Introduction to Nuclear Fusion

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Tokamak equilibrium

Tokamak Equilibrium

$$\nabla p = \vec{J} \times \vec{B}$$

$$\nabla \times \vec{B} = \mu_0 \vec{J}$$

$$\nabla \cdot \vec{B} = 0$$

- Consider the axisymmetric torus, the simplest, multi-dimensional configuration
- We shall derive the Grad–Shafranov equation for axisymmetric equilibria.
- This provides a complete description of toroidal equilibrium:

  radial pressure balance, toroidal force balance,

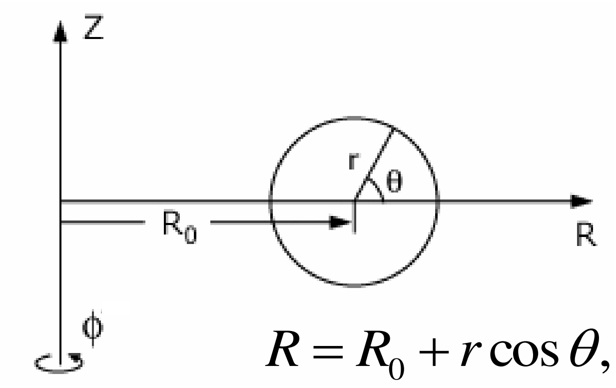
  β limits, q-profiles, etc.

Tokamak Equilibrium

- **The Grad-Shafranov Equation**

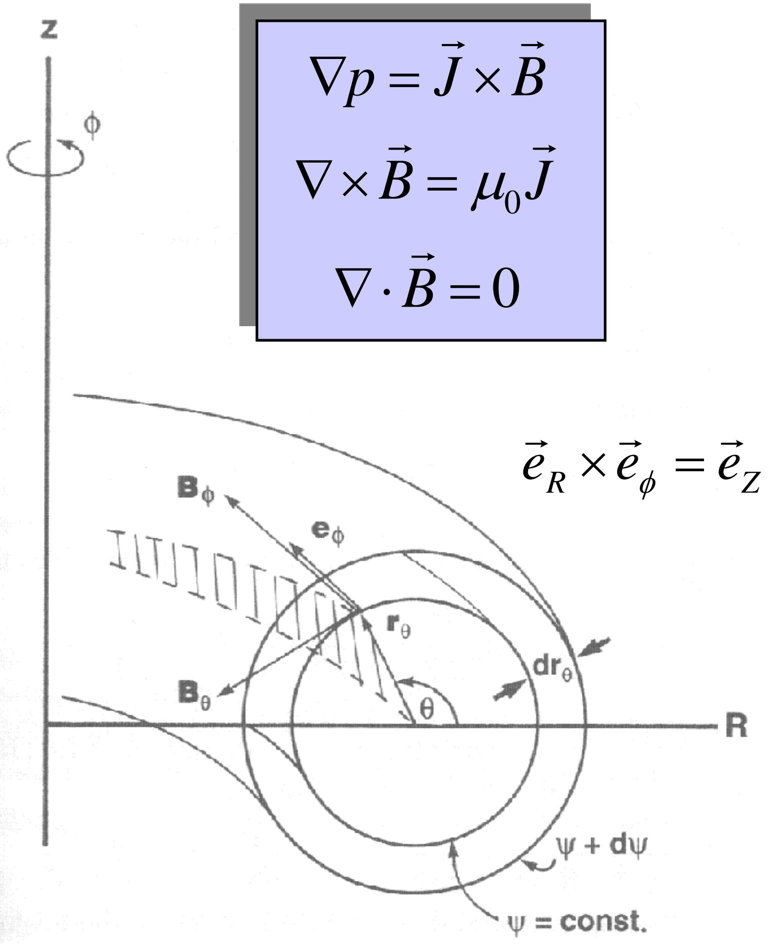
$$\Delta^{*}\psi = -\mu_0 R^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi}$$

