\langle V_\parallel^2 \rangle = \left\langle \frac{V_\perp^2}{2} \right\rangle = T/M = P/\rho,$$

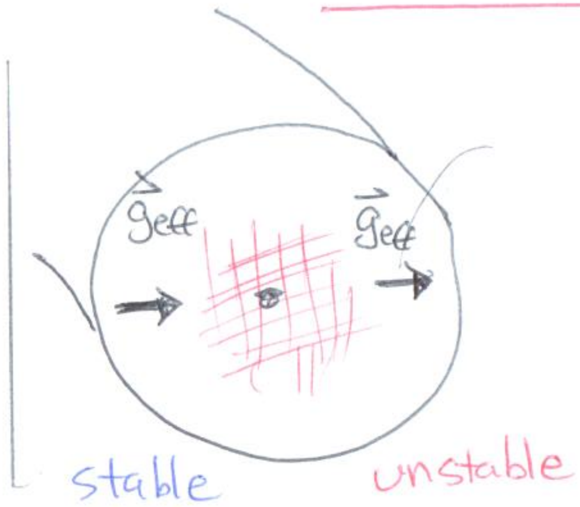
$$g_{eff} = \frac{2P}{\rho R_c} \quad (19.32)$$

* Noting that $s^{-1} = |\vec{\nabla} p| / p$,

$$\gamma \approx \left(2 |\vec{\nabla} p| / \rho R_c \right)^{1/2} \qquad (19.33)$$

$$\sim C_s / (s R_c)^{1/2} . \qquad (19.34)$$

i.e, both pressure gradient and bad curvature are necessary for instability



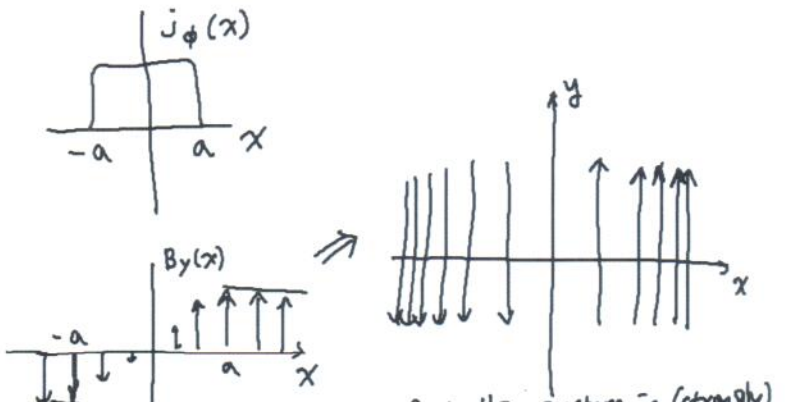
Resistive MHD Instability

Tearing Instability

- Ideal MHD instability is dangerous since it can grow in a short time scale (eg. $\gamma \sim c_s/\sqrt{L_p R}$ for pressure gradient driven interchange instability)

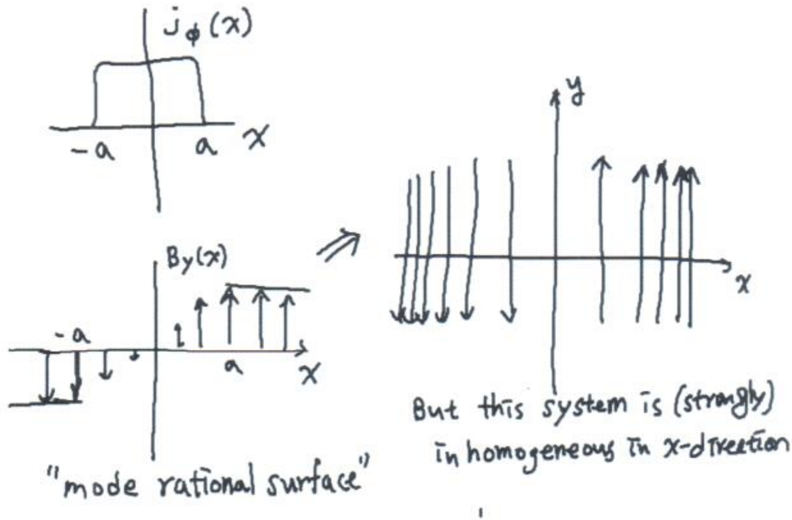
- Even when plasma is <u>stable</u> to ideal MHD instability, there could be slowly growing resistive MHD instability

