INVISCID FLOW
Week 10

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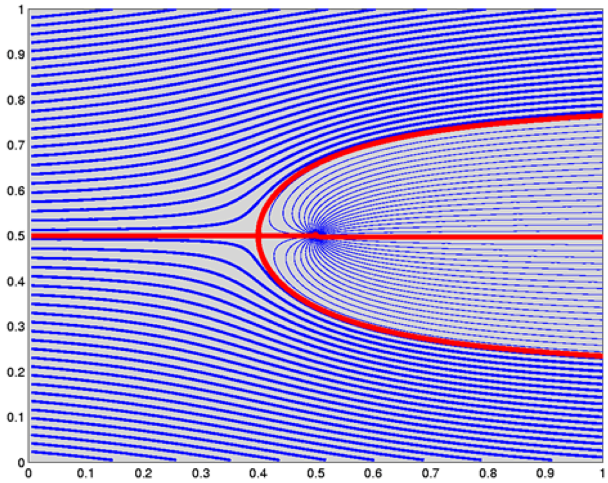
Panel Method

- Panel method (surface singularity method) is a technique for solving potential flow over 2-D and 3-D geometries.
  - Inviscid, incompressible and irrotational fluid
- Basic idea
  - Approximate the surface of a body (e.g., an airfoil) by a series of line segments (i.e., panels)
  - Distribute singularities (e.g., sources, vortices or doublets) on each panel
  - Derive the velocity components based on the contributions from all singularities
- Advantage
  - Need not to generate complex grids
  - Get results within a relatively short time
  - Capable of treating different kinds of geometries

Panel Method

- Main idea
  - Uniform flow + Source



  - We can mimic a complex flow with proper distributions of sources/sinks, vortices, doublets, and etc.

Panel Method

- There are many choices as to how to formulate a panel method
  - singularity solutions, variation within a panel, singularity strength and distribution, etc.
  - The simplest and practical one is based on a distribution of sources and vortices on the surface of the geometry.

$$\phi = \phi_\infty + \phi_s + \phi_v \tag{1}$$

  - where, ϕ is the total potential function and its three components are potentials corresponding to the free stream, the source/sink distribution, and the vortex distribution.
  - Here, source/sink and vortex have locally varying strengths $q(s)$ and $\gamma(s)$ where *s* is a coordinate which spans the complete surface of the body (e.g., airfoil) in any ways one wants.

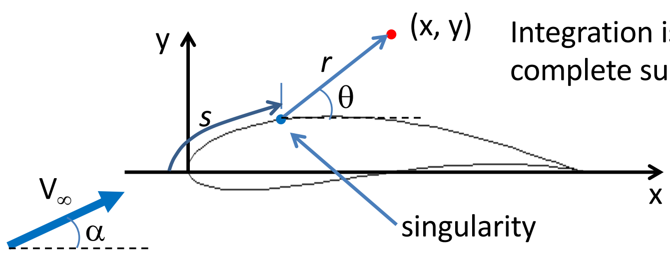
Panel Method

- The potentials created by the distribution of sources/sinks and vortices are given by (in polar coordinates):

$$\phi_s = \int \frac{q(s)}{2\pi} \ln r ds \qquad (2)$$

$$\phi_v = -\int \frac{\gamma(s)}{2\pi} \theta ds$$

where the various quantities are defined as below.



Integration is carried out along the complete surface of the airfoil.

Last Lecture

- Schwarz-Christoffel Transformation
  - Flow around a vertically aligned plate, with flow separation
- Panel Method
  - Introduction

Panel Method

- Using the superposition principle (recall, governing equation is Laplace equation which is linear), any such distribution of sources/sinks and vortices satisfies Laplace's equation may be used.

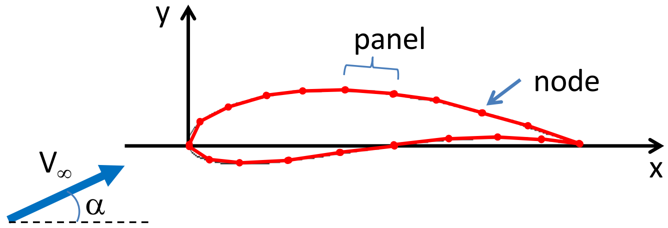
- But we need to find conditions for $q(s)$ and $\gamma(s)$ such that
  - the flow tangency boundary condition
  - the Kutta condition are satisfied.

- We have multiple options, in theory
  - Use the source distribution to satisfy flow tangency and the vortex distribution to satisfy the Kutta condition
  - Use arbitrary combinations of both sources/sinks and vortices to satisfy both boundary conditions simultaneously

Panel Method

- Method by Hess and Smith
  - Take the vortex strength to be constant over the whole airfoil
  - use the Kutta condition to determine its value
  - while allowing the source strength to vary from panel to panel so that, together with the constant vortex distribution, the flow tangency boundary condition is satisfied everywhere.