- Nonlinear
- Partial differential equation
- Grad and Rubin (1958), Shafranov (1960)
- Toroidal axisymmetric $\partial/\partial\Phi = 0$



$$R = R_0 + r\cos\theta,\quad Z = r\sin\theta$$

$$\nabla p = \vec{J}\times\vec{B}$$
$$\nabla\times\vec{B} = \mu_0\vec{J}$$
$$\nabla\cdot\vec{B} = 0$$



$$\vec{e}_R\times\vec{e}_\phi = \vec{e}_Z$$

J.P. Freidberg, "Ideal Magneto-Hydro-Dynamics", lecture note

A. A. Harms et al, "Principles of Fusion Energy", World Scientific (2000)

Tokamak Equilibrium

- **The Grad-Shafranov Equation**

Sequence of solution of the MHD equilibrium equations

1. The $\nabla\cdot\mathbf{B} = 0$
2. Ampere's law: $\mu_0\mathbf{J} = \nabla\times\mathbf{B}$
3. The momentum equation: $\mathbf{J}\times\mathbf{B} = \nabla p$

$$\nabla p = \vec{J}\times\vec{B}$$
$$\nabla\times\vec{B} = \mu_0\vec{J}$$
$$\nabla\cdot\vec{B} = 0$$

$$\vec{J}\times\vec{B} = \nabla p$$

$$\nabla p = \left(\vec{J}_\phi + \vec{J}_p\right)\times\left(\vec{B}_\phi + \vec{B}_p\right)$$

Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - Momentum equation $\vec{J} \times \vec{B} = \nabla p$

$$\nabla p = \left(\vec{J}_\phi + \vec{J}_p\right) \times \left(\vec{B}_\phi + \vec{B}_p\right)$$

- aim: to express each term with ψ and F $\qquad \psi = \frac{\psi_p}{2\pi}, \quad F = \frac{\mu_0 I_p}{2\pi}$

$$\nabla \cdot \vec{B} = 0 \longrightarrow \vec{B}_p = -\frac{1}{R} \vec{e}_\phi \times \nabla \psi$$

$$\mu_0 \vec{J}_p = \nabla \times \vec{B}_\phi \longrightarrow \vec{J}_p = -\frac{1}{\mu_0 R} \vec{e}_\phi \times \nabla F, \quad \vec{B}_\phi = \frac{F}{R} \vec{e}_\phi$$

$$\mu_0 \vec{J}_\phi = \left(\nabla \times \vec{B}\right)_\phi \longrightarrow \vec{J}_\phi = -\frac{1}{\mu_0 R} \Delta^* \psi \vec{e}_\phi$$

Tokamak Equilibrium

- **The Grad-Shafranov Equation**

  - The ▽·**B** Equation

    In cylindrical coordinates for toroidal axisymmetric fields ($\partial/\partial\Phi=0$)

$$\nabla\cdot\vec{B}=0 \qquad \frac{1}{R}\frac{\partial(RB_R)}{\partial R}+\frac{1}{R}\frac{\partial B_\phi}{\partial\phi}+\frac{\partial B_Z}{\partial Z}=0$$

$$\vec{B}\cdot\nabla\psi=0 \qquad B_R\frac{\partial\psi}{\partial R}+B_Z\frac{\partial\psi}{\partial Z}=0$$

Magnetic flux surface

$$B_R=-\frac{1}{R}\frac{\partial\psi}{\partial Z},\quad B_Z=\frac{1}{R}\frac{\partial\psi}{\partial R}$$

Ψ: Stream function for the poloidal magnetic field

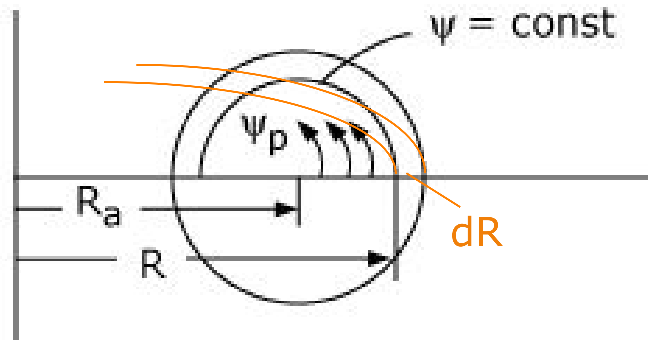
Tokamak Equilibrium

- **The Grad-Shafranov Equation**

  - The stream function ψ is closely related to the poloidal flux in the plasma.

