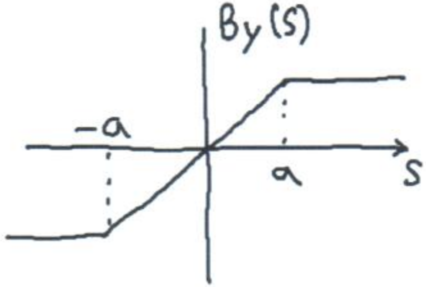
Fusion Plasma Theory II. 2019

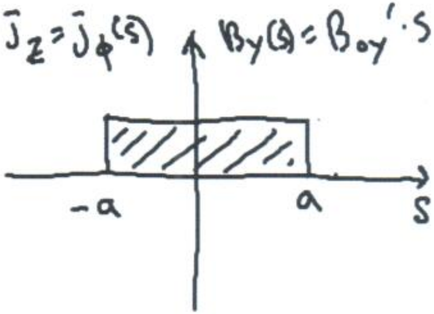
Week 7

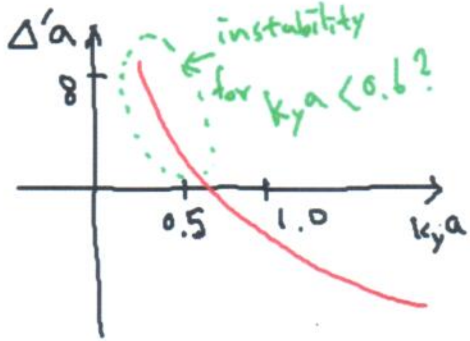


(1) Example from Goldston and Rutherford, pages 354-356

$\Rightarrow$

$$\Delta' a = \frac{2k_y a \left[\exp\left(-2k_y a\right) - 2k_y a + 1\right]}{\exp\left(-2k_y a\right) + 2k_y a - 1}$$





Since $k_y = m/r_s = nq/r_s$,
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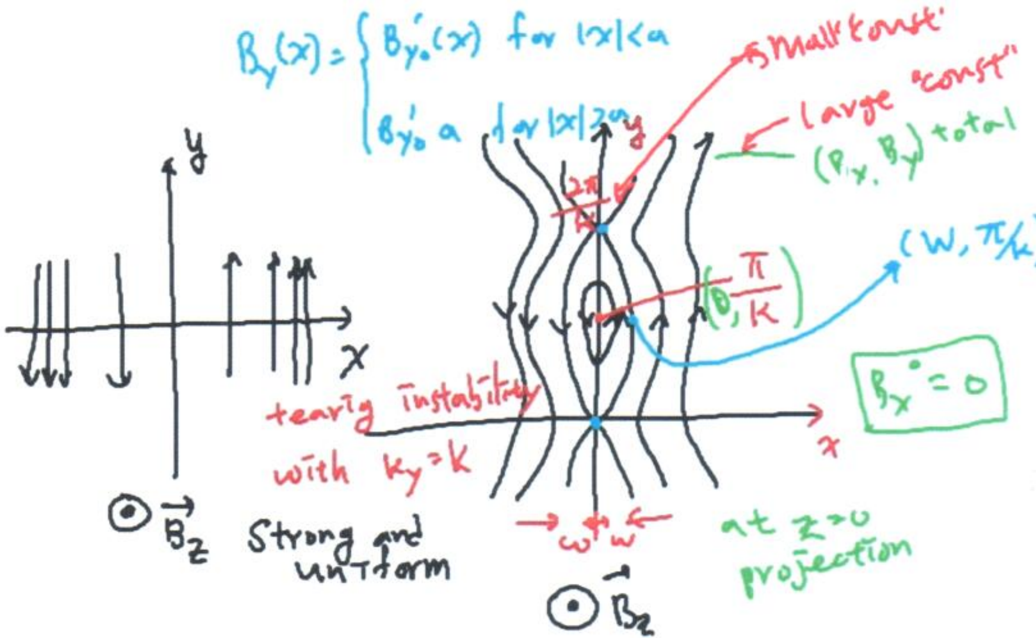
Structure of Magnetic Island

(Goldston and Rutherford section 20.6, page 357-)

From the slab "plasma current sheet" equilibrium

Total $\mathbf{B}$ follows



$B_y(x) = \begin{cases} B'_{y_0}(x) & \text{for } |x|<a \\ B'_{y_0}\, a & \text{for } |x|\geq a \end{cases}$
small "const"
large "const"
(B_x, B_y) total
$(W, \pi/k)$
$(0, \pi/k)$
$2\pi/k$
$B_x^{\circ} = 0$
tearing instability with $k_y = k$
$\odot\ \vec{B}_z$ Strong and uniform
$\odot\ \vec{B}_z$
at $z=0$ projection
w w
x
y

$$\frac{dx}{dl} = \frac{B_x}{B}, \quad \frac{dy}{dl} = \frac{B_y}{B}$$
$$\therefore \frac{dx}{dy} = \frac{B_x}{B_y} \tag{20.56}$$

$$B_y = B'_{y0}x + \cancel{\delta B_y}$$
$$B_x = \delta B_x = \bar{B}_x e^{\gamma t} \sin(ky) \tag{20.57}$$

$\Rightarrow$ integrate Equation (20.56) to get

$$\frac{1}{2}B'_{y0}x^2 + \frac{\bar{B}_x}{k}e^{\gamma t}\cos(ky) = \text{const.} \tag{20.58}$$

For large $|x|$, perturbation and distortion of **B** is small
For small $|x|$, perturbation and distortion of **B** is relatively large

For some values of "constant" which are small enough only a limited range of y can satisfy Equation (20.58)

$\Rightarrow$ Projection of **B** field lines "close on themselves"
$\Rightarrow$ "Magnetic Islands"

The surface which separates the closed field lines from the open field lines: "magnetic separatrix".