Panel Method



- Eq(1) can be discretized as:

$$\phi = V_\infty (x\cos\alpha + y\sin\alpha) + \sum_{j=1}^{N}\int_{panel_j}\left[\frac{q(s)}{2\pi}\ln r - \frac{\gamma}{2\pi}\theta\right]ds \qquad (3)$$

  – Integrations over each discrete panel on the surface of the airfoil → variation of source and vortex strength within each of the panels
  – Vortex strength was considered to be a constant
  – Source strength distribution within each panel? - major approximation of the panel method: <u>it is constant on each of the panels, too</u>
  – As the number of panels increases, N → ∞, the effect of this approximation will be reduced (of course this will increase the cost of the computation considerably)

Panel Method

- Therefore, we have (N + 1) unknowns to solve for in our problem
  – N panel source strengths q_i and the constant vortex strength γ.
  – We need (N + 1) independent equations
    - obtained by formulating the flow tangency boundary condition at each of the N panels,
    - and by enforcing the Kutta condition
  – The solution of the problem will require the inversion of a matrix of size (N +1) × (N +1)
  – Then, where should we impose those boundary conditions?
    - At the nodes?
    - At the points of the surface of actual airfoil?
    - At the midpoints of the panels?

Panel Method

- Velocities are infinite at the nodes of each panels → poor choice for boundary condition imposition
- The second option is reasonable, but difficult to implement in practice
- The last option might alter the geometry, but it is easy to implement and yields fairly accurate results for a reasonable number of panels.
- This location is also used for the imposition of the Kutta condition (on the last panels on upper and lower surfaces of the airfoil, assuming that their midpoints remain at equal distances from the trailing edge as the number of panels is increased).

Panel Method: Actual Practice

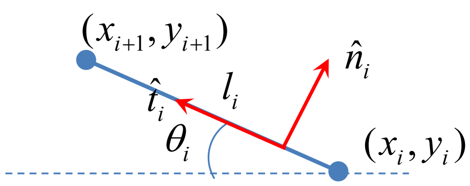
- Consider the i^{th} panel to be located between the i^{th} and $(i + 1)^{th}$ nodes, with its orientation to the x-axis given by

$$\sin\theta_i = \frac{y_{i+1} - y_i}{l_i}, \quad \cos\theta_i = \frac{x_{i+1} - x_i}{l_i}$$

where l_i is the length of the panel under consideration. Then, the normal and tangential vectors to this panel are given as below.

$$\hat{\mathrm{n}}_i = -\sin\theta_i \hat{\mathrm{i}} + \cos\theta_i \hat{\mathrm{j}}, \ \hat{\mathrm{t}}_i = -\sin\theta_i \hat{\mathrm{i}} + \cos\theta_i \hat{\mathrm{j}}$$

Tangential vector is oriented in the direction from i^{th} node to $(i+1)^{th}$ node, while normal vector (if the airfoil is traversed clockwise) points into the fluid.



Panel Method: Actual Practice

- Coordinates of the midpoint of the panel are given by

$$\bar{x}_i = \frac{x_i + x_{i+1}}{2}, \; \bar{y}_i = \frac{y_i + y_{i+1}}{2}$$

and the velocity components at these midpoints are given by

$$u_i = u(\bar{x}_i, \bar{y}_i), \; v_i = v(\bar{x}_i, \bar{y}_i)$$

- The flow-tangency boundary condition can be written as $\bar{u} \cdot \bar{n} = 0$.

$$-u_i \sin\theta_i + v_i \cos\theta_i = 0 \qquad \text{for } i = 1, \cdots, N$$

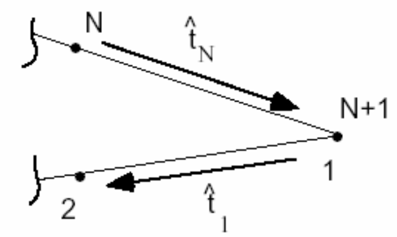
- The Kutta condition is simply given by

$$u_1 \cos\theta_1 + v_1 \sin\theta_1 = -u_N \cos\theta_N - v_N \sin\theta_N \tag{4}$$

where the negative signs are due to the fact that the tangential vectors at the first and last panels have nearly opposite directions.