$$
\begin{aligned}
\psi_p &= \int \vec{B}_p \cdot d\vec{A} \\
&= \int B_Z(R, Z=0) R\, d\phi\, dR \\
&= \int_0^{2\pi} d\phi \int_{R_a}^{R} R\, dR\, B_Z(R, Z=0) \\
&= \int_{R_a}^{R} 2\pi R \frac{1}{R} \frac{\partial \psi}{\partial R} dR \\
&= 2\pi\left[\psi(R,0) - \psi(R_a,0)\right] = 2\pi\psi
\end{aligned}
$$

$$\leftarrow \quad B_Z = \frac{1}{R}\frac{\partial \psi}{\partial R}$$



Poloidal flux on axis is zero

Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - The $\nabla\cdot$**B** Equation

$$\nabla\cdot\vec{B}=0 \qquad \frac{1}{R}\frac{\partial(RB_R)}{\partial R}+\frac{1}{R}\frac{\partial B_\phi}{\partial \phi}+\frac{\partial B_Z}{\partial Z}=0$$

$$\vec{B}\cdot\nabla\psi=0 \qquad B_R\frac{\partial\psi}{\partial R}+B_Z\frac{\partial\psi}{\partial Z}=0$$

Magnetic flux surface

$$B_R=-\frac{1}{R}\frac{\partial\psi}{\partial Z},\quad B_Z=\frac{1}{R}\frac{\partial\psi}{\partial R}$$

Ψ: Stream function for the poloidal magnetic field

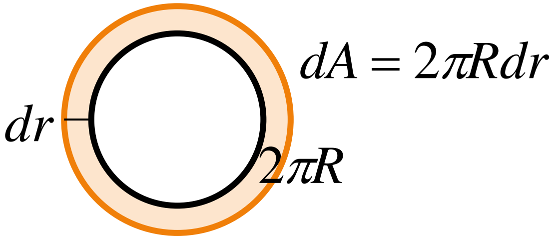
$$2\pi\psi=\psi_p=\int\vec{B}_p\cdot d\vec{A}$$

$$\vec{B}=B_\phi\vec{e}_\phi+\vec{B}_p$$

$$2\pi d\psi=2\pi(\psi+d\psi)-2\pi\psi=\int_{A+\delta A}\vec{B}_p\cdot d\vec{A}-\int_A\vec{B}_p\cdot d\vec{A}=\int_{\delta A}\vec{B}_p\cdot d\vec{A}\approx B_p 2\pi R dr$$

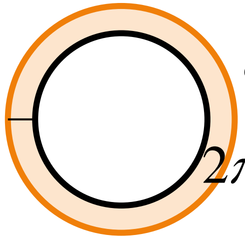
$$RB_p=\frac{d\psi}{dr}=|\nabla\psi|$$

$$\vec{B}_p=-\frac{1}{R}\vec{e}_\phi\times\nabla\psi$$



Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - Ampere's law (poloidal component)



$$\mu_0 \vec{J}_p = \nabla \times \vec{B}_\phi$$

$$\int_{\delta A} \nabla \times \vec{B}_\phi \cdot d\vec{A} = \oint \vec{B}_\phi \cdot d\vec{l} = 2\pi d(RB_\phi) = 2\pi d\psi \frac{\partial(RB_\phi)}{\partial\psi}$$

