Fusion Plasma Theory II. 2019

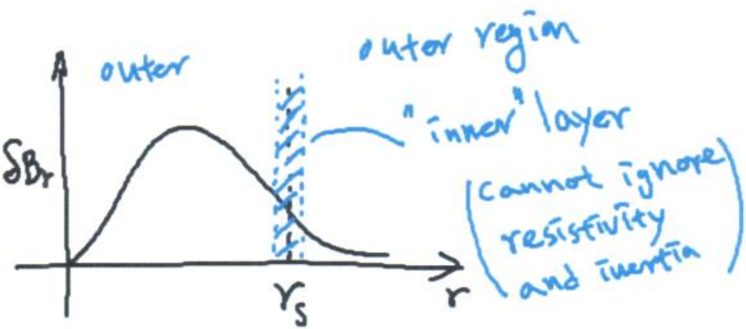
Week 6

Tearing Instability

The theory of tearing mode involves solving one set of equations over most of the plasma and another set in a resistive layer around the resonant surface.

$$q(r_s) = \frac{m}{n}$$

The complete eigenfunction then requires a matching of the resulting solutions and the condition for matching determines the growth rate eigenvalue.



(1) Outer region:

η, resistivity is very small, so it's ignored.
$\omega \propto \eta^\alpha$, $0 < \alpha < 1$ is expected since we consider a resistive instability.
Therefore, inertia $\rho\,\partial_t \delta \mathbf{u}$ is also negligible leaving " $\mathbf{j} \times \mathbf{B} = -\boldsymbol{\nabla} p$" $\therefore$ $\boldsymbol{\nabla} \times (\mathbf{j} \times \mathbf{B}) = 0$.

Since $\boldsymbol{\nabla} \cdot \mathbf{B} = 0$ and $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$, $(\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{j} - (\mathbf{j} \cdot \boldsymbol{\nabla})\mathbf{B} = 0$.

For high aspect ratio tokamak, $\epsilon = a/R \ll 1$, thus

$$B_{0,\theta} \sim r\frac{\partial}{\partial r} B_{0,\phi} \sim \epsilon B_\phi, \quad j_{0,\theta} \sim \epsilon j_{0,\phi}$$
$$\delta B_\phi \sim \epsilon \delta B_r \sim \epsilon \delta B_\theta, \quad \delta j_r \sim \delta j_\theta \sim \epsilon \delta j_\phi$$

$\Rightarrow$ only $\delta B_r, \delta B_\theta$ and δj_ϕ are important.
(Recall that this is a shear-Alfvénic fluctuation.)
$\Rightarrow (\mathbf{j} \cdot \boldsymbol{\nabla}) B_\phi \ll (\mathbf{B} \cdot \boldsymbol{\nabla}) j_\phi \;\Rightarrow \delta(\mathbf{B} \cdot \boldsymbol{\nabla} j_\phi) = 0.$

$\boldsymbol{\nabla} \cdot \mathbf{B} = 0 \Rightarrow \partial_r (r \delta B_r) + \partial_\theta \, \delta B_\theta = 0.$
It's convenient to introduce a function $\delta\psi$ which satisfies

$$\delta B_r = -\frac{1}{r} \frac{\partial}{\partial \theta} \delta\psi, \quad \delta B_\theta = \frac{\partial}{\partial r} \delta\psi.$$

Ampère's Law can be written as

$$\mu_0 \delta j_\phi = \nabla^2 \delta\psi = \frac{1}{r^2} \frac{\partial}{\partial r} r \frac{\partial}{\partial r} \delta\psi + \frac{1}{r^2} \frac{\partial^2}{\partial \theta^2} \delta\psi,$$

assuming $\delta\psi$ we obtain, $\propto e^{i(m\theta - n\phi)}$. Combining this with $\delta(\mathbf{B} \cdot \boldsymbol{\nabla} j_\phi) = 0$,

$$\frac{1}{\mu_0}\left(\frac{mB_\theta}{r}-\frac{nB_\phi}{R}\right)\nabla^2\delta\psi-\frac{m}{r}\left(\frac{dj_\phi}{dr}\right)\delta\psi=0,$$

or

$$\frac{1}{r}\frac{\mathrm{d}}{\mathrm{d}r}r\frac{\mathrm{d}}{\mathrm{d}r}\delta\psi=\frac{m^2}{r^2}\delta\psi-\frac{\frac{dj_{0,\phi}}{dr}}{\frac{B_\theta}{\mu_0}\left(1-\frac{nq(r)}{m}\right)}\delta\psi=0. \tag{114}$$