- Resistive MHD instability can occur due to the fact that the constraint of the plasma and the magnetic field being frozen together in ideal MHD is <u>relaxed</u>



But this system is (strongly) inhomogeneous in x-direction

"mode rational surface"



This is rather counter-intuitive since the resistivity is an example of "dissipation", and dissipation usually damps out perturbations. Indeed there will be a damping effect from the resistivity, but this is dominated by other effect which allows the perturbation to form a structure which can lower the magnetic free energy by tearing and reconnecting the perturbed magnetic field near the mode rational (resonant) surface without making too much field line bending.

Let's review the MHD equations and shear Alfvénic perturbation.

$$\rho\left(\frac{\partial}{\partial t}\mathbf{u}+(\mathbf{u}\cdot\boldsymbol{\nabla}\mathbf{u})\right)=-\boldsymbol{\nabla}p-\frac{1}{4\pi}\mathbf{B}\times(\boldsymbol{\nabla}\times\mathbf{B})$$
$$\frac{\partial}{\partial t}\mathbf{B}=\boldsymbol{\nabla}\times(\mathbf{u}\times\mathbf{B})+\frac{\eta}{\mu_0}\nabla^2\mathbf{B}$$

If we linearize and assume very strong $\mathbf{B}_0$, i.e., low-β, we obtain,

$$\rho_0\frac{\partial}{\partial t}\delta\mathbf{u}=+\frac{1}{4\pi}(\mathbf{B}\cdot\boldsymbol{\nabla})\delta\mathbf{B}$$
$$\frac{\partial}{\partial t}\delta\mathbf{B}=(\mathbf{B_0}\cdot\boldsymbol{\nabla})\delta\mathbf{u}+\frac{\eta}{\mu_0}\nabla^2\delta\mathbf{B}$$

Without resistivity,
this 2×2 system yields the dispersion relation for shear-Alfvén wave:

$$\frac{\partial}{\partial t}\rightarrow -i\omega,\;\; \mathbf{B}_0\cdot\boldsymbol{\nabla}\rightarrow iB_0\,k_\parallel$$
$$\Rightarrow \omega^2=k_\parallel^2 v_\mathrm{A}^2 \text{ with } v_\mathrm{A}^2\equiv\frac{B_0^2}{4\pi\rho}$$

Now, by including of the resistivity,
with $\nabla_\perp^2 \gg \nabla_\parallel^2$ $(\lambda_\parallel \gg \lambda_\perp)$ and $\nabla_\perp^2 = -k_\perp^2$, The dispersion relation becomes

$$\omega(\omega + i\frac{\eta}{\mu_0}k_\perp^2) = k_\parallel^2 v_\mathrm{A}^2$$

$\Rightarrow$ resistivity induces a very small damping $\propto$ "η".

The second counter-intuitive result is that the resistive instability can grow in much shorter time scale than the magnetic field diffusion time!
$\therefore$ It's still less dangerous than ideal MHD instability, but cannot be ignored.

$\boxed{\text{For } B_\phi = 5\text{T},\ n_e = 10^{20}\text{m}^{-3},\ T_e = 5\text{keV},\ a = 1\text{m}}$

Resistive diffusion time :	$\tau_{\rm R} = \mu_0 a^2/\eta \Rightarrow$	10 minutes
Alfvén transit time :	$\tau_{\rm A} = a/v_{\rm A} \Rightarrow$	0.1 μsec
$\gamma^{-1}_{\text{tearing mode}}(\sim \tau_{\rm R}^{3/5}\tau_{\rm A}^{2/5})$:	~ 70 ms $\Rightarrow$	"Relevant"

This problem is non-trivial, and a prime example of elaborate boundary layer problem.

1. Solve ideal MHD equation away from the mode rational surface. (Outer solution)
   Here both inertia ($\sim \omega$) and resistive diffusion($\sim \eta\partial_x^2$) are negligible.
   Only the quantity Δ' characterizes the free energy associated with $j_\phi(r)$.

