Fusion Plasma Theory II. 2019

Week 8

Nonlinear Evolution of Tearing Mode

The following resistive MHD equations should describe the resistive tearing mode.

$$\rho \frac{d}{dt}\mathbf{u} = -\boldsymbol{\nabla} p + \mathbf{j} \times \mathbf{B}$$
$$\frac{\partial}{\partial t}\mathbf{B} = \boldsymbol{\nabla} \times (\mathbf{u} \times \mathbf{B}) + \frac{\eta}{\mu_0}\nabla^2\mathbf{B}$$

Heuristic Derivation (Wesson 7.3. page 356)

As the magnetic island grows, the flow pattern around it slows down the growth of island and associated δB_r. Therefore, the inertia becomes negligible, and the force balance is maintained while magnetic island grows.

That is,

$$\rho \frac{d\mathbf{u}}{dt} = -\nabla p + \mathbf{j} \times \mathbf{B} \tag{125}$$

$$\frac{\partial}{\partial t}\delta B_r = \nabla \times (\delta \mathbf{u} \times \mathbf{B}_0)\big|_r + \frac{\eta}{\mu_0}\nabla^2 \delta B_r \tag{126}$$

in the region near island. Also, in this region,

$$\frac{\partial^2}{\partial r^2} = \frac{\partial^2}{\partial x^2} \gg k_y^2.$$

By integrating Eq. (126) radially over an island width, we obtain

$$w\frac{\partial}{\partial t}\delta B_r = \frac{\eta}{\mu_0}\frac{\partial \delta B_r}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2}.$$

Since "δB_r"$= -im$"$\delta\psi$"$/r$ and $w \propto \delta B_r^{1/2}$,

$$\frac{dw}{dt} \simeq \frac{\eta}{2\mu 0}\frac{1}{\delta B_r}\frac{\partial \delta B_r}{\partial r}\bigg|_{r_s-w/2}^{r_s+w/2} = \frac{\eta}{2\mu_0}\frac{\frac{\partial}{\partial r}\delta\psi}{\delta\psi}\bigg|_{r_s-w/2}^{r_s+w/2} = \frac{\eta}{2\mu_0}\Delta'(w)$$

Here, $\Delta'(w)$ is defined by a jump condition from $r - r_s = \pm w/2$. Therefore, for a thin island,

$$w \simeq \frac{\eta}{2\mu_0}\Delta'(0)t.$$

We recall from the linear theory (in the linear phase), an exponential growth in time is predicted.

$$\text{i.e.} \quad \delta B_r \sim e^{\gamma t}, \quad w \sim e^{\gamma t/2} \ :$$

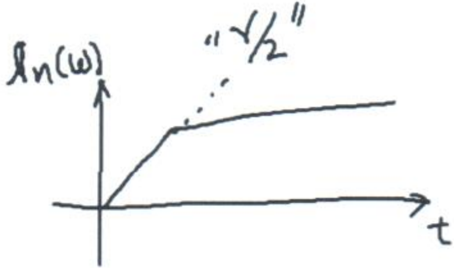
Now, from the nonlinear theory (in the nonlinear phase), we predict an algebraic growth in time

$$w \sim t, \quad \delta B_r \sim t^2. \qquad :$$

Note that the linear definition of Δ' was

$$\Delta' \equiv \left.\frac{\delta\psi'}{\delta\psi}\right|_{r_s+0} - \left.\frac{\delta\psi'}{\delta\psi}\right|_{r_s-0}$$

from the ideal MHD region.



More Formal Derivation of Nonlinear Evolution of Magnetic Island:

By taking a curl ($\nabla \times$) of

$$\nabla p \simeq \mathbf{j} \times \mathbf{B},$$

and using the high aspect ratio tokamaks ordering, we obtain

$$0 \simeq \boldsymbol{\nabla} \times (\mathbf{j} \times \mathbf{B}) \simeq (\mathbf{B} \cdot \boldsymbol{\nabla})\mathbf{j}. \text{ Therefore } \mathbf{j} \simeq j(\psi)\mathbf{e}_\phi$$

Then, the Ohm's law can be written as

$$\frac{\partial \psi}{\partial t} + \mathbf{B} \cdot \boldsymbol{\nabla}\Phi = \eta j(\psi) = \frac{\eta}{\mu_0}\nabla_\perp^2 \psi$$

where the 1st term of LHS is the induction term, and the 2nd term is the electrostatic part of $E_\parallel$. Here Φ is the electrostatic potential, (not a toroidal angle).
We note that a flux surface average gets rid of the 2nd term on the LHS (because it is a perfect derivative).

$$\left\langle \frac{\partial \psi}{\partial t} \right\rangle = \frac{\eta}{\mu_0} \left\langle \frac{\partial^2}{\partial x^2} \psi \right\rangle = \frac{\eta}{\mu_0}\Delta' \psi \tag{127}$$

where

$$\langle f \rangle = \frac{\oint dl\, f / |\boldsymbol{\nabla}\psi|}{\oint dl / |\boldsymbol{\nabla}\psi|} \text{ denotes the flux surface average.}$$

This integration can be explicitly evaluated, as shown in a pioneering paper [P.H. Rutherford, Phys. Fluids 16, 1903 (1973)].

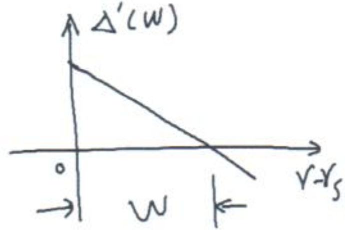
$$\frac{dw}{dt} = 1.22\frac{\eta}{\mu_0}\Delta'(w). \tag{128}$$

By integrating in time

$$w \simeq 1.22\frac{\eta}{\mu_0}\Delta'(0)t \tag{129}$$

So-called "Rutherford regime" refers to slow growth of an island, for a thin island.
As island grows it modifies the T_e profile and J_ϕ profile consequently, and Δ' evaluated at $r = r_s \pm w/2$ is also modified.
Island will growth until $\Delta'(w) = 0$, because

$$\Delta'(w) \simeq \Delta'(0) - \alpha w.$$



"Island width saturation"

This classical tearing mode theory still requires $\Delta'(0) > 0$ for an unstable tearing mode and an existence of magnetic island.
However, there have been many tokamak experiments which have reported the observation results of magnetic perturbation suggesting the existence of magnetic island even when $\Delta' < 0$.