Panel Method: Actual Practice

- The Kutta condition
  - The flow must leave the trailing edge smoothly
  - We satisfy the Kutta condition approximately by equating velocity components tangential to the panels adjacent to the trailing edge on the upper and lower surface



$$\bar{V} \cdot \hat{t}\big|_1 = -\bar{V} \cdot \hat{t}\big|_N$$

$$(u_1\hat{i} + v_1\hat{j}) \cdot (-\cos\theta_1\hat{i} + \sin\theta_1\hat{j}) = -(u_N\hat{i} + v_N\hat{j}) \cdot (-\cos\theta_N\hat{i} + \sin\theta_N\hat{j})$$

$$u_1 \cos\theta_1 + v_1 \sin\theta_1 = -u_N \cos\theta_N - v_N \sin\theta_N$$

Panel Method: Actual Practice

- Now, the velocity at the midpoint of each panel are computed by superposition of the contributions of all sources and vortices located at the midpoint of every panel (including itself).

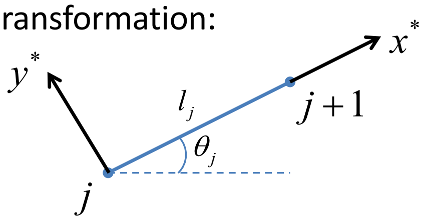
- Since the velocity induced by the source or vortex on a panel is proportional to the source or vortex strength in that panel, q_i and γ can be pulled out of the integral in Eq(3).

$$u_i = V_\infty \cos\alpha + \sum_{j=1}^{N} q_j u_{sij} + \gamma \sum_{j=1}^{N} u_{vij},\ v_i = V_\infty \sin\alpha + \sum_{j=1}^{N} q_j v_{sij} + \gamma \sum_{j=1}^{N} v_{vij} \quad (5)$$

where u_{sij}, v_{sij}, u_{vij}, v_{vij} are the velocity components at the midpoint of i^th^ panel induced by a source/a vortex of unit strength at j^th^ panel.

Panel Method: Actual Practice

- In a local coordinate system tangential and normal to the panel, we can perform the integrals in Eq(3) by noticing that the local velocity components can be expanded into absolute ones according to the following transformation:



$$u = u^* \cos\theta_j - v^* \sin\theta_j$$
$$v = u^* \sin\theta_j + v^* \cos\theta_j \quad (6)$$

- Now, the local velocity components at the midpoint of the i^th^ panel due to a unit-strength **source distribution** on the j^th^ panel can be written as

$$u^*_{sij} = \frac{1}{2\pi}\int_0^{l_j} \frac{x^* - t}{(x^* - t)^2 + y^{*2}}\,dt,\ v^*_{sij} = \frac{1}{2\pi}\int_0^{l_j} \frac{y^*}{(x^* - t)^2 + y^{*2}}\,dt \quad (7)$$

where (x^*, y^*) is the coordinates of the midpoint of i^th^ panel in the local coordinate system of j^th^ panel.

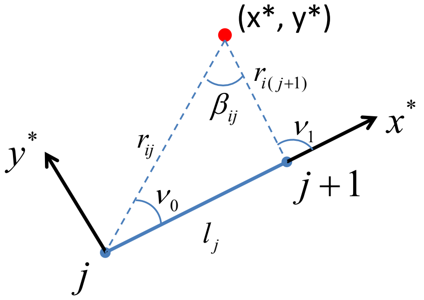
Panel Method: Actual Practice

- By integrating Eq(7),

$$u^*_{sij} = -\frac{1}{2\pi} \ln\left[(x^* - t)^2 + y^{*2} \right]^{1/2} \Bigg|_{t=0}^{t=l_j}$$
$$v^*_{sij} = \frac{1}{2\pi} \tan^{-1} \frac{y^*}{x^* - t} \Bigg|_{t=0}^{t=l_j} \qquad (8)$$

Panel Method: Actual Practice

- Simple geometric interpretation of the results



r_{ij} distance from the midpoint of i^th panel to j^th node

β_{ij} angle by the j^th panel at the midpoint of i^th panel

$$u^*_{sij} = -\frac{1}{2\pi} \ln \frac{r_{i(j+1)}}{r_{ij}}$$

$$v^*_{sij} = \frac{\nu_1 - \nu_0}{2\pi} = \frac{\beta_{ij}}{2\pi}$$

- When the point of interest approaches the midpoint of the panel from the *outside* of the body (i.e., airfoil), this angle, $\beta_{ii} \to \pi$.
- However, when the midpoint of the panel is approached from the inside of the airfoil, $\beta_{ii} \to -\pi$.
- Since we are interested in the flow outside of the airfoil only, we will always take $\beta_{ii} = \pi$.