$$\int_{\delta A} \mu_0 \vec{J}_p \cdot d\vec{A} = \mu_0 J_p 2\pi R dr$$

$$2\pi d\psi \frac{\partial(RB_\phi)}{\partial\psi} = \mu_0 J_p 2\pi R dr$$

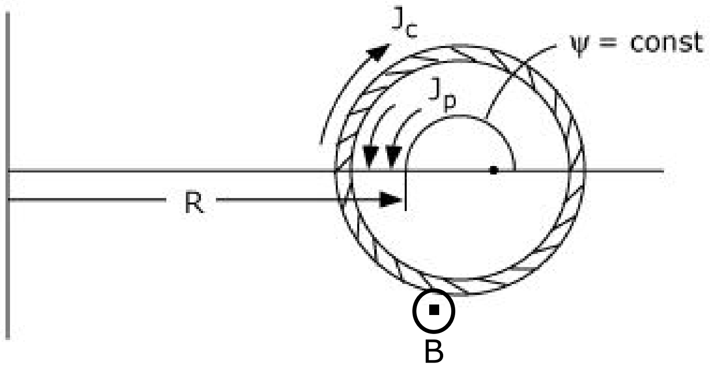
$$\Rightarrow \frac{\partial(RB_\phi)}{\partial\psi} = \frac{\mu_0 J_p R}{(d\psi / dr)} = \frac{\partial F}{\partial\psi} \quad \leftarrow \quad F(\psi) \equiv RB_\phi$$

$$\Rightarrow \mu_0 J_p R = \frac{\partial F}{\partial\psi}\frac{d\psi}{dr} = \frac{\partial F}{\partial\psi}|\nabla\psi| = \frac{\partial F}{\partial r} = |\nabla F(\psi)|$$

$$J_p = \frac{|\nabla F|}{\mu_0 R} \qquad \vec{J}_p = -\frac{1}{\mu_0 R}\vec{e}_\phi \times \nabla F, \quad B_\phi = \frac{F}{R}$$

Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - Interpretation of $F(\psi)$



$$
\begin{aligned}
I_p &= \int \vec{J}_p \cdot d\vec{A} \\
&= \int J_Z(R, Z=0) R\, d\phi\, dR \\
&= \int_0^{2\pi} d\phi \int_0^R R\, dR\, J_Z(R, Z=0) \\
&= \int_0^{2\pi} d\phi \int_0^R R\, dR \frac{1}{\mu_0 R}\frac{\partial}{\partial R}(RB_\phi) \\
&= \int_0^R 2\pi R \frac{1}{\mu_0 R}\frac{\partial F}{\partial R} dR \\
&= 2\pi\left[F(R,0) - F(0,0)\right]/\mu_0 \\
&= 2\pi F(\psi)/\mu_0
\end{aligned}
$$

$$\leftarrow J_Z = \frac{1}{\mu_0}\frac{1}{R}\frac{\partial}{\partial R}\left(RB_\phi\right), \quad F(\psi) \equiv RB_\phi$$

$$R = 0 \rightarrow F(\psi) = 0$$

$I_p(\psi)$ is the total poloidal current and the toroidal field coil currents passing through the circle $\psi(R, 0)$ = const.

Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - Ampere's law (toroidal component)

$$\mu_0 \vec{J} = \nabla \times \vec{B}$$

$$J_\phi = \frac{1}{\mu_0}\left(\frac{\partial B_R}{\partial Z} - \frac{\partial B_Z}{\partial R}\right) = \frac{1}{\mu_0}\left(-\frac{1}{R}\frac{\partial^2 \psi}{\partial Z^2} - \frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right)\right)$$

$$\equiv -\frac{1}{\mu_0 R}\Delta^* \psi \qquad \leftarrow \quad B_R = -\frac{1}{R}\frac{\partial \psi}{\partial Z}, \quad B_Z = \frac{1}{R}\frac{\partial \psi}{\partial R}$$

$$\Delta^* \psi \equiv R\frac{\partial}{\partial R}\left(\frac{1}{R}\frac{\partial \psi}{\partial R}\right) + \frac{\partial^2 \psi}{\partial Z^2}$$

elliptic operator

Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - Momentum equation $\vec{J}\times\vec{B}=\nabla p$

$$\nabla p=\left(\vec{J}_\phi+\vec{J}_p\right)\times\left(\vec{B}_\phi+\vec{B}_p\right)$$

