INVISCID FLOW
Week 4

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Basic Conservation Laws – Viscosity Coefficients

- Let's find a physical meaning of two parameters, λ and μ
- Consider a simple shear flow in a incompressible fluid where the velocity is defined as u = u(y) and v = w = 0.

$$\sigma_{ij} = -p\delta_{ij} + \lambda\delta_{ij}\frac{\partial u_k}{\partial x_k} + \mu\left(\frac{\partial u_j}{\partial x_i} + \frac{\partial u_i}{\partial x_j}\right) \Rightarrow \begin{aligned} &\sigma_{12} = \sigma_{21} = \mu\frac{du}{dy} \\ &\sigma_{11} = \sigma_{22} = \sigma_{33} = -p \\ &\sigma_{13} = \sigma_{31} = \sigma_{23} = \sigma_{32} = 0 \end{aligned}$$

- from Newton's law of viscosity, the proportionality factor (μ) between the shear stress and the velocity gradient in a simple shear flow is the dynamic viscosity
  – Kinematic viscosity υ = μ/ρ

Basic Conservation Laws – Viscosity Coefficients

- λ is usually referred as a second viscosity coefficient
- Let's consider the average of normal stress components as

$$-\bar{p} = \frac{1}{3}(\sigma_{11} + \sigma_{22} + \sigma_{33})$$

  - This is mechanical pressure, i.e., coming from hydrostatics and/or from the motion of a fluid
  - Different from the thermodynamic pressure
  - σ_{ii} = trace of $\boldsymbol{\sigma}$

$$-\bar{p} = -p + \lambda \frac{\partial u_k}{\partial x_k} + \frac{2}{3}\mu \frac{\partial u_k}{\partial x_k} = -p + (\lambda + \frac{2}{3}\mu)\frac{\partial u_k}{\partial x_k}$$

  - Difference between the thermodynamic pressure and the mechanical pressure is proportional to the divergence of the velocity vector

Basic Conservation Laws – Viscosity Coefficients

$$p - \bar{p} = (\lambda + \frac{2}{3}\mu)\frac{\partial u_k}{\partial x_k} = \mathrm{K}\frac{\partial u_k}{\partial x_k}$$ K: bulk viscosity

- What is bulk viscosity?
  - mechanical pressure is a measure of the translational energy of the molecules only
  - thermodynamic pressure is a measure of the total energy, which includes vibrational and rotational modes of energy as well as the translational mode
  - in a flow field, it is possible to have energy transferred from one mode to another
  - bulk viscosity is a measure of this transfer of energy from the translational mode to the other modes
  - If the fluid is a monatomic gas, the only mode of molecular energy is the translational mode. Then, the mechanical pressure and thermodynamic pressure are the same: $\lambda = -\frac{2}{3}\mu$ Stoke's relation

Basic Conservation Laws – Navier-Stokes Equation

$$\rho\frac{\partial u_j}{\partial t}+\rho u_k\frac{\partial u_j}{\partial x_k}=\frac{\partial \sigma_{ij}}{\partial x_i}+\rho f_j \qquad \sigma_{ij}=-p\delta_{ij}+\lambda\delta_{ij}\frac{\partial u_k}{\partial x_k}+\mu\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)$$

- Momentum conservation eqn + Newtonian fluid constitutive eqn → Navier-Stokes Eqn

$$\frac{\partial \sigma_{ij}}{\partial x_i}=\frac{\partial}{\partial x_i}\left[-p\delta_{ij}+\lambda\delta_{ij}\frac{\partial u_k}{\partial x_k}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]$$

$$=-\frac{\partial p}{\partial x_j}+\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial u_k}{\partial x_k}\right)+\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]$$

$$\rho\frac{\partial u_j}{\partial t}+\rho u_k\frac{\partial u_j}{\partial x_k}=-\frac{\partial p}{\partial x_j}+\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial u_k}{\partial x_k}\right)+\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]+\rho f_j$$

Basic Conservation Laws – Navier-Stokes Equation

- In incompressible (constant density) and constant viscosity flow,

$$\rho\frac{\partial u_j}{\partial t}+\rho u_k\frac{\partial u_j}{\partial x_k}=-\frac{\partial p}{\partial x_j}+\frac{\partial}{\partial x_j}\left(\lambda\frac{\partial u_k}{\partial x_k}\right)+\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]+\rho f_j$$

$$\frac{\partial}{\partial x_i}\left[\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]=\mu\left[\frac{\partial}{\partial x_j}\left(\frac{\partial u_i}{\partial x_i}\right)+\frac{\partial^2 u_j}{\partial x_i \partial x_i}\right]=\mu\frac{\partial^2 u_j}{\partial x_i \partial x_i}$$

$$\rho\frac{\partial u_j}{\partial t}+\rho u_k\frac{\partial u_j}{\partial x_k}=-\frac{\partial p}{\partial x_j}+\mu\frac{\partial^2 u_j}{\partial x_i \partial x_i}+\rho f_i$$

