# Kalman Filtering

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## Kalman Filtering w/ No Uncertainty

- Linear discrete plant dynamics with measurement (cf. EKF, UKF):

$$\boxed{x_{k+1} = F_k x_k + G_k u_k, \quad y_k = H_k x_k}$$

where $x_k \in \Re^n$ is state, $u_k \in \Re^p$ input, and $y_k \in \Re^m$ measurement output.

- If $F_k, G_k, u_k$ known (impractical: to be relaxed), state estimator:

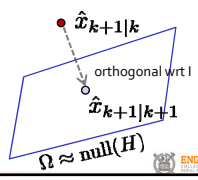
$$\boxed{\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} + G_k u_k}$$

where $\hat{x}_{k+1|k}$ is **prediction** of $x_{k+1}$ given "best" estimate $\hat{x}_{k|k}$ of $x_k$ propagated via dynamics over $[k, k+1]$ (can't do any better than this).

- Now, suppose measurement $y_{k+1}$ given at $k+1$. Then, how to update $\hat{x}_{k+1|k}$ using this information?

- First of all, the estimate $\hat{x}_{k+1|k+1}$ of $x_{k+1}$ should be consistent with this information $y_{k+1} = H_{k+1} x_{k+1}$, i.e.,

$$\boxed{\hat{x}_{k+1|k+1} \in \Omega := \{x \in \Re^n \mid y_{k+1} = H_{k+1} x\}}$$



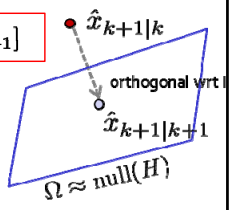
# Kalman Filtering w/ No Uncertainty

- Estimate $\hat{x}_{k+1|k+1}$ of $x_{k+1}$ should consistent w/ $y_{k+1} = H_{k+1}x_{k+1}$, i.e.,

$$\hat{x}_{k+1|k+1} \in \Omega := \{x \in \Re^n \mid y_{k+1} = H_{k+1}x\}$$

- Optimal estimate $\hat{x}_{k+1|k+1} \Rightarrow$ **correction** of $\hat{x}_{k+1|k}$ into its closest point on $\Omega$ with Euclidean norm.
- Using $\hat{x}_{k+1|k+1} - \hat{x}_{k+1|k} = H_{k+1}^T\alpha$ and $y_{k+1} = H_{k+1}\hat{x}_{k+1|k+1}$,

$$\boxed{\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + H_{k+1}^T(H_{k+1}H_{k+1}^T)^{-1}\left[y_{k+1} - \hat{y}_{k+1}\right]}$$




where $\hat{y}_{k+1} = H_{k+1}\hat{x}_{k+1|k}$ (best estimated output).

- Kalman filtering w/ no uncertainty: with $\hat{x}_{0|0}$,
  1. Plant: $x_{k+1} = F_kx_k + G_ku_k$ w/ measurement $y_k = H_kx_k$
  2. **Prediction** (propagation): $\hat{x}_{k+1|k} = F_k\hat{x}_{k|k} + G_ku_k$
  3. Measurement (output): $y_{k+1} = H_{k+1}x_{k+1}$
  4. Estimated measurement: $\hat{y}_{k+1} = H_{k+1}\hat{x}_{k+1|k}$
  5. **Correction** (update): $\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + H_{k+1}^T(H_{k+1}H_{k+1}^T)^{-1}\left[y_{k+1} - \hat{y}_{k+1}\right]$



# Kalman Filtering w/ Process Noise

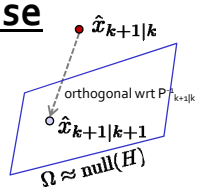
- Plant dynamics with measurement and process noise $v_k$:

$$\boxed{x_{k+1} = F_kx_k + G_ku_k + v_k, \quad y_k = H_kx_k}$$

where $v_k \in \Re^n$ zero mean Gaussian w/ $E[v_k] = \bar{v}_k = 0$ and covariance $E[(v_k - \bar{v}_k)(v_k - \bar{v}_k)^T)] = V_k \in \Re^{n\times n}$ (e.g., uncertainty in actuation $u_k$, modeling $F_k, G_k$, unmodeled friction/slip, discretization).

- Now, $x_k$ becomes RV $\Rightarrow$ need to estimate its mean and also covariance too, i.e., starting from $(\hat{x}_{0|0}, P_{o|o})$,
  - **Prediction** $(\hat{x}_{k+1|k}, P_{k+1|k})$: by propagating $(\hat{x}_{k|k}, P_{k|k})$ via plant dynamics with uncertainty $V_k$ due to process noise.
  - **Correction** $(\hat{x}_{k+1|k+1}, P_{k+1|k+1})$: by using $r_{k+1} = y_{k+1} - \hat{y}_{k+1}$ with uncertainty $S_{k+1}$ of $r_{k+1}$ also taken into account.
- **Prediction:**
  - State (mean) prediction: $\boxed{\hat{x}_{k+1|k} = F_k\hat{x}_{k|k} + G_ku_k}$
  - Uncertainty (covariance) propagation: $\boxed{P_{k+1|k} = F_kP_{k|k}F_k^T + V_k}$ where $P_{k+1|k} = E[(\hat{x}_{k+1|k} - x_{k+1})(\hat{x}_{k+1|k} - x_{k+1})^T]$, i.e., uncertainty from perfect estimate $x_{k+1}$ (w/ $v_k$ independent from $x_k, x_{k|k}$).

# Kalman Filtering w/ Process Noise




- Plant dynamics with measurement and process noise $v_k$:

$$\boxed{x_{k+1} = F_k x_k + G_k u_k + v_k, \quad y_k = H_k x_k}$$

- **Prediction:**
  - State (mean) prediction: $\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} + G_k u_k$
  - Uncertainty (covariance) propagation: $P_{k+1|k} = F_k P_{k|k} F_k^T + V_k$
- Now, suppose measurement $y_{k+1}$ is given. Then, how to update $(\hat{x}_{k+1|k}, P_{k+1|k})$ using this information?
- The estimate $\hat{x}_{k+1|k+1}$ should again be consistent with $y_{k+1} = H_{k+1} x_{k+1}$:

$$\boxed{\hat{x}_{k+1|k+1} \in \Omega := \{x \in \Re^n \mid y_{k+1} = H_{k+1} x\}}$$

  Yet, we shouldn't weigh all channels of $x$ equally as some channel may be more uncertain than others $\Rightarrow$ different metric other than $I$.
- **Mahalanobis metric** $\boxed{P^{-1}_{k+1|k}}$: more weight and updating action for channels with smaller $P_{k+1|k}$ (i.e., high certainty).