$$=\left(\vec{J}_\phi-\frac{1}{\mu_0 R}\vec{e}_\phi\times\nabla F\right)\times\left(\vec{B}_\phi-\frac{1}{R}\vec{e}_\phi\times\nabla\psi\right)$$

$$=-\vec{J}_\phi\times\left(\frac{1}{R}\vec{e}_\phi\times\nabla\psi\right)-\left(\frac{1}{\mu_0 R}\vec{e}_\phi\times\nabla F\right)\times\vec{B}_\phi$$

$$=-\frac{1}{R}\vec{e}_\phi\left(\vec{J}_\phi\cdot\nabla\psi\right)+\nabla\psi\left(\vec{J}_\phi\cdot\frac{1}{R}\vec{e}_\phi\right)+\vec{B}_\phi\times\left(\frac{1}{\mu_0 R}\vec{e}_\phi\times\nabla F\right)$$

$$=\frac{\vec{J}_\phi}{R}\nabla\psi+\frac{1}{\mu_0 R}\vec{e}_\phi\left(\vec{B}_\phi\cdot\nabla F\right)-\nabla F\left(\vec{B}_\phi\cdot\frac{1}{\mu_0 R}\vec{e}_\phi\right)$$

$$=\frac{J_\phi}{R}\nabla\psi-\frac{B_\phi}{\mu_0 R}\nabla F$$

Symmetry →

$$\vec{J}_p=-\frac{1}{\mu_0 R}\vec{e}_\phi\times\nabla F$$

$$\vec{B}_p=-\frac{1}{R}\vec{e}_\phi\times\nabla\psi$$

$$B_\phi=\frac{F(\psi)}{R}$$

$$J_\phi=-\frac{1}{\mu_0 R}\Delta^*\psi$$

← $A\times(B\times C)=B(A\cdot C)-C(A\cdot B)$

Tokamak Equilibrium

- **The Grad-Shafranov Equation**
  - Momentum equation $\vec{J} \times \vec{B} = \nabla p$

Symmetry

$$\vec{J}_p = -\frac{1}{\mu_0 R} \vec{e}_\phi \times \nabla F$$

$$\vec{B}_p = -\frac{1}{R} \vec{e}_\phi \times \nabla \psi$$

$$B_\phi = \frac{F(\psi)}{R}$$

$$J_\phi = -\frac{1}{\mu_0 R} \Delta^* \psi$$

$$\nabla p = \left(\vec{J}_\phi + \vec{J}_p\right) \times \left(\vec{B}_\phi + \vec{B}_p\right) = \frac{J_\phi}{R} \nabla \psi - \frac{B_\phi}{\mu_0 R} \nabla F$$

$$\begin{aligned} J_\phi &= R \frac{\nabla p}{\nabla \psi} + \frac{B_\phi}{\mu_0} \frac{\nabla F}{\nabla \psi} \\ &= R \frac{dp}{d\psi} + \frac{B_\phi}{\mu_0} \frac{dF}{d\psi} \\ &= R \frac{dp}{d\psi} + \frac{F(\psi)}{\mu_0 R} \frac{dF}{d\psi} \\ &= -\frac{1}{\mu_0 R} \Delta^* \psi \end{aligned}$$

Tokamak Equilibrium

- **The Grad-Shafranov Equation**

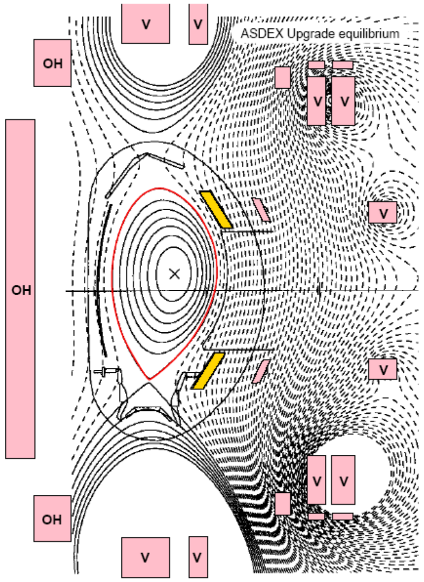
$$\Delta^{*}\psi = -\mu_0 R^2 \frac{dp}{d\psi} - F\frac{dF}{d\psi}$$