This equation has a singularity at $r=r_s$, where $q(r_s)=m/n$.
It's invalid at $r=r_s$.
Therefore, we need to solve for this equation for $r>r_s$ and $r<r_s$ separately.
We can impose that

$$\lim_{r\to r_s\text{from left}}\delta\psi(r)=\lim_{r\to r_s\text{from right}}\delta\psi(r)$$

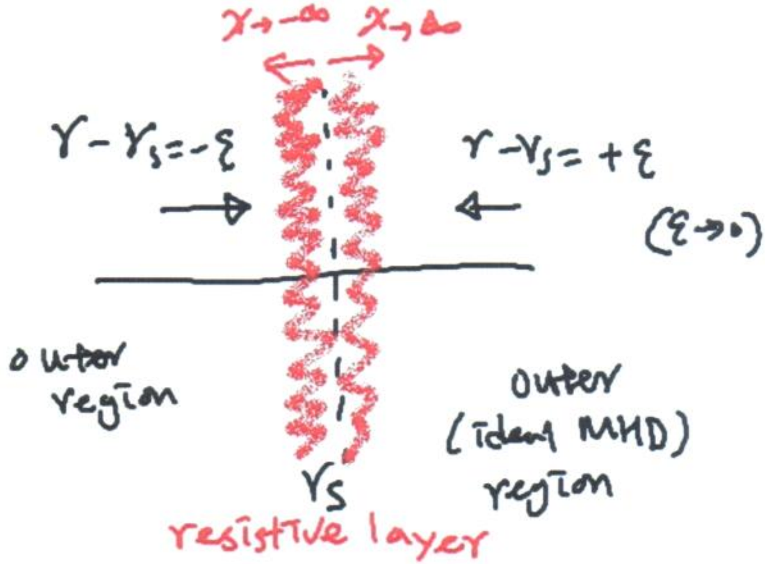
but

$$\lim_{r\to r_s\text{from left}}\frac{\partial}{\partial r}\delta\psi\neq\lim_{r\to r_s\text{from right}}\frac{\partial}{\delta r}\delta\psi.$$

Since we are dealing with linear equations (in $\delta\psi$), we need to match $\dfrac{d\delta\psi/dr}{\delta\psi}$:

$$\Delta' \equiv \lim_{\epsilon\to 0}\left[\left[\frac{\frac{d}{dr}\delta\psi}{\delta\psi}\right]_{r=r_s+\epsilon} - \left[\frac{\frac{d}{dr}\delta\psi}{\delta\psi}\right]_{r=r_s-\epsilon}\right] \tag{115}$$

Boundary Analysis consists of an asymptotic matching of
Δ' from outer solutions (which characterizes the destabilizing effect of dj_ϕ/dr) to
Δ' from inner solutions (which depends on ω and η).



(2) Resistive Layer

(only for near $r = r_s$, where $q(r_s) = m/n$)
From the Ohm's law,

$$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j}, \text{ we obtain } -\frac{\partial}{\partial t}\mathbf{B} + \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) = \frac{\eta}{\mu_0} \boldsymbol{\nabla} \times (\boldsymbol{\nabla} \times \mathbf{B})$$

By linearizing, we get

$$-\frac{\partial}{\partial t}\delta B_r + \mathbf{B}_0 \cdot \boldsymbol{\nabla} \delta u_r = -\frac{\eta}{\mu_0}\nabla^2 \delta B_r.$$

Assuming "perturbed quantities" $\propto \exp\left[\gamma t + i(m\theta - n\phi)\right]$,

$$\delta B_r = -\frac{im\delta\psi}{r} \Rightarrow \frac{d^2}{dr^2}\delta\psi = \frac{mu_0\gamma}{\eta}\delta\psi + \frac{\mu_0 B_{0,\theta}}{\eta}\left(1 - \frac{nq(r)}{m}\right)\delta u_r. \tag{116}$$

(Here $|d\delta\psi/dr| \gg |\delta\psi/r|$ has been used assuming fast radial variation of $\delta\psi$ in the narrow resistive layer.)

$$\Delta'_{\text{inside}} = \int_{-\infty}^{\infty} d(r{-}r_s)\ \frac{\frac{d^2}{dr^2}\delta\psi}{\delta\psi} = \frac{\mu_0\gamma}{\eta}\int_{-\infty}^{\infty} d(r{-}r_s)\left(1 + \frac{B_{0,\theta}}{r}\left(1 - \frac{nq(r)}{m}\right)\frac{\delta u_r}{\delta\psi}\right) \tag{117}$$