# Kalman Filtering w/ Process Noise

- The estimate $\hat{x}_{k+1|k+1}$ should again be consistent with $y_{k+1} = H_{k+1} x_{k+1}$:

$$\boxed{\hat{x}_{k+1|k+1} \in \Omega := \{x \in \Re^n \mid y_{k+1} = H_{k+1} x\}}$$

- Optimal estimate $\hat{x}_{k+1|k+1}$: **correction** of $\hat{x}_{k+1|k}$ into its closest point on $\Omega$ with Mahalanobis norm $P^{-1}_{k+1|k}$.
- Using $\hat{x}_{k+1|k+1} - \hat{x}_{k+1|k} = P_{k+1|k} H_{k+1}^T \alpha$ ($\perp$ null$(H)$ w.r.t. $P^{-1}_{k+1|k}$), $y_{k+1} = H_{k+1} \hat{x}_{k+1|k+1}$, and $\hat{y}_{k+1} = H_{k+1} \hat{x}_{k+1|k}$ (best estimated output):

$$\boxed{\begin{aligned} &\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}] \\ &K_{k+1} = P_{k+1|k} H_{k+1}^T S_{k+1}^{-1}, \quad S_{k+1} = H_{k+1} P_{k+1|k} H_{k+1}^T \end{aligned}}$$

  - **Residual covariance** $S_{k+1}$: uncertainty in $r_{k+1}$ (solely due to $\hat{y}_{k+1}$);
  - **Kalman gain** $K_{k+1}$: more update action for more uncertain state with more certain measurement information.
- Uncertainty update (reduction):

$$\begin{aligned} P_{k+1|k+1} &= E\left[(\hat{x}_{k+1|k+1} - x_{k+1})(\hat{x}_{k+1|k+1} - x_{k+1})^T\right] \\ &\boxed{= P_{k+1|k} - P_{k+1|k} H_{k+1}^T S_{k+1}^{-1} H_{k+1} P_{k+1|k}} \end{aligned}$$

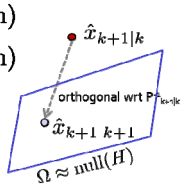
# Kalman Filtering w/ Process Noise

- Plant dynamics with measurement and process noise $v_k$:

$$\boxed{x_{k+1} = F_k x_k + G_k u_k + v_k, \quad y_k = H_k x_k}$$

- **Prediction:**

$$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} + G_k u_k \qquad \text{(state prediction)}$$
$$P_{k+1|k} = F_k P_{k|k} F_k^T + V_k \qquad \text{(uncertainty propagation)}$$




- **Measurement:** $y_{k+1} = H_{k+1} x_k$ and $\hat{y}_{k+1} = H_{k+1} \hat{x}_{k+1|k}$
- **Correction:**

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}] \qquad \text{(state correction)}$$
$$P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k} H_{k+1}^T S_{k+1}^{-1} H_{k+1} P_{k+1|k} \qquad \text{(uncertainty reduction)}$$

  - Redisual variance: $S_{k+1} = H_{k+1} P_{k+1|k} H_{k+1}^T$ $(= E[r_{k+1} r_{k+1}^T])$
  - Kalman gain: $K_{k+1} = P_{k+1|k} H_{k+1}^T S_{k+1}^{-1}$
  - Uncertainty always reduced with certain measurement info $y_k$.
  - $H = I \Rightarrow P_{k+1|k+1} \to 0$, i.e., perfect estimation with $y_{k+1} = x_{k+1}$.



# Kalman Filtering

- Plant dynamics with process noise $v_k$ and measurement noise $w_k$:

$$\boxed{x_{k+1} = F_k x_k + G_k u_k + v_k, \quad y_k = H_k x_k + w_k}$$

$w_k \in \Re^n$ zero mean Gaussian w/ $E[w_k] = \bar{w}_k = 0$ and covariance $E[(w_k - \bar{w}_k)(w_k - \bar{w}_k)^T)] = W_k \in \Re^{n \times n}$.

- **Prediction** (same as before):

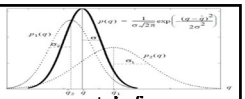
$$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} + G_k u_k \qquad \text{(state prediction)}$$
$$P_{k+1|k} = F_k P_{k|k} F_k^T + V_k \qquad \text{(uncertainty propagation)}$$

- **Measurement:** $y_{k+1} = H_{k+1} x_k + w_k$ and $\hat{y}_{k+1} = H_{k+1} \hat{x}_{k+1|k}$
  - Both $y_{k+1}$ and $\hat{y}_{k+1}$ are now RVs with uncertainty, $W_{k+1}$ and $\hat{W}_{k+1}$.
  - Given $y_{k+1}$, real measurement would likely distributed by $N(y_{k+1}, W_{k+1})$.
  - For $\hat{y}_{k+1}$, its covariance given by

$$\hat{W}_{k+1} = E\left[(\hat{y}_{k+1} - H_k x_{k+1})(\hat{y}_{k+1} - H_{k+1} x_{k+1})^T\right] = H_{k+1} P_{k+1|k} H_{k+1}^T$$

  - Given $N(y_{k+1}, W_{k+1})$ and $N(\hat{y}_{k+1}, \hat{W}_{k+1})$, **most like output** $y_{k+1}^*$?