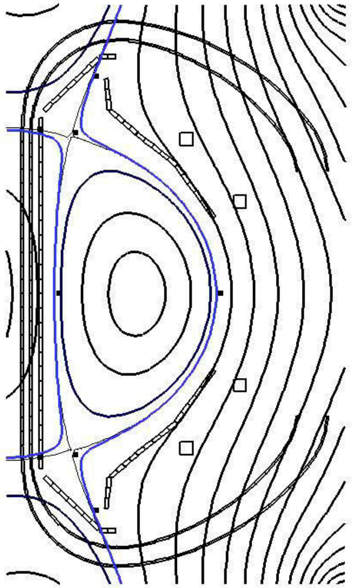
↑ $\mathbf{J_\Phi \times B_p}$: strength of $\mathbf{B_p}$

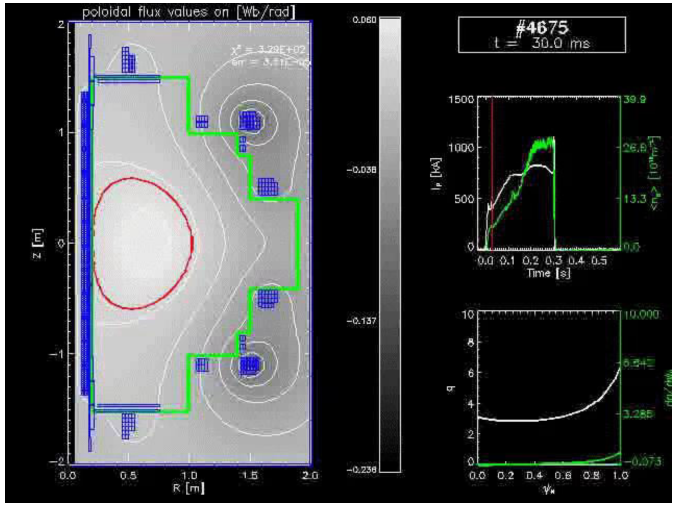
↑ ∇p: plasma load on a magnetic flux surface

↑ $\mathbf{J_p \times B_\Phi}$: strength of $\mathbf{B_\Phi}$

- BCs: provided by the transformer-induced poloidal magnetic field outside the plasma
- In practice, the G-S equation is solved numerically to find the geometrical location of the magnetic surfaces and the radial distribution of the axial current density in a way that is consistent with the experimentally measured pressure profiles (p) and the externally applied field (F).







Force balance in a tokamak

Tokamak Equilibrium

- **The Grad-Shafranov Equation**

$$\Delta^* \psi = -\mu_0 R^2 \frac{dp}{d\psi} - F \frac{dF}{d\psi}$$

↑ $\mathbf{J}_\Phi \times \mathbf{B}_p$: strength of $\mathbf{B}_p$

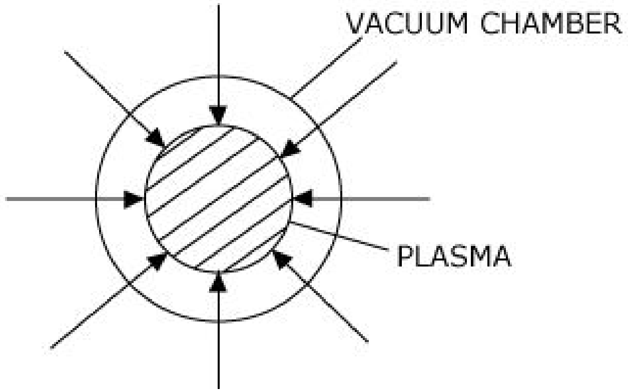
↑ ∇p: plasma load on a magnetic flux surface