In outer region,

$$\frac{1}{r}\frac{d}{dr}r\frac{d}{dr}\delta\psi = \frac{m^2}{r^2}\delta\psi - \frac{\frac{dj_{0,\phi}}{dr}}{\frac{B_\theta}{\mu_0}\left(1 - \frac{nq(r)}{m}\right)}\delta\psi = 0 \tag{118}$$

δu_r can be obtained from

$$\rho \frac{\partial}{\partial t} \mathbf{u} = \mathbf{j} \times \mathbf{B} - \boldsymbol{\nabla} p. \tag{119}$$

By taking $\boldsymbol{\nabla}\times$ of Eq. (119)

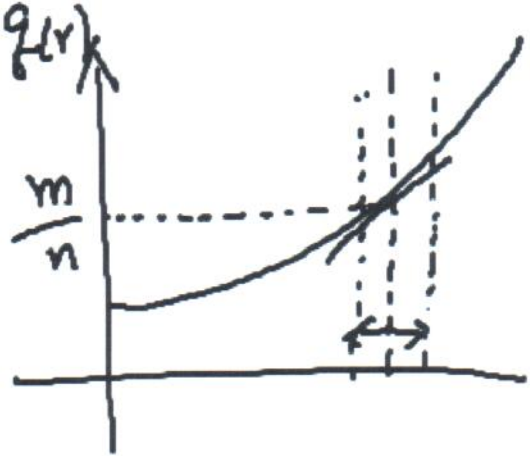
$$\boldsymbol{\nabla} \times \rho \frac{\partial}{\partial t} \mathbf{u} = \boldsymbol{\nabla} \times (\mathbf{j} \times \mathbf{B})$$

By taking ϕ-component and linearizing, we get

$$(\boldsymbol{\nabla} \times \rho \frac{\partial}{\partial t} \delta \mathbf{u})_\phi = \frac{\gamma \rho}{m} i r \frac{d^2}{dr^2} \delta u_r$$

where

$$\frac{\gamma \rho r^2}{m^2} \frac{d^2}{dr^2} \delta u_r = \frac{B_\theta}{\mu_0} \left(1 - \frac{n}{m} q(r)\right) \frac{d^2}{dr^2} \delta \psi - \left(\frac{d j_{0,\phi}}{dr}\right) \delta \psi \tag{120}$$



q(r)
m
n

Tearing Instability

$$\Delta' = 2.12\,\frac{\mu_0 \gamma d}{\eta}$$

"η" is given in Equation (6.8.17) of Wesson.

This leads to a growth rate γ,

$$\gamma = 0.55 \left(\frac{\eta}{\mu_0}\right)^{3/5} \left(\frac{m B_\theta q'}{(\mu_0 \rho)^{1/2} r q}\right) \Delta'^{\,4/5}$$

$\gamma > 0$ for $\Delta' > 0$.

Δ' depends on the equilibrium current profile.
Δ' should be calculated from the outer region ideal MHD equation.

With respect to $\tau_{\mathrm{R}} = \dfrac{\mu_o a^2}{\eta}$ and $\tau_{\mathrm{A}} = \dfrac{a}{B_\phi/(\mu_0 \rho)^{1/2}}$,

$$\gamma = \frac{0.55}{\tau_{\mathrm{R}}^{3/5} \tau_{\mathrm{A}}^{2/5}} \left(n \frac{a}{R} \frac{a q'}{q}\right)^{2/5} (a\Delta')^{4/5}$$

Note the hybrid time scale between $\tau_{\mathrm{R}} \sim 10\mathrm{min}$ and $\tau_{\mathrm{A}} \sim 0.1\mu\mathrm{s} \Rightarrow \gamma^{-1} \sim 70\mathrm{ms}$.

"d" $\sim a(\tau_A/\tau_R)^{2/5} \sim 0.15\mathrm{mm}$: resistive layer width.
$\tau_{\mathrm{R}}/\tau_{\mathrm{A}} \equiv S$ $(\propto 1/\eta)$ is Lundquist number or magnetic Reynolds number.

For present day tokamak core, $S > 10^7$ and even larger in solar system. Nonlinear MHD simulations, on the other hand, can treat $S \leq 10^6$ up to now.

