Signals and Systems Lecture 1. March. 3. 2016.

- Go over Lecture 1 ppt (10~15 min)
- Syllabus (5 min)

- Rules

① Don't lie
Don't Cheat
Don't Copy
Do your HW by yourself. – on/off grading
not based on final
Results.

② workload

you have 60 hr / week

4전공 1교양
55 hrs 5 hrs

$$\begin{array}{r}14\\4\overline{)55}\end{array}$$

14 hours / week

→ 3 hours of lecture
11 hours for HW.

③ HW.
- Reading ☆ very Important
- Examples
- Problem sets

) will be in your exam.

④ Exam ~4 Questions
- ~~from~~ HW problem
- ~~from~~ modied HW problem.
- Intuitive. problem
- Hard problem.

⑤ Notation: "f" instead of "ω" ⇐ write down.

⑥ Textbook: Buy English version. Return. Korean translated book. There is no hope if you ~~dare~~ are not fluent in reading.

⑦ Langage : Target 85% English + 15% Korean.
"But" we may change to Korean if everyone agrees.

⑧ Matlab : very important.
tutorial (This week / Next week)
→ Not mandatory. Solve your HW.

⑨ Organization of the course
- Signal & Systems: LTI
- Fourier transform.
- Sampling.
- Discrete time domain FT.

⑩ ~~Matlab example.~~ 초안지.

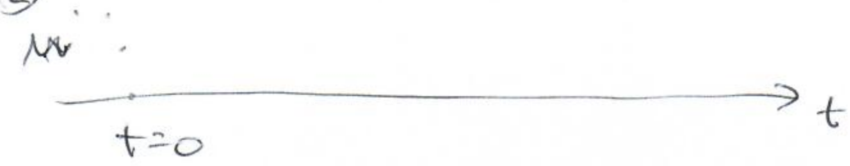
Chapter 1

Signals & Systems.

1.1. Continuous time & Discrete-time Signals

①

②



③

④

Which one is not likely ~~to~~ to be a signal!

- signal: a fn of indep. variables

  t: indep. variable.

  f(t): signal.

CT- signal: $f(t) = \sin t$

dt - signal $f(n) = \sin a\pi n$

indep. variable

$\boxed{n: \text{Integer}}$.

Q: Digital?

Q: CT → DT ??

3/3
Stop!

- Signal energy & power.

total emgergy over $t_1 \le t \le t_2$
or $n_1 \le n \le n_2$

$$\int_{t_1}^{t_2} |x(t)|^2 dt$$

→ Can be complex

$$\sum_{n=n_1}^{n_2} |x(n)|^2$$

time averged power $\frac{1}{t_2-t_1}\int_{t_1}^{t_2} |x(t)|^2 dt$

$$\frac{1}{n_2-n_1+1}\sum_{n=n_1}^{n_2} |x(n)|^2$$

We will use $x(n)$ instead of $x[n]$

$$E_\infty \triangleq \int_{\infty}^{\infty} |x(t)|^2 dt$$
$$\triangleq \sum_{n=-\infty}^{\infty} |x(n)|^2$$

Can be infinite.

$$P_\infty \triangleq \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} |x(t)|^2 dt$$
$$\triangleq \lim_{N\to\infty} \frac{1}{2N+1}\sum_{n=-\infty}^{\infty} |x(n)|^2$$

- Classes of signals

① $E_\infty < \infty \longrightarrow P_\infty = 0.$ Ex) $x(t) = \begin{cases} 1 & 0 \le t \le 1 \\ 0 & \end{cases}$

② $P_\infty < \infty$ & $E_\infty = \infty$ Ex) $x(n) = 4$

③ $P_\infty = \infty$ & $E_\infty = \infty$ $x(t) = t$

1.2. transformations of indep. variables

- Time shift : $x(t-t_0)$

  $x(n-n_0)$

- Time reversal $x(-t)$

  $x(-n)$

- Time scaling $x(2t)$: fast

  $x(t/2)$: slow.