# Kalman Filtering

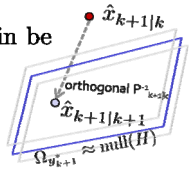


- Merge measurement info $N(y_{k+1}, W_{k+1})$ and estimated measurement info $N(\hat{y}_{k+1}, \hat{W}_{k+1})$ using product of Gaussians (cf. Th.8.2.1) $\Rightarrow$ **most like measurement**: $y^*_{k+1} = \text{product}(y_{k+1}, \hat{y}_{k+1})$, which is still Gaussian with

$$\boxed{\begin{aligned} y^*_{k+1} &= \hat{y}_{k+1} + \hat{W}_{k+1} S^{-1}_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}] && \text{(mean)} \\ W^*_{k+1} &= \hat{W}_{k+1} - \hat{W}_{k+1} S^{-1}_{k+1} \hat{W}_{k+1} && \text{(covariance)} \end{aligned}}$$

where $S_{k+1} = \hat{W}_{k+1} + W_{k+1} = H_{k+1} P_{k+1|k} H^T_{k+1} + W_{k+1}$, i.e., combined uncertainty in $r_{k+1} = \hat{y}_{k+1} - y_{k+1}$ (residual variance).

- With $y^*_{k+1}$ as best measurement, estimate $\hat{x}_{k+1|k+1}$ should again be consistent w/ that information:



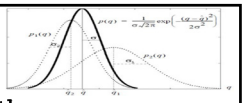

$$\hat{x}_{k+1|k+1} \in \Omega_{y^*_{k+1}} := \{x \in \Re^n \mid y^*_{k+1} = H_{k+1} x\}$$

- Optimal estimate $\hat{x}_{k+1|k+1}$: **correction** of $\hat{x}_{k+1|k}$ into its closest point on $\Omega_{y^*_{k+1}}$ with Mahalanobis norm $P^{-1}_{k+1|k}$:

$$\boxed{\begin{aligned} \hat{x}_{k+1|k+1} &= \hat{x}_{k+1|k} + P_{k+1|k} H^T_{k+1} \hat{W}^{-1}_{k+1} \cdot [y^*_{k+1} - \hat{y}_{k+1}] \\ &= \hat{x}_{k+1|k} + P_{k+1|k} H^T_{k+1} S^{-1}_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}] \end{aligned}}$$

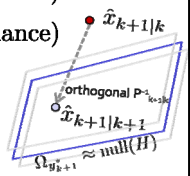


# Kalman Filtering



- **Most like measurement**: $y^*_{k+1} = \text{product}(y_{k+1}, \hat{y}_{k+1})$ with

$$\begin{aligned} y^*_{k+1} &= \hat{y}_{k+1} + \hat{W}_{k+1} S^{-1}_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}] && \text{(mean)} \\ W^*_{k+1} &= \hat{W}_{k+1} - \hat{W}_{k+1} S^{-1}_{k+1} \hat{W}_{k+1} && \text{(covariance)} \\ S_{k+1} &= H_{k+1} P_{k+1|k} H^T_{k+1} + W_{k+1} && \text{(residual variance)} \end{aligned}$$




- Optimal estimate $\hat{x}_{k+1|k+1}$:

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + P_{k+1|k} H^T_{k+1} S^{-1}_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}]$$

which is in the same form as before (yet, different residual variance $S_{k+1} = H_{k+1} P_{k+1|k} H^T_{k+1} + W_{k+1}$ instead of $S_{k+1} = H_{k+1} P_{k+1|k} H^T_{k+1}$).

- Uncertainty update (reduction):

$$\begin{aligned} P_{k+1|k+1} &= E\left[(\hat{x}_{k+1|k+1} - x_{k+1})(\hat{x}_{k+1|k+1} - x_{k+1})^T\right] \\ &= E\left[\left((I - P_{k+1|k} H^T_{k+1} S^{-1}_{k+1})(\hat{x}_{k+1|k} - x_{k+1})\right)(\ldots)^T\right] \\ &= P_{k+1|k} - P_{k+1|k} H^T_{k+1} S^{-1}_{k+1} H_{k+1} P_{k+1|k} \end{aligned}$$

which is again in the same form as before.

# Kalman Filtering

- Plant dynamics with process noise $v_k$ and measurement noise $w_k$:

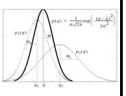
$$\boxed{x_{k+1} = F_k x_k + G_k u_k + v_k, \quad y_k = H_k x_k + w_k}$$

- **Prediction:**

$$\hat{x}_{k+1|k} = F_k \hat{x}_{k|k} + G_k u_k \qquad \text{(state prediction)}$$

$$P_{k+1|k} = F_k P_{k|k} F_k^T + V_k \qquad \text{(uncertainty propagation)}$$

- **Measurement:** $y_{k+1} = H_{k+1} x_k + w_{k+1}$ and $\hat{y}_{k+1} = H_{k+1} \hat{x}_{k+1|k}$

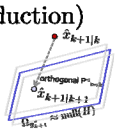


- **Correction:**

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}] \qquad \text{(state correction)}$$

$$P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k} H_{k+1}^T S_{k+1}^{-1} H_{k+1} P_{k+1|k} \qquad \text{(uncertainty reduction)}$$

  - Redisual variance: $S_{k+1} = H_{k+1} P_{k+1|k} H_{k+1}^T + W_{k+1}$
  - Kalman gain: $K_{k+1} = P_{k+1|k} H_{k+1}^T S_{k+1}^{-1}$ direction with state uncertainty X weighted gain of measurement




  - $K_{k+1}$ automatically and optimally adjusting, incorporating measurement uncertainty and state estimate uncertainty.
  - With $H_k = I$: 1) If $W_{k+1} = 0 \Rightarrow K_{k+1} = I$ and $P_{k+1|k+1} = 0$; 2) If $W_{k+1} = \infty \Rightarrow K_{k+1} = 0$ and $P_{k+1|k+1} = P_{k|k}$.



# Kalman Filtering: Example

# Kalman Filtering: SLAM



# Extended Kalman Filtering

- Standard KF assumes linear plant dynamics with linear measurement model:

$$\boxed{x_{k+1} = F_k x_k + G_k u_k + v_k, \quad y_k = H_k x_k + w_k}$$

- Yet, of course, most real systems are nonlinear with:

$$\boxed{x_{k+1} = f_k(x_k, u_k, v_k), \quad y_k = h_k(x_k) + w_k}$$

  where $u_k$ input, $v_k$ process noise, $y_k$ sensor reading, $w_k$ measurement noise.