Panel Method: Actual Practice

- Similarly, for the velocity field induced by **the vortex** on jth panel at the midpoint of ith panel,

$$u_{vij}^{*} = -\frac{1}{2\pi}\int_{0}^{l_j}\frac{y^{*}}{(x^{*}-t)^{2}+y^{*2}}dt = \frac{\beta_{ij}}{2\pi} \quad (9)$$

$$v_{vij}^{*} = -\frac{1}{2\pi}\int_{0}^{l_j}\frac{x^{*}-t}{(x^{*}-t)^{2}+y^{*2}}dt = \frac{1}{2\pi}\ln\frac{r_{i(j+1)}}{r_{ij}} \quad (10)$$

- Flow-tangency boundary condition, using Eq(5) and (6), can be written as

$$\sum_{j=1}^{N} A_{ij}q_j + A_{i(N+1)}\gamma = b_i \qquad \boxed{b_i = V_\infty \sin(\theta_i - \alpha)}$$

$$A_{ij} = -u_{sij}\sin\theta_i + v_{sij}\cos\theta_i$$

$$= -u_{sij}^{*}(\cos\theta_j\sin\theta_i - \sin\theta_j\cos\theta_i) + v_{sij}^{*}(\sin\theta_j\sin\theta_i + \cos\theta_j\cos\theta_i)$$

$$\therefore 2\pi A_{ij} = \sin(\theta_i - \theta_j)\ln\frac{r_{i(j+1)}}{r_{ij}} + \cos(\theta_i - \theta_j)\beta_{ij}$$

Panel Method: Actual Practice

- Similarly for the vortex strength,

$$2\pi A_{i(N+1)} = \sum_{j=1}^{N}\cos(\theta_i - \theta_j)\ln\frac{r_{i(j+1)}}{r_{ij}} - \sin(\theta_i - \theta_j)\beta_{ij}$$

- The flow-tangency boundary condition gives us N-equations. We need an additional one provided by the Kutta condition in order to obtain a system that can be solved. According to Eq(4),

$$\sum_{j=1}^{N} A_{(N+1)j}q_j + A_{(N+1)(N+1)}\gamma = b_{N+1}$$

Panel Method: Actual Practice

- After similar manipulations we obtain,

$$2\pi A_{(N+1)j} = \sum_{k=1,N} \sin(\theta_k - \theta_j)\beta_{kj} - \cos(\theta_k - \theta_j)\frac{r_{k(j+1)}}{r_{kj}} \quad (11)$$

$$2\pi A_{(N+1)(N+1)} = \sum_{k=1,N} \sum_{j=1}^{N} \sin(\theta_k - \theta_j)\frac{r_{k(j+1)}}{r_{kj}} + \cos(\theta_k - \theta_j)\beta_{kj}$$

$$b_{N+1} = -V_\infty \cos(\theta_1 - \alpha) - -V_\infty \cos(\theta_N - \alpha)$$

- These various expressions set up a matrix problem of the kind $Ax = b$.

$$\begin{bmatrix} A_{11} & \dots & A_{1i} & \dots & A_{1N} & A_{1,N+1} \\ \vdots & & \vdots & & \vdots & \vdots \\ A_{i1} & \dots & A_{ii} & \dots & A_{iN} & A_{i,N+1} \\ \vdots & & \vdots & & \vdots & \vdots \\ A_{N1} & \dots & A_{Ni} & \dots & A_{NN} & A_{N,N+1} \\ A_{N+1,1} & \dots & A_{N+1,i} & \dots & A_{N+1,N} & A_{N+1,N+1} \end{bmatrix} \begin{bmatrix} q_1 \\ \vdots \\ q_i \\ \vdots \\ q_N \\ \gamma \end{bmatrix} = \begin{bmatrix} b_1 \\ \vdots \\ b_i \\ \vdots \\ b_N \\ b_{N+1} \end{bmatrix}$$

Panel Method: Actual Practice

- Finally, once you have solved the system for the unknowns of the problem, it is easy to construct the tangential velocity at the midpoint of each panel according to the following formula

$$V_{ti} = V_\infty \cos(\theta_i - \alpha) + \sum_{j=1}^{N} \frac{q_j}{2\pi}\left[\sin(\theta_i - \theta_j)\beta_{ij} - \cos(\theta_i - \theta_j)\ln\frac{r_{i(j+1)}}{r_{ij}}\right]$$

$$+\frac{\gamma}{2\pi}\sum_{j=1}^{N}\left[\sin(\theta_i - \theta_j)\ln\frac{r_{i(j+1)}}{r_{ij}} + \cos(\theta_i - \theta_j)\beta_{ij}\right]$$

- And we can calculate the pressure coefficient at the midpoint of each panel according to the following formula.

$$C_p(\bar{x}_i, \bar{y}_i) = 1 - \frac{V_{ti}^2}{V_\infty^2}$$

The force and moment coefficients can be calculated from pressure distribution!