- combined:

  $x(at+b)$

- periodic signal

  $x(t) = x(t+T)$

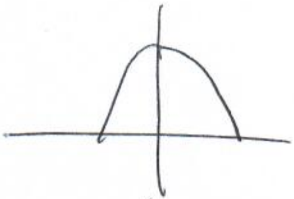
  period. "fundamental period"

  eg. $\sin(t)$

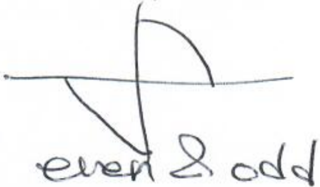
  $x(n) = x(n+N)$

- Even & odd signals

  even $x(-t) = x(t)$



  odd $x(-t) = -x(t)$



→ A real signal is sum of even & odd sig

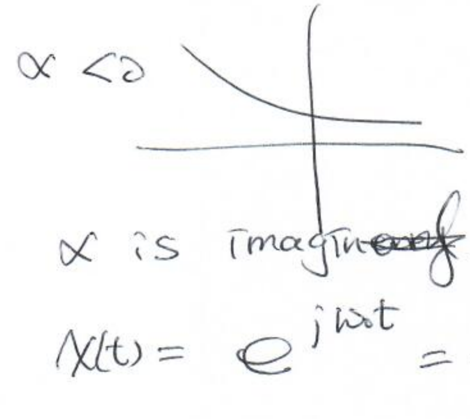
→ Demonstrate this.

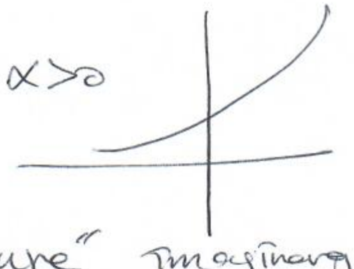
1.3. Exponentials & sinusoidal signals

- Complex exponential: $x(t)=Ce^{\alpha t}$

~~to~~ Very simple looking! what does it mean?
↑ ask to students

If α & C are real.

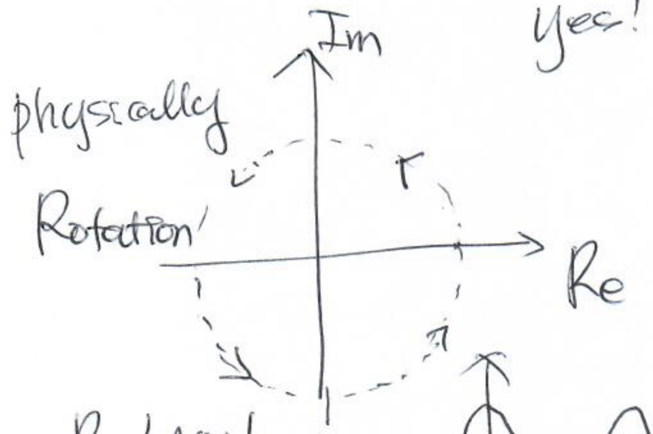




If α is ~~imaginary~~ imaginary (or "pure" imaginary)

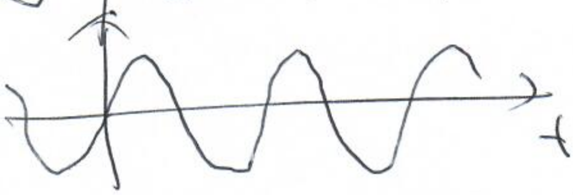
$x(t)= e^{j\omega_0 t} = e^{j2\pi f_0 t} = \cos 2\pi f_0 t + j \sin 2\pi f_0 t$

yes! we will use "f"



Re{x(t)}:

Im{x(t)}:



periodicity. $e^{j2\pi f_0 t} = e^{j2\pi f_0 (t+T)}$

when $e^{j2\pi f_0 T}=1$

$2\pi f_0 T = 2\pi n$ (n: nonzero integer)