- **Extended Kalman filtering (EKF):**
  - State prediction (i.e., $\hat{x}_{k+1|k}$) and correction (i.e., $\hat{x}_{k+1|k+1}$) using <u>nonlinear</u> plant dynamics and measurement model.
  - Uncertainty propagation (i.e., $P_{k+1|k}$) and update (i.e., $P_{k+1|k+1}$) using <u>linearized</u> (i.e., approximate) plant/measurement models.
  - Assume mean propagates still to mean, while uncertainty propagates via linearized model.
  - Generally not true for nonlinear mapping $\Rightarrow$ may even diverge if approximation/linearization error too large (cf. UKF).

# Extended Kalman Filtering

- Nonlinear plant and measurement models with process/measurement noise:

$$\boxed{x_{k+1} = f_k(x_k, u_k, v_k), \quad y_k = h_k(x_k) + w_k}$$

- **Prediction**:
  - State prediction (mean propagation via nonlinear plant model):

$$\boxed{\hat{x}_{k+1|k} = f_k(\hat{x}_{k|k}, u_k, 0)}$$

  - Uncertainty propagation via linearized plant dynamics:
    * Define $\tilde{x}_{k+1|k} = \hat{x}_{k+1|k} - x_{k+1}$ and $\tilde{x}_{k|k} = \hat{x}_{k|k} - x_k$. Then, $P_{k+1|k} = E[(\hat{x}_{k+1|k} - x_{k+1})(\hat{x}_{k+1|k} - x_{k+1})^T] = E[\tilde{x}_{k+1|k}\tilde{x}_{k+1|k}^T]$ and $P_{k|k} = E[(\hat{x}_{k|k} - x_k)(\hat{x}_{k|k} - x_k)^T] = E[\tilde{x}_{k|k}\tilde{x}_{k|k}^T]$.
    * Linearization at $\hat{x}_{k|k}$:

$$\begin{aligned}\tilde{x}_{k+1|k} &\approx \left.\frac{\partial f_k}{\partial x}\right|_{(\hat{x}_{k|k}, u_k)} \tilde{x}_{k|k} + \left.\frac{\partial f_k}{\partial v}\right|_{(\hat{x}_{k|k}, u_k)} v_k \\ &= F_k(\hat{x}_{k|k}, u_k)\tilde{x}_{k|k} + G_{vk}(\hat{x}_{k|k}, u_k)v_k\end{aligned}$$

    * Covariance propagation (via linearized plant dynamics):

$$\boxed{P_{k+1|k} = F_k P_{k|k} F_k^T + G_{vk} V_k G_{vk}^T}$$



# Extended Kalman Filtering

- Nonlinear plant and measurement models with process/measurement noise:

$$\boxed{x_{k+1} = f_k(x_k, u_k, v_k), \quad y_k = h_k(x_k) + w_k}$$

- **Prediction**:

$$\hat{x}_{k+1|k} = f_k(\hat{x}_{k|k}, u_k, 0) \qquad \text{(state prediction)}$$

$$P_{k+1|k} = F_k P_{k|k} F_k^T + G_{vk} V_k G_{vk}^T \qquad \text{(covariance propagation)}$$

- **Measurement:**
  - $y_{k+1} = h_{k+1}(x_{k+1}) + w_k$ (real) and $\hat{y}_{k+1} = h_{k+1}(\hat{x}_{k+1|k})$ (estimate).
  - Linearize measurement around $\hat{x}_{k+1|k}$:

$$\hat{y}_{k+1|k} - y_{k+1} \approx \left.\frac{\partial h_{k+1}}{\partial x}\right|_{\hat{x}_{k+1|k}} \tilde{x}_{k+1|k} + w_k = H_{k+1}(\hat{x}_{k+1|k})\tilde{x}_{k+1|k} + w_k$$

  - Residual of covariance of $r_{k+1} = y_{k+1} - \hat{y}_{k+1}$:

$$S_{k+1} = E[(y_{k+1} - \hat{y}_{k+1})(y_{k+1} - \hat{y}_{k+1})^T] = H_{k+1}P_{k+1|k}H_{k+1}^T + W_{k+1}$$

  - **Kalman gain:** $K_{k+1} = P_{k+1|k}H_{k+1}^T S_{k+1}^{-1}$.
  - State correction: $\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}]$
  - Uncertainty update: $P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k}H_{k+1}^T S_{k+1}^{-1} H_{k+1} P_{k+1|k}$.

# Extended Kalman Filtering

- Nonlinear plant and measurement models with process/measurement noise:

$$\boxed{x_{k+1} = f_k(x_k, u_k, v_k), \quad y_k = h_k(x_k) + w_k}$$

- **Prediction**:

$$\hat{x}_{k+1|k} = f_k(\hat{x}_{k|k}, u_k, 0) \qquad \text{(state prediction)}$$
$$P_{k+1|k} = F_k P_{k|k} F_k^T + G_{vk} V_k G_{vk}^T \qquad \text{(covariance propagation)}$$

where $F_k = \partial f_k / \partial x|_{(\hat{x}_{k|k}, u_k)}$ and $G_{vk} = \partial f_k / \partial v|_{(\hat{x}_{k|k}, u_k)}$

- **Measurement:** $y_{k+1} = h_{k+1}(x_{k+1}) + w_k$ and $\hat{y}_{k+1} = h_{k+1}(\hat{x}_{k+1|k})$.

- **Correction:**

$$\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}] \qquad \text{(state correction)}$$
$$P_{k+1|k+1} = P_{k+1|k} - P_{k+1|k} H_{k+1}^T S_{k+1}^{-1} H_{k+1} P_{k+1|k} \qquad \text{(uncertainty update)}$$

where $H_{k+1} = \partial h_{k+1} \partial x|_{\hat{x}_{k+1|k}}$.

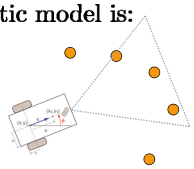
  - Redisual variance: $S_{k+1} = H_{k+1} P_{k+1|k} H_{k+1}^T + W_{k+1}$
  - Kalman gain: $K_{k+1} = P_{k+1|k} H_{k+1}^T S_{k+1}^{-1}$



# EKF Localization

- Consider WMR with state $x = (x_r, y_r, \theta_r)$ and linear/angular velocity inputs $u = [u_1, u_2]$. Then, **nonlinear discrete-time** kinematic model is:

$$\boxed{x_{k+1} = \begin{pmatrix} x_{r,k} + u_{1,k}\Delta t \cos\theta_{r,k} \\ y_{r,k} + u_{1,k}\Delta t \sin\theta_{r,k} \\ \theta_{r,k} + u_{2,k}\Delta t \end{pmatrix} + \begin{pmatrix} v_{x,k} \\ v_{y,k} \\ v_{r,k} \end{pmatrix}}$$



where $\Delta t > 0$ sampling rate, $v_k \in \Re^3$ zero-mean Gaussian process noise (e.g., discretization error, actuator noise, friction, etc.)