The half-width "w" of the magnetic island is determined by the defining the value of a constant separatrix.

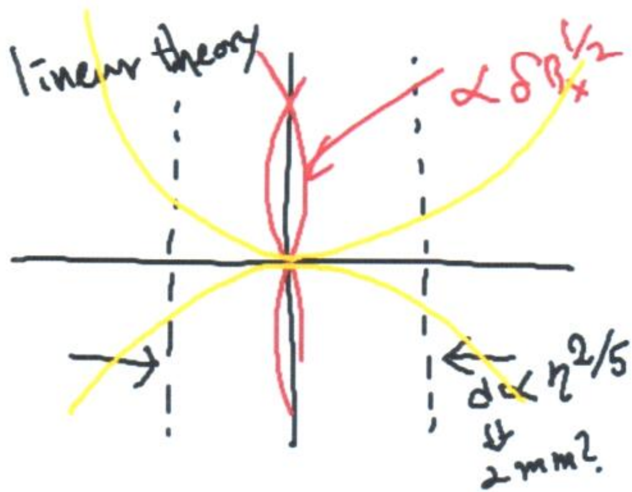
$$\text{at } (w, \frac{\pi}{k}) \quad \frac{1}{2}B'_{y0}w^2 - \frac{\bar{B}_x}{k}e^{\gamma t} = \frac{\bar{B}_x}{k}e^{\gamma t} \quad \text{at } (0, \frac{2\pi}{k})$$

$$\Rightarrow \; w = 2\left(\frac{\bar{B}_x}{kB'_{y0}}\right)^{1/2} e^{\gamma t/2} \propto \delta B_x^{1/2}$$

On the other hand, from the linear tearing mode theory

$$d \propto \eta^{2/5} \propto \delta B_x^0$$

where d is the resistive layer width.
The linear theory assumes $\delta B_x, \xi, \delta\psi, \cdots$ are infinitesimally)negligibly extremely) small!



γ linear growth rate assume $\bar{B}_x$ is roughly a constant in x.

However, magnetic islands with $w \sim 10\,\mathrm{cm}$ have been observed from experiments.
Slab current sheet equilibrium was motivated from a consideration that:

$$
\begin{aligned}
r &\to x = r - r_s \\
r\theta &\to y \\
R\phi &\to z \\
q &= \frac{B_\phi}{B_\theta}\frac{r}{R} \\
q(r_s) &= \frac{m}{n} \\
k_\theta &= \frac{m}{r} \\
k_\phi &= -\frac{n}{R} \\
k_\parallel &= \frac{\mathbf{k}\cdot\mathbf{B}}{|\mathbf{B}|} = \frac{m}{r}\frac{B_\theta}{B} - \frac{n}{R}\frac{B_\phi}{B} \\
&= \frac{B_\theta}{rB}\left(m - nq(r)\right) \\
q(r) &= q(r_s) + (r - r_s)\left(\frac{\partial q}{\partial r}\right)(r_s) + \cdots \\
&= \frac{m}{n} + \left(\frac{\partial q}{\partial r}\right)(r - r_s)
\end{aligned}
$$

Structure of Magnetic Island in Tokamak Geometry

From the slab "plasma curren sheet" equilibrium, we have

$$\chi = \theta - \frac{n}{m}\phi$$

χ is the "binormal angle" orthogonal to both r and coordinate along $\mathbf{B}$.

$$\delta \mathbf{B} \propto \exp(im\chi)$$

and $\mathbf{B}$ in binormal direction is given by

$$B^* = B_\theta \left(1 - \frac{n}{m} q(r)\right).$$

After an expression,

$$B^* = -B_\theta \frac{q'}{q}\bigg|_{r_s} x$$

where $x = r - r_s$. $\mathbf{B}$ field lines satisfy,

$$\frac{dr}{d\chi} = \frac{B_r}{B^*} \tag{7.2.2}$$

$$B^* = -B_\theta \frac{q'}{q}\bigg|_{r_s} x$$

$$B_r = \delta B_r = \bar{B}_r e^{\gamma t} \sin(m\chi)$$

$\Rightarrow$ integrate Eq. (7.2.2) to get

$$x^2 + \frac{w^2}{2}\cos(m\chi) = \frac{w^2}{2}\cos(m\chi_0)$$

Here "w" is defined as the half-width, while it is defined as a full-width in Wesson.

$$w = 2\left(\frac{rq\delta\bar{B}_r}{mq'B_\theta}\right)^{1/2} e^{\gamma t/2} \propto (\delta B_r)^{1/2}$$

Note that $w \searrow$ with $\hat{s} \nearrow$, $m \nearrow$ and $B_\theta \nearrow$.

From linear tearing mode theory, the resistive layer width $d \propto \eta^{2/5}$, and $\delta\bar{B}_r/B_0$ was assumed to be infinitesimally small.
$\therefore w \propto (\delta\bar{B}_r)^{1/2} < d$ should be satisfied for linear theory to be valid.

But w can exceed "d" easily in practice as δB_r grows!
$\Rightarrow$ linear theory of tearing mode should be modified.
Nonlinear Evolution of Magneitc Island should be applied