fundamental period $T_{fund} = \frac{1}{f_0}$

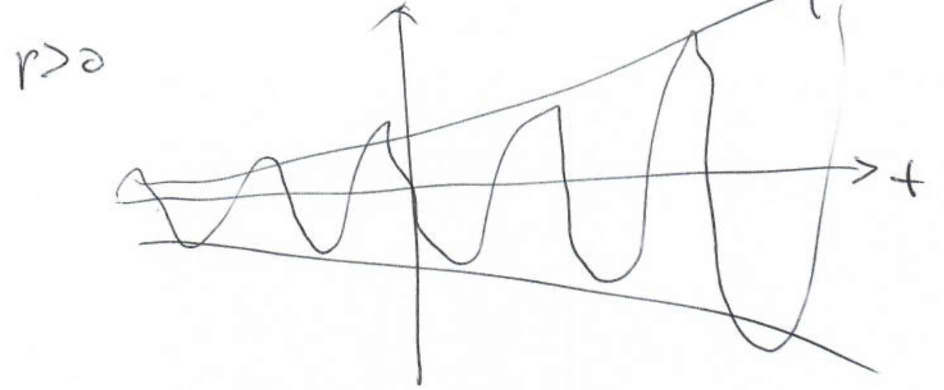
- If α & C are both complex

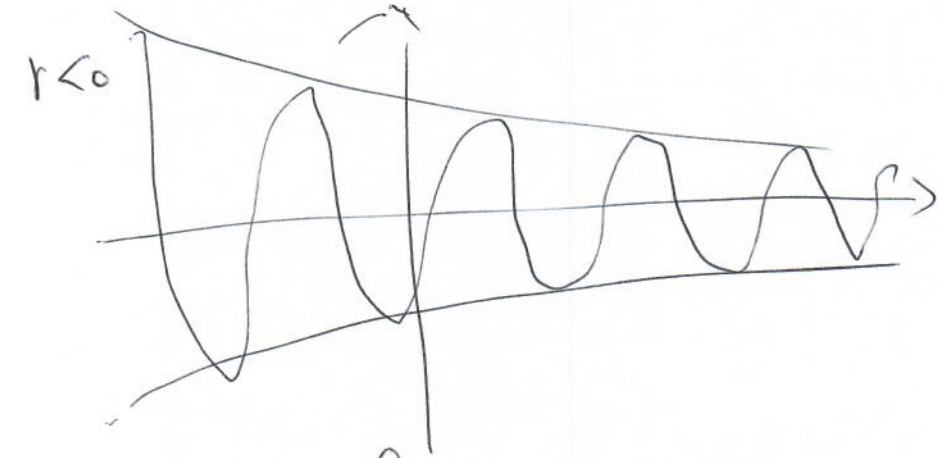
let's write $C = |C| e^{j\theta}$

$\alpha = r + j2\pi f_0$

$$Ce^{\alpha t} = |C| e^{j\theta} e^{(r+j2\pi f_0)t}$$

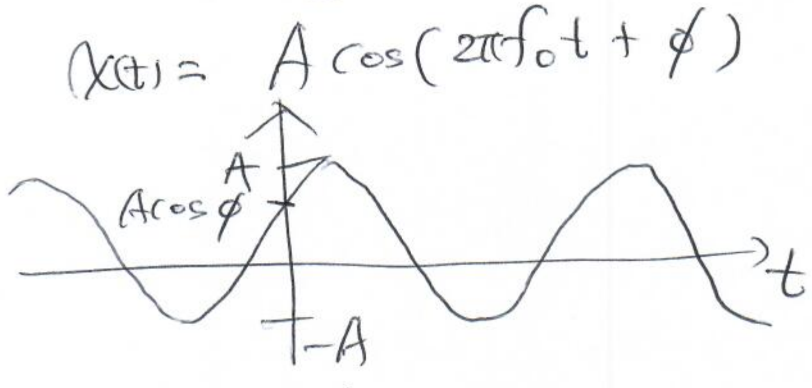
$$= |C| e^{rt} e^{j(2\pi f_0 t + \underbrace{\theta}_{\text{phase offset.}})}$$





- sinusoidal signal

$$x(t) = A\cos(2\pi f_0 t + \phi)$$



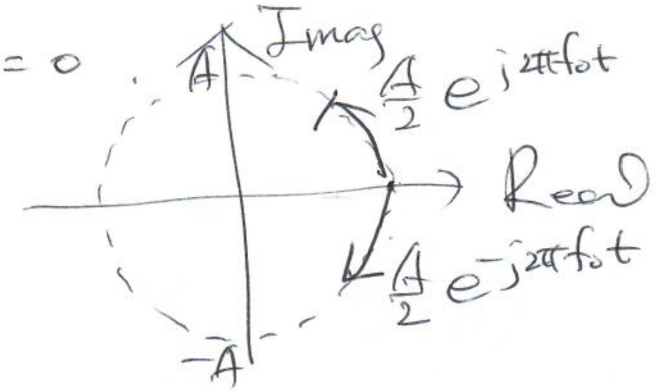
$$T_{\text{fund}} = \frac{1}{f_0}$$

if t = sec ϕ : radian ω_0 : radian/s $= 2\pi f_0$

f_0 : cycle/sec or Hz

$$A\cos(2\pi f_0 t+\phi) = \frac{A}{2}\left(e^{j\phi}e^{j2\pi f_0 t} + e^{-j\phi}e^{j-2\pi f_0 t}\right) \quad ⑥$$