- WMR is equipped with **range/bearing sensors** and, at each $k$, it can sense $p_k$ landmarks among $n_L$ surrounding/stationary andmarks.

- Assume **data-association** (i.e., which sensing is associated with which landmark) is somehow done, i.e., following association map $a_k$ is known:

$$\boxed{a_k : \{1, 2, ..., p_k\} \mapsto \{1, 2, ..., n_L\}}$$

- Then, at each $k$, WMR has $p_k$ active measurements, $h_j(x_k, a_k(j)) + w_{j,k}$, $j = 1, ..., p_k$, each associated with $a_k(j)$-th landmark.

# EKF Localization

- Measurement equation with $p_k$ measurements at $k$ is given by:

$$\boxed{y_k = \begin{pmatrix} h_1(x_k, a_k(1)) \\ h_2(x_k, a_k(2)) \\ \vdots \\ h_{p_k}(x_k, a_k(p_k)) \end{pmatrix} + w_k, \quad h_j(x_k, a_k(j)) = \begin{pmatrix} r_k^j(p_k^r, p_{a_k(j)}^L) \\ \theta_k^j(p_k^r, p_{a_k(j)}^L) \end{pmatrix}}$$

where, with $l_j = a_p(j)$, $r_k^j = \sqrt{(x_{r,k} - x_{l_j})^2 + (y_{r,k} - y_{l_j})^2}$ (i.e., range) and $\theta_k^j = \text{atan2}(y_{r,k} - y_{l_j}, x_{r,k} - x_{l_j}) - \theta_{r,k}$ (i.e., bearing).

- **State prediction:** with $\Delta t = 1$ for simplicity,

$$\boxed{\hat{x}_{k+1|k} = \begin{pmatrix} \hat{x}_{r,k|k} + u_{1,k} \cos \hat{\theta}_{r,k|k} \\ \hat{y}_{r,k|k} + u_{1,k} \sin \hat{\theta}_{r,k|k} \\ \hat{\theta}_{r,k|k} + u_{2,k} \end{pmatrix}}$$

- Linearization of state equation for uncertainty propagation:

$$\tilde{x}_{k+1|k} = \hat{x}_{k+1|k} - x_{k+1} = \begin{bmatrix} 1 & 0 & \mathrm{s}\,\hat{\theta}_{r,k} u_{1,k} \\ 0 & 1 & \mathrm{c}\,\hat{\theta}_{r,k} u_{1,k} \\ 0 & 0 & 1 \end{bmatrix} \tilde{x}_{k|k} + v_k = \boxed{F_k \tilde{x}_{k|k} + v_k}$$



# EKF Localization

- Estimated measurements: for $j = 1, ..., p_k$,

$$\boxed{\hat{y}_{j,k+1} = \begin{pmatrix} \sqrt{(\hat{x}_{r,k+1|k} - x_{l_j})^2 + (\hat{y}_{r,k+1|k} - y_{l_j})^2} \\ \text{atan2}(\hat{y}_{r,k+1|k} - y_{l_j}, \hat{x}_{r,k+1|k} - x_{l_j}) - \hat{\theta}_{r,k+1|k} \end{pmatrix}}$$

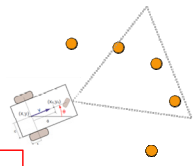
- Linearization measurement equation for uncertainty propagation:

$$r_{k+1}^j = \hat{y}_{j,k+1} - y_{j,k+1} = \begin{bmatrix} \frac{\hat{x}_{r,k+1|k} - x_{l_j}}{\hat{r}_{k+1|k}^j} & \frac{\hat{y}_{r,k+1|k} - y_{l_j}}{\hat{r}_{k+1|k}^j} & 0 \\ \frac{-\sin \hat{\theta}_k^j}{\hat{r}_{k+1|k}^j} & \frac{\cos \hat{\theta}_k^j}{\hat{r}_{k+1|k}^j} & -1 \end{bmatrix} \tilde{x}_{k+1|k} + w_{k+1}^j$$

$$\boxed{= H_{k+1}^j \tilde{x}_{k+1|k} + w_k^j}$$

thus, stacking up these equations for $y = [y_1, y_2, ..., y_{p_k}]$,

$$r_{k+1} = H_{k+1} \tilde{x}_{k+1|k} + w_k$$



- **Measurement update:**

$$\boxed{\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}[y_{k+1} - \hat{y}_{k+1}]}$$

with Kalman gain $K_{k+1} = P_{k+1|k} H_{k+1}^T S_{k+1}^{-1}$.

# Simple EKF-SLAM

- Estimated measurements: for $j = 1, ..., p_k$,

$$\boxed{\hat{y}_{j,k+1} = \begin{pmatrix} \sqrt{(\hat{x}_{r,k+1|k} - x_{l_j})^2 + (\hat{y}_{r,k+1|k} - y_{l_j})^2} \\ \text{atan2}(\hat{y}_{r,k+1|k} - y_{l_j}, \hat{x}_{r,k+1|k} - x_{l_j}) - \hat{\theta}_{r,k+1|k} \end{pmatrix}}$$

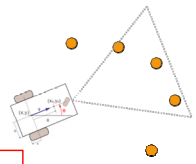
- Linearization measurement equation for uncertainty propagation:

$$r^j_{k+1} = \hat{y}_{j,k+1} - y_{j,k+1} = \begin{bmatrix} \frac{\hat{x}_{r,k+1|k} - x_{l_j}}{\hat{r}^j_{k+1|k}} & \frac{\hat{y}_{r,k+1|k} - y_{l_j}}{\hat{r}^j_{k+1|k}} & 0 \\ \frac{-\sin\hat{\theta}^j_k}{\hat{r}^j_{k+1|k}} & \frac{\cos\hat{\theta}^j_k}{\hat{r}^j_{k+1|k}} & -1 \end{bmatrix} \tilde{x}_{k+1|k} + w^j_{k+1}$$

$$\boxed{= H^j_{k+1}\tilde{x}_{k+1|k} + w^j_k}$$

thus, stacking up these equations for $y = [y_1, y_2, ..., y_{p_k}]$,

$$r_{k+1} = H_{k+1}\tilde{x}_{k+1|k} + w_k$$



- **Measurement update:**

$$\boxed{\hat{x}_{k+1|k+1} = \hat{x}_{k+1|k} + K_{k+1}[y_{k+1} - \hat{y}_{k+1}]}$$

with Kalman gain $K_{k+1} = P_{k+1|k}H^T_{k+1}S^{-1}_{k+1}$.