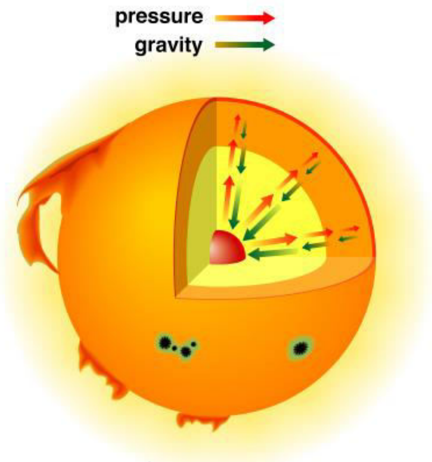
↑ $\mathbf{J}_p \times \mathbf{B}_\Phi$: strength of $\mathbf{B}_\Phi$

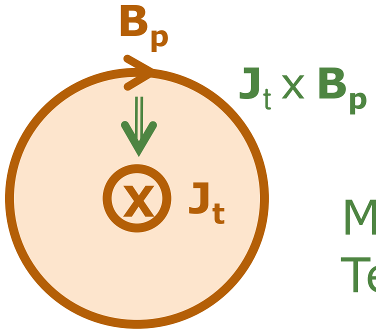
What kind of forces does a plasma have regarding equilibrium?

Tokamak Equilibrium

- **Basic Forces Acting on Tokamak Plasmas**
  - Radial force balance

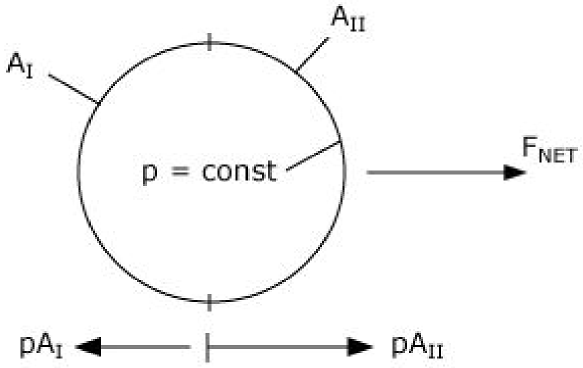






Magnetic pressure,
Tension force

  - Toroidal force balance: Tire tube force

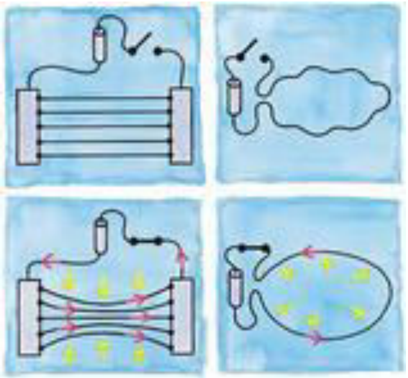


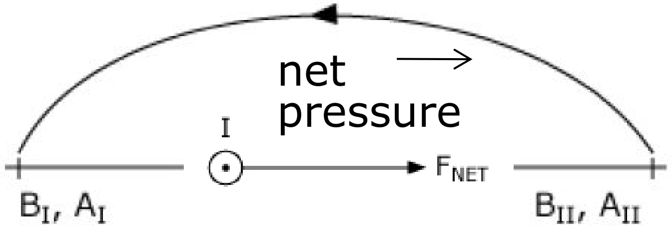
$$F_{NET} \sim -e_R(pA_I - pA_{II})$$

J.P. Freidberg, "Ideal Magneto-Hydro-Dynamics", lecture note

Tokamak Equilibrium

- **Basic Forces Acting on Tokamak Plasmas**
  - Toroidal force balance: Hoop force