if $\phi = 0$.



- Eugery & power in complex exp. sinusoidal

$$P_\infty = \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}\left|e^{j2\pi f_0 t}\right|^2 dt = 1$$

i.e finite averege power.

$$E_\infty = \int_{-\infty}^{\infty}\left|e^{j2\pi f_0 t}\right|^2 dt = \infty$$

i.e. infinite total energy.

- Harmonically related complex exponentials

→ sets of periodic exponential with a common period "T_0"

$$e^{j2\pi f T_0} = 1$$

$2\pi f T_0 = 2\pi k$, k = ~~none zero~~ integer.

$$f_0 = \frac{1}{T_0}$$

$\phi_k(t) = e^{j2\pi k f_0 t}$, k: integer. (including '0')

⇒ forms basis functions (later).

- Discrete-time Complex exponential & sinusoidal signal ⑦

$$x[n] = C\alpha^n = Ce^{\beta n} \quad \text{where} \quad \alpha = e^{\beta}$$

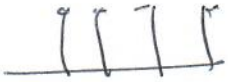
If C and α are real

~~$|\alpha| > 1$ vs $|\alpha| < 1$~~

$\alpha > 1$

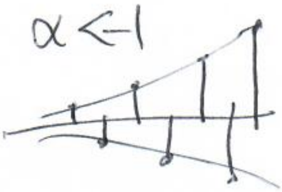
$0 < \alpha < 1$

$\alpha = 1$



$-1 \alpha < 0$

$\alpha < -1$



If C and α are complex

$C = |C|e^{j\theta} \quad \alpha = |\alpha|e^{j2\pi f_0}$

$C\alpha^n = |C||\alpha|^n\{\cos(2\pi f_0 n + \theta) + j\sin(2\pi f_0 n + \theta)$

$\alpha > 1$

$0 < \alpha < 1$

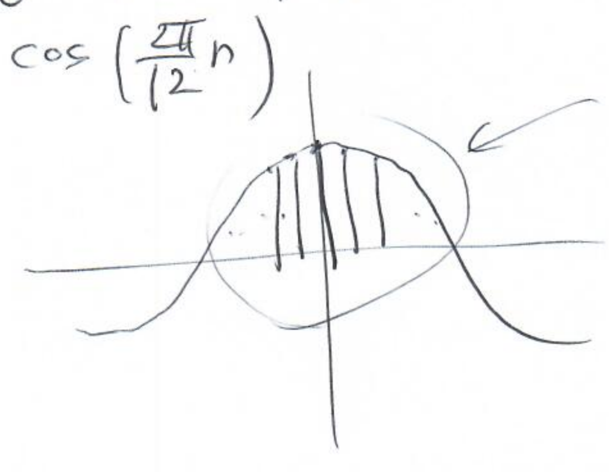
- Sinusoidal signal

β is pure Imaginery (i.e. $|\alpha| = 1$)

$x(n) = Ce^{j2\pi f_0 n} = |C|e^{j(2\pi f_0 n + \theta)}$

$E_\infty = \infty \quad P_\infty < \infty$

ex) $\cos\left(\frac{2\pi}{12}n\right)$



How many points?

Chapter 2

LTI

3月15日 Start ①

Chapter 2 LTI Systems

2.0 LTI ?

What is linear ?