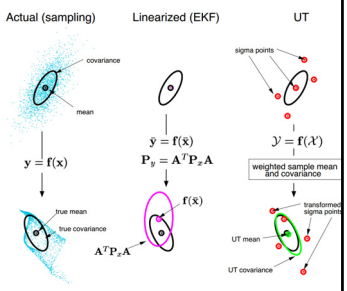


# EKF: Example 2

# Unscented Transformation

- EKF propagates mean to mean via nonlinear equation, and uncertainty via linear approximation $\Rightarrow$ **distortion** via nonlinear mapping not considered.
- Linearization (EKF) vs Unscented Transformation (UKF)
  - Consider $y = g(x)$ w/ RV $x \approx (\bar{x}, P_x)$ propagate via nonlinear $g$.
  - Mean mapping + linearization (EKF):

$$\boxed{\bar{y}^{\text{EKF}} = g(\bar{x}), \quad P_y^{\text{EKF}} = \left[\frac{\partial g}{\partial x}\right]_{\bar{x}} P_x \left[\frac{\partial g}{\partial x}\right]_{\bar{x}}^T}$$




  - **Unscented Transformation (UKF):**
    1. Define $2n+1$ sigma points $x_i^\sigma \in \Re^n$:

$$\boxed{x_o^\sigma = \bar{x}, \quad x_i^\sigma = \bar{x} \pm \left[\sqrt{(n+\lambda)P_x}\right]_i}$$

    2. Propagate $x_i^\sigma$ directly via $g$: $\mathcal{Y}_i = g(x_i^\sigma)$.
    3. Estimate $(\bar{y}, P_y)$ using $\mathcal{Y}_i$ via

$$\boxed{\bar{y} \approx \sum W_i^m \mathcal{Y}_i, \quad P_y \approx \sum W_i^c \left[(\mathcal{Y}_i - \bar{y})(\mathcal{Y}_i - \bar{y})^T\right]}$$

    with $W_o^m = \frac{\lambda}{n+\lambda}$, $W_o^c = \frac{\lambda}{n+\lambda} + (1-\alpha^2+\beta)$, $W_i^m = W_i^c = \frac{1}{2(n+\lambda)}$, $\lambda = \alpha^2(n+k) - n$ ($\alpha \approx 0$, $k = 0$, $\beta = 2$).
    4. 3rd order accurate for Gaussian $x$, second-order for non-Gaussian.



# Unscented Kalman Filtering

- Nonlinear plant dynamics and measurement models w/ process/sensing noise:

$$\boxed{x_{k+1} = f_k(x_k, u_k, v_k), \quad y_k = h_k(x_k, w_k)}$$

- **Augmented state** for propagation: $x_k^a := (x_k, v_k, w_k)$.
- **Initialization:**

$$\bar{x}_o^a = E[x_o^a] = (\bar{x}_o, 0, 0)$$
$$P_o^a = E[(x_o^a - \bar{x}_o^a)(x_o^a - \bar{x}_o^a)^T] = \text{diag}[P_{xo}, V_k, W_k]$$

- **Iteration:** given $\bar{x}_k^a = (\bar{x}_{k|k}, 0, 0)$ and $P_k^a = \text{diag}[P_{x,k|k}, 0, 0]$,
- **Sigma points** generation:

$$x_{i,k}^{a\sigma} := \{\bar{x}_k^a, \bar{x}_k^a \pm \left[\sqrt{(n+\lambda)P_k^a}\right]_i\}, \quad x_{i,k}^{a\sigma} =: (x_{i,k}^\sigma, v_{i,k}^\sigma, w_{i,k}^\sigma)$$

- **Prediction** by propagating $(2n+1)$-sigma points via nonlinear map $f_k$:

$$\boxed{\begin{aligned} \mathcal{X}_{i,k+1|k}^x &= f_k(x_{i,k}^\sigma, u_k, v_{i,k}^\sigma) \\ \bar{x}_{k+1|k} &= \sum W_i^m \mathcal{X}_{i,k+1|k}^x \\ P_{x,k+1|k} &= \sum W_i^c [\mathcal{X}_{i,k+1|k}^x - \bar{x}_{k+1|k}][\mathcal{X}_{i,k+1|k}^x - \bar{x}_{k+1|k}]^T \end{aligned}}$$

# Unscented Kalman Filtering

- Nonlinear plant dynamics and measurement models w/ process/sensing noise:

$$x_{k+1} = f_k(x_k, u_k, v_k), \quad y_k = h_k(x_k, w_k)$$

- **Sigma points** generation:

$$x_{i,k}^{a\sigma} := \{\bar{x}_k^a, \bar{x}_k^a \pm \left[\sqrt{(n+\lambda)P_k^a}\right]_i\}, \quad x_{i,k}^{a\sigma} =: (x_{i,k}^{\sigma}, v_{i,k}^{\sigma}, w_{i,k}^{\sigma})$$

- **Measurement estimate** by propagating sigma points via $h_{k+1}$:

$$\boxed{\begin{aligned} \mathcal{Y}_{i,k+1|k} &= h_{k+1}(\mathcal{X}_{i,k+1|k}^{x}, w_{i,k}^{\sigma}) \\ \hat{y}_{k+1} &= \sum W_i^m \mathcal{Y}_{i,k+1|k} \end{aligned}}$$