Summary of tearing mode analysis in the resistive layer

Ohm's law:

$\mathbf{E} + \mathbf{u} \times \mathbf{B} = \eta \mathbf{j}$
Linearized, near $r = r_s \Rightarrow$

$$\frac{d^2}{ds^2}\delta\psi = \frac{\mu_0 \gamma}{\eta}\delta\psi - \frac{\mu_0 B_\theta}{\eta}\frac{q'}{q} s\,\delta u_r$$

where

$$s \equiv r - r_s, \;\; 1 - \frac{nq(r)}{m} \simeq -\left.\frac{q'}{q}\right|_{r_s} s.$$

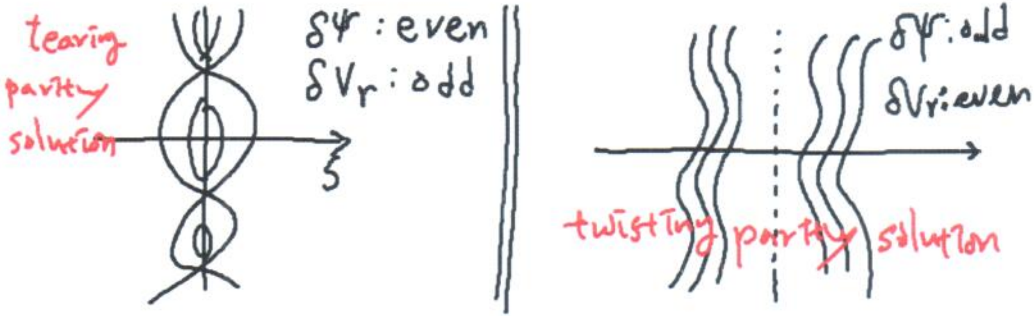
Momentum Equation (or Equation of Motion):

$\rho \frac{d\mathbf{u}}{dt} = -\boldsymbol{\nabla} p + \mathbf{j} \times \mathbf{B}$

$$\frac{d^2}{ds^2}\delta u_r - \left(\frac{B_\theta^2 m^2 q'^2}{\rho\eta\gamma r^2 q^2}\right) s^2\,\delta u_r = -\frac{B_\theta m^2 q'}{\rho\eta r^2 q} s\,\delta\psi + \left(\frac{d}{dr} j_\phi\right)\delta\psi$$

Look for a tearing parity (for which $\delta\psi$ is an even funcion w.r.t. $s = 0$).

$$\gamma \propto S^{-3/5}\tau_{\mathrm{A}}^{-1}$$



as $S \nearrow$, the linear gorwth rate of tearing mode $\searrow$.

We are considering just one pair of n and m for the tearing mode analysis.

We'll learn that

$$a\Delta' \propto -2m$$

for large values of m.
$\Rightarrow$ We usually worry about the low-n and low-m modes.
In reality, only several pairs of (n, m) such as $(1,1)$, $(1,2)$, $(2,3)$, $(3,4)$, $(2,5)$, $(3,5)$
etc are unstable.

One can find the correct scalings of γ and the resistive layer width δ, with respect to η, Δ' and so on from the dimensional analysis! (without performing an integration.)

One should note that

$$\frac{d^2}{ds^2}\delta\psi \sim \frac{d}{ds}\left[\frac{\left[\frac{d}{ds}\delta\psi\right]_{r_s-\epsilon}^{r_s+\epsilon}}{\delta\psi}\right]\delta\psi \sim \frac{\Delta'}{\delta}\delta\psi$$

$$\text{from Ohm's law} \Rightarrow \sim \frac{\mu_0\gamma}{\eta}\delta\psi. \tag{121}$$

Also,

$$\frac{d^2}{ds^2}\delta u_r \sim \frac{\delta u_r}{\delta^2}$$

$$\text{from equation of motion} \Rightarrow \sim \left(\frac{B_\theta^2 m^2 q'^2}{\rho\eta\gamma r^2 q^2}\right)\delta^2\,\delta u_r. \tag{122}$$

Equations (121) and (122) relate γ and δ (unknowns) to γ, Δ' etc. From Equation (121),

$$\eta\Delta' \propto \delta\gamma \tag{123}$$

From Equation(122) $\Rightarrow$

$$\delta^4 \sim \left(\frac{B_\theta^2 m^2 q'^2}{\rho\eta\gamma r^2 q^2}\right)^{-1} \tag{124}$$

Now using Equations (123) and (124) $\Rightarrow$

$$\begin{aligned}\gamma &\sim \eta^{3/5}\Delta'^{4/5}\\ \delta &\sim \eta^{2/5}\Delta'^{1/5}\end{aligned}$$

are obtained.

Physical meaning of Equation (123):

$$\underset{(a)}{\eta}\ \underset{(b)}{\Delta'} \propto \underset{(c)}{\delta\gamma}$$

(a) Free energy can only be released in the presence of "η"

(b) Free energy source

(c) Has a dimension of velocity in radial direction
which quantifies the rate of energy release

We can calculate Δ' explicitly form the outer region using ideal MHD equations, for simple $j_\phi(r)$ profiles.