- If we can ignore the effect of viscosity,

$$\rho\frac{\partial u_j}{\partial t}+\rho u_k\frac{\partial u_j}{\partial x_k}=-\frac{\partial p}{\partial x_j}+\rho f_i$$

Euler Equation → Inviscid Flow

Basic Conservation Laws – Energy Equations

- The first term at RHS of energy conservation eqn,

$$\rho\frac{\partial e}{\partial t}+\rho u_k\frac{\partial e}{\partial x_k}=\sigma_{ij}\frac{\partial u_j}{\partial x_i}-\frac{\partial q_j}{\partial x_j}$$

Work done by the surface stresses

$$\sigma_{ij}\frac{\partial u_j}{\partial x_i}=\left[-p\delta_{ij}+\lambda\delta_{ij}\frac{\partial u_k}{\partial x_k}+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right]\frac{\partial u_j}{\partial x_i}$$

$$=-p\frac{\partial u_k}{\partial x_k}+\lambda\left(\frac{\partial u_k}{\partial x_k}\right)^2+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_j}{\partial x_i}$$

reversible transfer of energy due to compression

Dissipation function, Φ

Basic Conservation Laws – Energy Equations

$$\Phi=\lambda\left(\frac{\partial u_k}{\partial x_k}\right)^2+\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_j}{\partial x_i}$$

- This is a measure of the rate at which mechanical energy is being converted into thermal energy → Dissipation function
- For an incompressible flow,

$$\Phi=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\frac{\partial u_j}{\partial x_i}$$

$$=\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\left[\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}-\frac{\partial u_i}{\partial x_j}\right)+\frac{1}{2}\left(\frac{\partial u_j}{\partial x_i}+\frac{\partial u_i}{\partial x_j}\right)\right]$$

$$=\frac{1}{2}\mu\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)^2$$

→ Positive definite → dissipation function always works to increase irreversibly the internal energy of an incompressible fluid

Basic Conservation Laws – Energy Equations

- Using the dissipation function and Fourier's law, the energy eqn becomes

$$\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = -p \frac{\partial u_k}{\partial x_k} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j} \right) + \Phi$$

$$\Phi = \lambda \left(\frac{\partial u_k}{\partial x_k} \right)^2 + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_j}{\partial x_i}$$

Basic Conservation Laws – Governing Equations

- For newtonian fluids, in summary

$$\frac{\partial \rho}{\partial t} + \frac{\partial}{\partial x_k}(\rho u_k) = 0$$

$$\rho \frac{\partial u_j}{\partial t} + \rho u_k \frac{\partial u_j}{\partial x_k} = -\frac{\partial p}{\partial x_j} + \frac{\partial}{\partial x_j}\left(\lambda \frac{\partial u_k}{\partial x_k} \right) + \frac{\partial}{\partial x_i}\left[\mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \right] + \rho f_j$$

$$\rho \frac{\partial e}{\partial t} + \rho u_k \frac{\partial e}{\partial x_k} = -p \frac{\partial u_k}{\partial x_k} + \lambda \left(\frac{\partial u_k}{\partial x_k} \right)^2 + \mu \left(\frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \frac{\partial u_j}{\partial x_i} + \frac{\partial}{\partial x_j}\left(k \frac{\partial T}{\partial x_j} \right)$$

$$p = p(\rho, T) \ \text{ e.g., } p = \rho R T \ \text{ for ideal gas}$$

$$e = e(\rho, T) \text{ e.g., } e = C_v T$$

  - Seven unknowns: p, ρ, e, T, and u_j

Basic Conservation Laws – Boundary Conditions

- Navier-Stokes equations are, mathematically, a set of three elliptic, second-order PDEs. The appropriate type of boundary conditions are therefore Dirichlet or Neumann conditions on a closed boundary
  - Dirichlet condition: one prescribes the value of a variable at the boundary, e.g. $u(x) = \text{constant}$
  - Neumann condition: one prescribes the gradient normal to the boundary of a variable at the boundary, e.g. $\partial u/\partial n = \text{constant}$.
- Physically, this usually amounts to specifying the velocity on all solid boundaries.
  - No-slip condition except for a few special cases