2. Solve resistive MHD equation near mode rational surface.
   Here both $\eta\partial_x^2$ and inertia should be kept.
   (Inner solution depends on ω and η.)

3. Asymptotic matching (at an intermediate region not just a position):
   $x \to \pm 0$ behavior of outer solution
   $x \to \pm\infty$ behavior of inner solution

Ideal MHD breaks down badly near mode rational surface, and we need to consider magnetic field diffusion $(\eta/\mu_0)\,\partial_x^2\,\delta\mathbf{B}$.

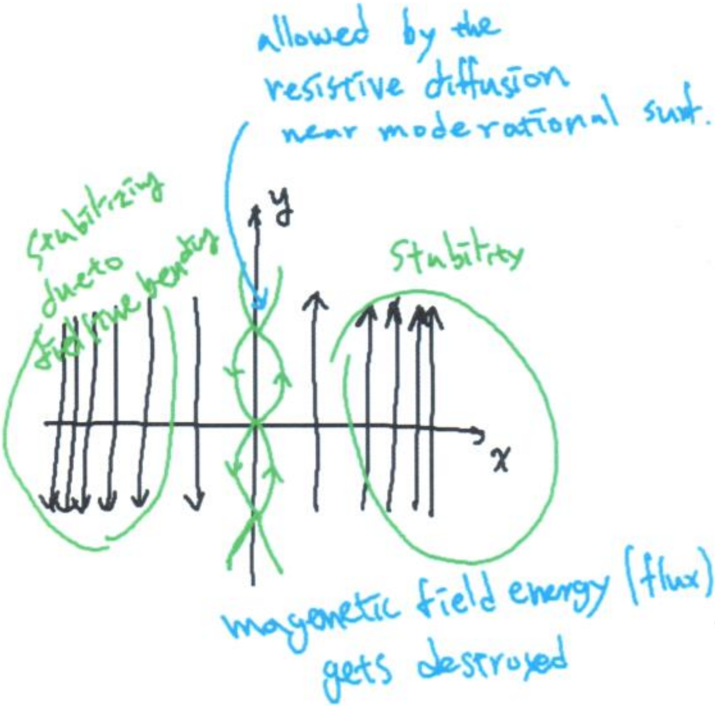
The growth rate of classical tearing mode is as follows.

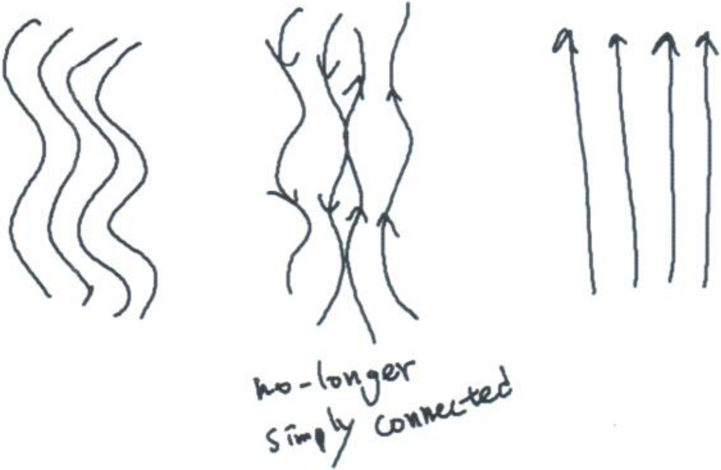
$$\gamma \propto \eta^{3/5} \Delta'^{4/5}$$

(Unstable when $\Delta' > 0$.) We can see that $\gamma \nearrow$ as $\Delta' \nearrow$.
This is one of the classic results from plasma theory.
[Furth, Killeen and Rosenbluth, Phys. Fluids 6, 459 (1963)]

Resistive diffusion term $(\eta/\mu_0)\,\partial_x^2\,\delta\mathbf{B}$ is non-negligible despite $\eta \ll 1$, since

$$\left|\frac{\partial}{\partial x}\right| \nearrow \text{ as } x \to 0.$$


allowed by the resistive diffusion near mode rational surf.
Stabilizing due to field line bending
y
x
Stability
magnetic field energy (flux) gets destroyed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