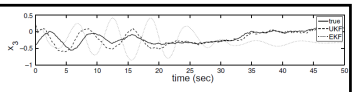
- **Measurement update** (same form as EKF):

$$\boxed{\begin{aligned} \bar{x}_{k+1|k+1} &= \bar{x}_{k+1|x} + K_{k+1} \cdot [y_{k+1} - \hat{y}_{k+1}] \\ P_{k+1|k+1} &= P_{k+1|k} - K_{k+1} P_{\tilde{y}_{k+1}\tilde{y}_{k+1}} K_{k+1}^T \end{aligned}}$$

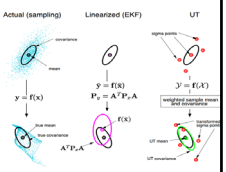
where $K_{k+1} := P_{\tilde{x}_{k+1|k}\tilde{y}_{k+1}} P_{\tilde{y}_{k+1}\tilde{y}_{k+1}}^{-1}$ is the **Kalman gain** with $P_{\tilde{y}_{k+1}\tilde{y}_{k+1}} = \sum W_i^c [\mathcal{Y}_{i,k+1|k} - \hat{y}_{k+1}][\mathcal{Y}_{i,k+1|k} - \hat{y}_{k+1}]^T$ (i.e., residual variance), and $P_{\tilde{x}_{k+1|k}\tilde{y}_{k+1}} = \sum W_i^c [\mathcal{X}_{i,k+1|k}^{x} - \bar{x}_{k+1|k}][\mathcal{Y}_{i,k+1|k} - \hat{y}_{k+1}]^T$ (i.e., cross-variance).



# EKF vs UKF




- EKF utilizes mean-to-mean propagation and linearized equation for uncertainty propagation ⇒ nonlinearity of the mapping (e.g., distortion) not properly considered, only approximated uncertainty propagation.
- UKF propagates opportunistically-chosen $(2n+1)$ sigma points directly via nonlinear mapping to estimate mean and covariance of mapped points.
  - Sampling-based method.
  - 3rd order accurate for GRV, 2nd order for non-Gaussian.
  - Better prediction/covariance accuracy than EFK.
  - No need to compute Jacobian (e.g., complex $f_k, h_k$)




  - Same measurement update form with Kalman gain $K_{k+1}$, i.e.,

$$\boxed{K_{k+1}^{\text{UKF}} = P_{\tilde{x}_{k+1|k}\tilde{y}_{k+1}} P_{\tilde{y}_{k+1}\tilde{y}_{k+1}}^{-1} \approx K_{k+1}^{\text{EKF}} = P_{k+1|k} H_{k+1}^T S_{k+1}^{-1}}$$

$P_{\tilde{y}_{k+1}\tilde{y}_{k+1}} = E[(\hat{y}_{k+1} - y_{k+1})(\hat{y}_{k+1} - y_{k+1})^T] = H_{k+1} P_{k+1|k} H_{k+1}^T + W_k = S_{k+1}$ and $P_{\tilde{x}_{k+1|k}\tilde{y}_{k+1}} = E\left[(\hat{x}_{k+1|k} - x_{k+1})[H_{k+1}(\hat{x}_{k+1|k} - x_{k+1}) + w_k]^T\right] = P_{k+1|k} H_{k+1}^T$.

  - EKF may become inconsistent (i.e., spurious update with over-confidence) due to fake information generated by different linearization points.

# Unscented Transformation



# Vehicle Kinematic Modeling

- Vehicle equation:

$$\dot{x}_i = V_i \cos\phi_i, \quad \dot{y}_i = V_i \sin\phi_i, \quad \dot{\phi}_i = w_i$$

- Vehicle model with measurement:

$$\dot{\hat{x}}_i = V_{im} \cos\hat{\phi}_i, \quad \dot{\hat{y}}_i = V_{im} \sin\hat{\phi}_i, \quad \dot{\hat{\phi}}_i = w_{im}$$

where $V_{im} = V_i + w_{Vi}$ and $w_{im} = w_i + w_{wi}$ are measurement corrupted by Gaussian sensor noise $w_{Vi}, w_{wi}$.

- Esitmation errors: $\tilde{x}_i = \hat{x}_i - x_i$, $\tilde{y}_i = \hat{y}_i - y_i$, $\tilde{\phi}_i = \hat{\phi}_i - \phi_i$.
- Linearized discrete error state equation:

$$\begin{pmatrix} \tilde{x}_{i,k+1} \\ \tilde{y}_{i,k+1} \\ \tilde{\phi}_{i,k+1} \end{pmatrix} = \begin{bmatrix} 1 & 0 & -V_{im}\sin\hat{\phi}_i\Delta t \\ 0 & 1 & V_{im}\cos\hat{\phi}_i\Delta t \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \tilde{x}_{i,k} \\ \tilde{y}_{i,k} \\ \tilde{\phi}_{i,k} \end{pmatrix} + \begin{bmatrix} \cos\hat{\phi}_i\Delta t & 0 \\ \sin\hat{\phi}_i\Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{pmatrix} w_{Vi,k} \\ w_{wi,k} \end{pmatrix}$$

i,e.,

$$\tilde{x}_{i,k+1} = \Phi_{i,k+1,k}\tilde{x}_{i,k} + G_{i,k}w_{i,k}$$

- Stacked error state propagation equation

# Vehicle Kinematic Modeling

- Vehicle equation: $\dot{x}_i = V_i \cos\phi_i$, $\dot{y}_i = V_i \sin\phi_i$, $\dot{\phi}_i = w_i$.
- Vehicle model with measurement:

$$\dot{\hat{x}}_i = V_{im}\cos\hat{\phi}_i, \quad \dot{\hat{y}}_i = V_{im}\sin\hat{\phi}_i, \quad \dot{\hat{\phi}}_i = w_{im}$$

  where $V_{im} = V_i + w_{Vi}$ and $w_{im} = w_i + w_{wi}$ are measurement corrupted by Gaussian sensor noise $w_{Vi}, w_{wi}$.
- Esitmation errors: $\tilde{x}_i = \hat{x}_i - x_i$, $\tilde{y}_i = \hat{y}_i - y_i$, $\tilde{\phi}_i = \hat{\phi}_i - \phi_i$.
- Linearized discrete error state equation:

$$\begin{pmatrix} \tilde{x}_{i,k+1} \\ \tilde{y}_{i,k+1} \\ \tilde{\phi}_{i,k+1} \end{pmatrix} = \begin{bmatrix} 1 & 0 & -V_{im}\,\mathrm{s}\,\hat{\phi}_i\Delta t \\ 0 & 1 & V_{im}\,\mathrm{c}\,\hat{\phi}_i\Delta t \\ 0 & 0 & 1 \end{bmatrix} \begin{pmatrix} \tilde{x}_{i,k} \\ \tilde{y}_{i,k} \\ \tilde{\phi}_{i,k} \end{pmatrix} + \begin{bmatrix} \mathrm{c}\,\hat{\phi}_i\Delta t & 0 \\ \mathrm{s}\,\hat{\phi}_i\Delta t & 0 \\ 0 & \Delta t \end{bmatrix} \begin{pmatrix} w_{Vi,k} \\ w_{wi,k} \end{pmatrix}$$