What is TI ?

examples of LTI system ~~y=x(t)~~ $y(t) = x(t)$

" of L but not TI $x(t^2]$

" nonlinear but TI $x^2(t)$

" nonlinear & not TI $x^2(t^2)$

interms of { math →
electrical →
Life. → vending machine
Neuron
girl friend / boy fried
~~Lunch Coupon~~. Magic

2.1. DT LTI system : ~~"Magic"~~ → Convolution Sum

$$x(n) = \cdots + x(-3)\delta(n+3) + x(-2)\delta(n+2) + x(-1)\delta(n+1)$$

$$+ x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + \cdots$$

Show Fig 2.1

$$= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Sifting property of discrete-time unit impulse
체를 거르다.

→ a discrete time signal can be represented by sum of shifted $\delta(n)$

$$x(n) \longrightarrow \boxed{\mathcal{H}_d} \longrightarrow y(n)$$

$$y(n) = \mathcal{H}_d[x(n)] = \mathcal{H}_d\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

If h(n) is linear.

Superposition → $= \sum_{k=-\infty}^{\infty} \mathcal{H}(x(k)\delta(n-k))$

Scaling → $= \sum_{k=-\infty}^{\infty} x(k)\mathcal{H}(\delta(n-k))$

$$= \sum_{k=-\infty}^{\infty} x(k)h_k(n)$$

If h(n) is linear & time invariant

$$= \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$

where h(n) is unit impulse response function of the system.

i.e. $\delta(n) \rightarrow \boxed{\mathcal{H}} \rightarrow h(n)$

$\delta(n-k) \rightarrow \boxed{\mathcal{H}} \rightarrow h(n-k)$ ∵ time-invariant

∴ In LTI systems

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k)$$, very important.

So we have a name for this : Convolution.

$$y(n) \triangleq x(n) * h(n)$$

More importantly, an LTI system is completely characterized by an "impulse"

h(n): Impulse response function

for any known input, we can calculate the output of the LTI system if we know impulse response fn

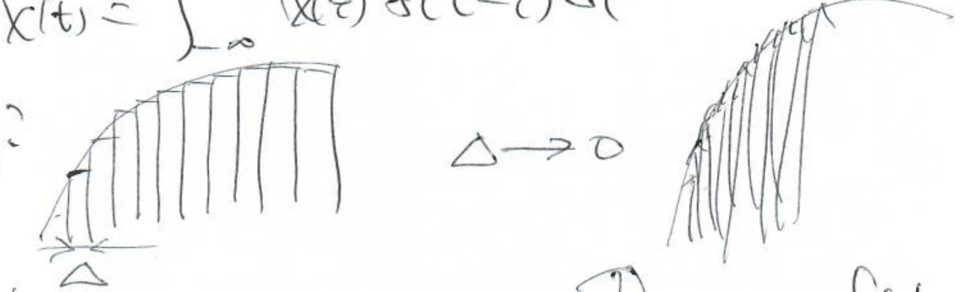
- Examples of Convolution by hand
  by computer program,
  This is very important!

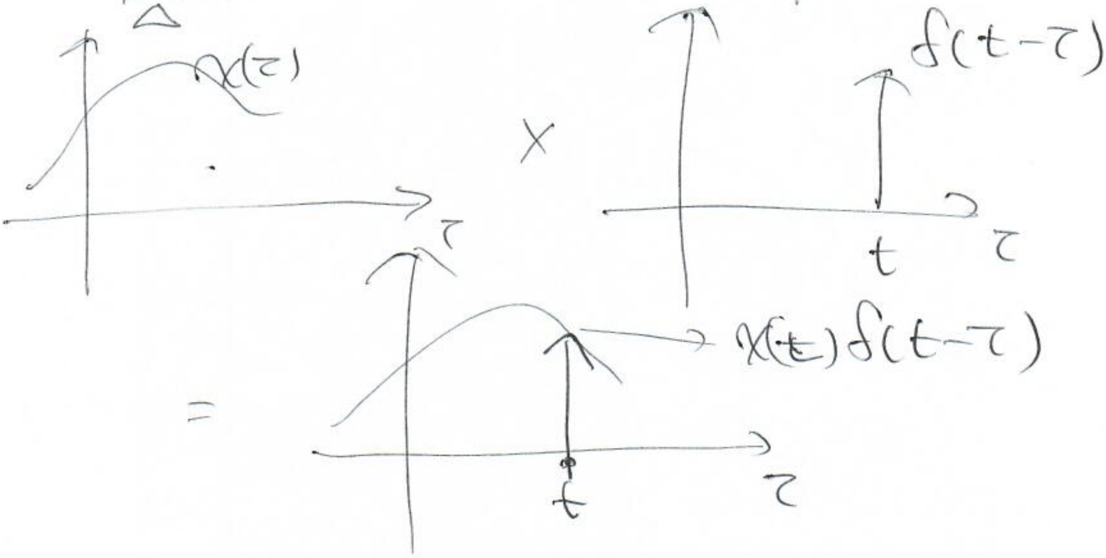
- 2.2. CT LTI system : Convolution integral