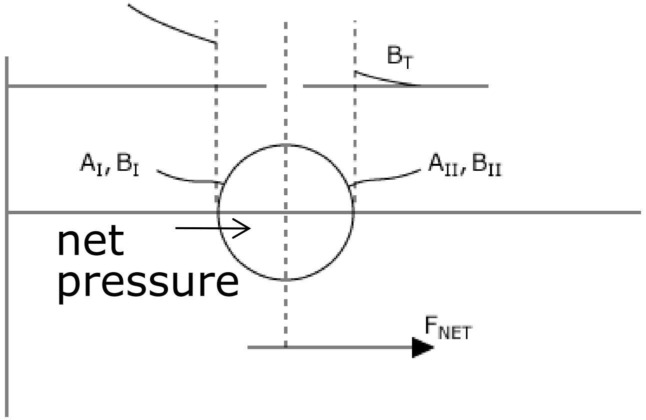
$$F_{NET} \sim e_R (B_I^2 A_I - B_{II}^2 A_{II}) / 2\mu_0$$

$$\phi_I = \phi_{II}$$

$$B_I > B_{II}, \quad A_I < A_{II}$$

$$B_I^2 A_I > B_{II}^2 A_{II}$$

  - Toroidal force balance: 1/*R* force



$$B_I > B_{II}, \quad A_I < A_{II}$$

$$B_I^2 A_I > B_{II}^2 A_{II}$$

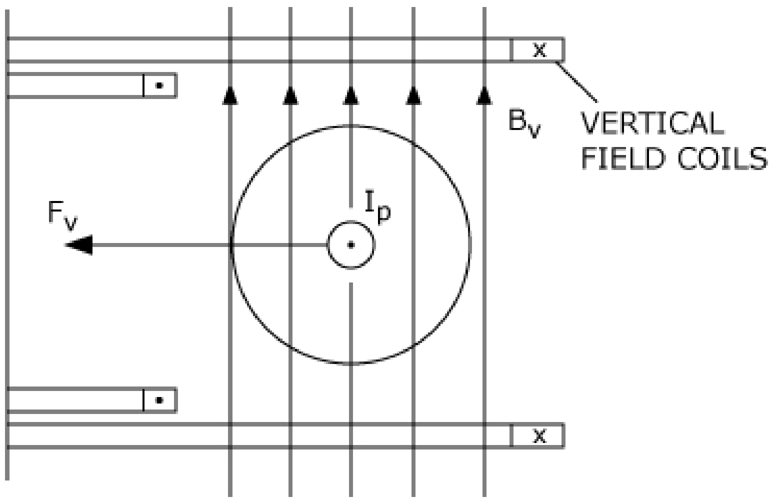
$$F_{NET} \sim e_R (B_I^2 A_I - B_{II}^2 A_{II}) / 2\mu_0 = 2\pi^2 a^2 \frac{B^2}{2\mu_0}$$

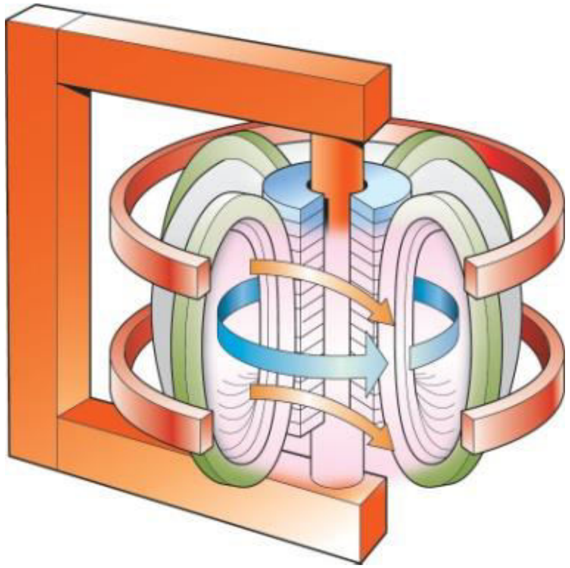
J.P. Freidberg, "Ideal Magneto-Hydro-Dynamics", lecture note

Tokamak Equilibrium

- **Basic Forces Acting on Tokamak Plasmas**
  - External coils required to provide the force balance

$\vec{I}_p \times \vec{B}_v$





$$F_v = BIL = 2\pi R_0 I_p B_v$$

$$B_\phi > B_\theta > B_v$$

How about vertical movement?

References

Lesch, Astrophysics, IPP Summer School (2008)