- Relative pose measurement: $z_{23} = \left[\ C^T(\phi_2)[p_3 - p_2]; \quad \phi_3 - \phi_2\ \right] \in \Re^3$, with $p_i = [x_i; y_i] \in \mathrm{E}(2)$ and $C(\phi_i) \in \mathrm{SO}(2)$.
- Real relative measurement:

  $\hat{z}_{23} = \left[\ C^T(\hat{\phi}_2)[\hat{p}_3 - \hat{p}_2]; \quad \hat{\phi}_3 - \hat{\phi}_2\ \right] + n_{23}$ (ideal) $z_{23} = \left[\ C^T(\phi_2)[p_3 - p_2]; \quad \phi_3 - \phi_2\ \right]$ $\Re^3$, with $p_i = [x_i; y_i] \in \mathrm{E}(2)$ and $C(\phi_i) \in \mathrm{SO}(2)$.



# Geometry of Lagrange Multiplier

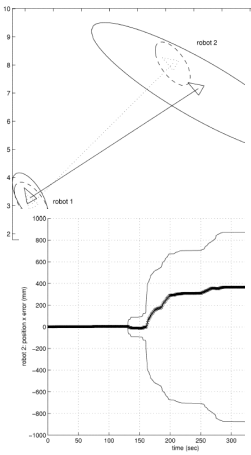



Fig. 4. Position $x$ error for robot 2 when no relative pose measurements are available. The robot integrates the rotational and translational velocity measured by the encoders in order to estimate its position (DR). The two bounding lines determine the $3\sigma$ region of confidence for the position $x$ error, and they are calculated based on the covariance of the position $x$ estimate. The flatline portions of the region of confidence (constant position uncertainty) correspond to time intervals that the robot was moving very slowly.

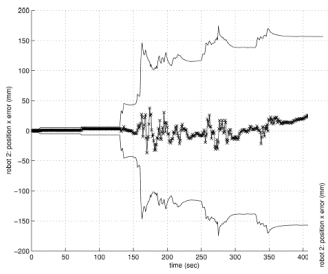



Fig. 5. Position $x$ error for robot 2 when the robots continuously relative position and orientation. The position uncertainty is almost one magnitude smaller compared to the case of simple DR. Note the diffe scale between this figure and Figs. 4 and 6. The two bounding lines d the $3\sigma$ region of confidence for the position $x$ error, and they are c based on the covariance of the position $x$ estimate. The flatline portio region of confidence (constant position uncertainty) correspond to time that the robot was moving very slowly.

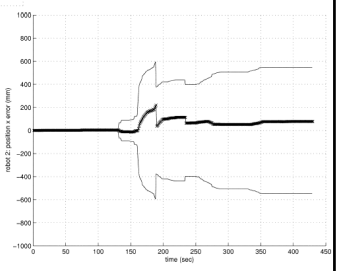



Fig. 6. Position $x$ error for robot 2 when the group collects relative position and orientation measurements intermittently (during the time instants $t = 189$ s and $t = 234$ s). The rest of the time, each of the robots dead reckons its position. Note the sharp decrease in uncertainty for $t = 189$ s and $t = 234$ s. The two bounding lines determine the $3\sigma$ region of confidence for the position $x$ error and they are calculated based on the covariance of the position $x$ estimate. The flatline portions of the region of confidence (constant position uncertainty) correspond to time intervals that the robot was moving very slowly.

# Geometry of Lagrange Multiplier

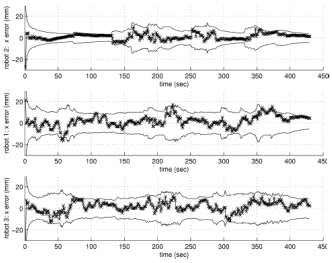



Fig. 9. Position $x$ error for robots 2, 1, and 3 when robot 2 receives absolu positioning information and continuously measures relative position ar orientation with respect to robots 1 and 3. The bounding lines around tl positioning errors determine the $3\sigma$ regions of confidence for the positic $x$ error, and they are calculated based on the covariance of the position estimates. For all three robots, the position error is bounded.

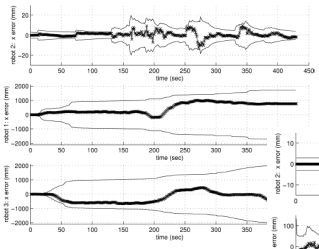



Fig. 11. Position $x$ error for robots 2, 1, and 3 when robot 2 rece positioning information and there is *no communication* with rol The bounding lines around the positioning errors determine the of confidence for the position $x$ error, and they are calculated b covariance of the position $x$ estimates. The position error is boun robot 2, while for robots 1 and 3, it grows continuously while move.

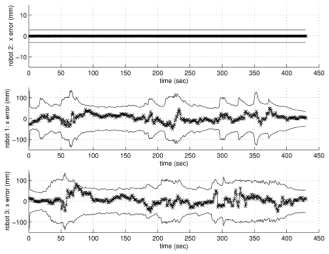



Fig. 12. Position $x$ error for robots 2, 1, and 3 when robot 2 is standing still while robots 1 and 3 continuously measure their relative position and orientation with respect to robot 2. The bounding lines around the positioning errors determine the $3\sigma$ regions of confidence for the position $x$ error, and they are calculated based on the covariance of the position $x$ estimates. Since robot 2 is not moving, its position uncertainty is constant. For the other two robots that use robot 2 as a landmark, the position error is bounded.