Similarly to DT case

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau$$

why?





$$x(t) \rightarrow \boxed{H} \rightarrow y(t)$$

$$y(t) = H(\ \ x(t)\ )$$

$$= H\left(\int_{-\infty}^{\infty} x(\tau)\,\delta(t-\tau)\,d\tau \right)$$

$$\xrightarrow{\text{Superposition}} \quad = \int_{-\infty}^{\infty} H\big(x(\tau)\,\delta(t-\tau)\big)\,d\tau$$

$$\xrightarrow{\text{Scaling}} \quad = \int_{-\infty}^{\infty} x(\tau)\,H\big(\delta(t-\tau)\big)\,d\tau$$

$$\xrightarrow{\text{Time Invariance}} = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$\triangleq x(t) * h(t)$$

) Convolution Integral.

Show examples (Fig 2.17, 19)

2.3. properties of LTI systems

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n)$$

$$y(t) = \int_{\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

✣ impulse response fully characterize the system.
(only in LTI) ~~what if it is not linear or T.I~~

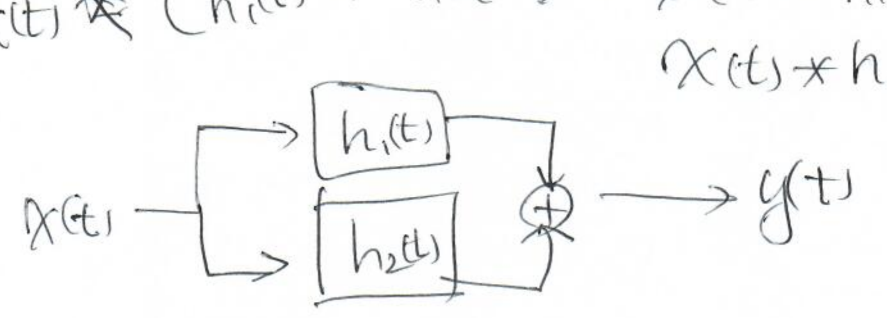
2.3.1 Commutative property

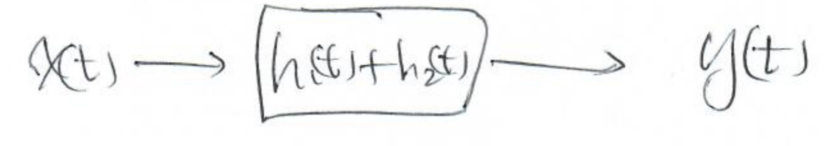
$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

→ This means we can choose ~~total~~ whichone to reverse/shifts in convolution.

2.3.2. Distributive property.

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$





2.3.3 Associative property.

$$(x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$$

$$x(t) \rightarrow \boxed{h_1(t)} \rightarrow \boxed{h_2(t)} \rightarrow y(t)$$

$$x(t) \rightarrow \boxed{h_1(t) * h_2(t)} \rightarrow y(t)$$

Commutative →

$$x(t) \rightarrow \boxed{h_2(t)} \rightarrow \boxed{h_1(t)} \rightarrow y(t)$$

When ? in LTI What if it is not LTI ?

2.3.4. Memory

memoryless $h(n) = k\delta(n)$
$h(t) = k\delta(t)$

otherwise the system has memory

2.3.5 Invertibility

Identity system ; $\delta(n)$ or $\delta(t)$

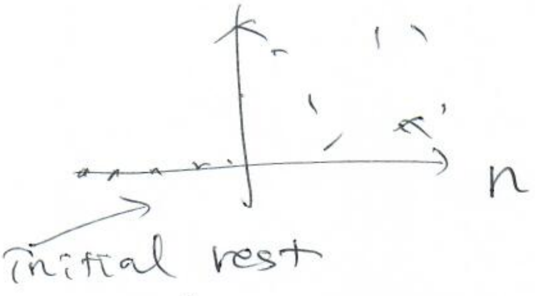
if invertible ~~h(x) h(t)~~ $h(t) * h(t) = \delta(t)$

Stop

Show ex 2.12 ($h(n) = u(n)$
$h_1(n) = \delta(n) - \delta(n-1)$)

2.3.6. Causality : $y(n)$ must not depend on $x(k)$ $k > n$

$h(n) = 0$ for $n < 0$



in causal system $y(n) = \sum_{k=-\infty}^{n} x(k)h(n-k)$

$= \sum_{k=0}^{\infty} h(k)x(n-k)$

or in CT $h(t) = 0$ for $t < 0$

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_0^{\infty} h(\tau)x(t-\tau)d\tau.$$

2.3.7 Stability

BIBO

$|x(n)| \leq B$ for all n

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k) x(n-k) \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h(k)| \, |x(n-k)|$$

$$\leq B \sum_{k=-\infty}^{\infty} |h(k)| \quad ($$

$\Rightarrow$ if $\sum_{k=-\infty}^{\infty} |h(k)| < \infty \rightarrow$ stable system

(sufficient cond.)

(absolutely summable) & necessary cond.

(Prob2,49).

2.3.8 Unit step response.

$$S(n) = u(n) * h(n)$$

$$= h(n) * u(n)$$

$n-k \geq 0$

$$= \sum_{k=-\infty}^{\infty} h(k) \cdot u(n-k) \quad n \geq k$$

$$= \sum_{k=-\infty}^{n} h(k)$$

h(n) can be recovered by

$s(n) - s(n-1)$

In CT

$$s(t) = u(t) * h(t)$$

$$= \int_{-\infty}^{t} h(\tau) d\tau$$

$$h(t) = \frac{ds(t)}{dt} = s'(t)$$

2.4. Causal LTI system by differential & difference equations

"here first"

2.4.1 linear Constant-Coefficient Differential equations.

a large # of system can be written as

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- a_k & b_k are constant e.g.

$$i(t) = \frac{x(t) - y(t)}{R}$$

$$i(t) = C \frac{dy(t)}{dt}$$

$$RC \frac{dy(t)}{dt} + y(t) = x(t)$$

- a_k & b_k constant: R, C does not change over time.
- We can show this is a linear equation.

(not time invariant yet: eg.) Cap. may have initial charge.

- How do we solve this?

assume $RC = 1$

→ need auxillary condition.

$$y(t) = \underbrace{y_p(t)}_{\text{particular solution}} + \underbrace{y_h(t)}_{\text{natural resp. or homogeneous solution}}$$

→ different auxillary Condition. results in diffent solutions. (e.g. Cap initial values)

→ one option is "initial rest"

i.e.) $x(t) = 0$ for $t \le t_0$
$y(t) = 0$ for $t \le t_0$
→ meaning cap has no charge initially.

Then LCCDE will be time invariant & causal

Constraint : $y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^k y(t_0)}{dt^k} = 0$.

→ We will learn how to solve this
In chapter 4 & 9

2.4.2 linear constant coefficient difference Eq.

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Again if initial rest (i.e. $x(n)=0$ for $n<n_0$
then $y(n)=0$ ")
the system is LTI & causal.

Additional things in discrete-time :

$$y(n) = \frac{1}{a_0} \left\{ \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k) \right\}$$

→ recursive equation.

when $N \ge 1$ causal LTI system has
an impulse response of infinite duration

→ infinite impulse response sys.
I I R

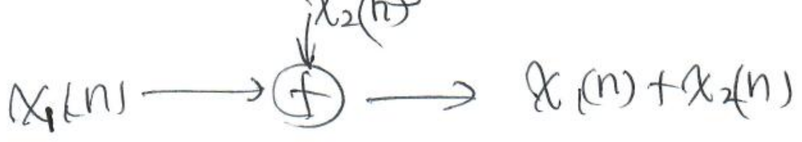
when $N = 0$
$$y(n) = \sum_{k=0}^{M} \left(\frac{b_k}{a_0}\right) x(n-k)$$

→ finite impulse response system

→ wait for Chapter 8&10 F I R

2.4.3. Block diagram.

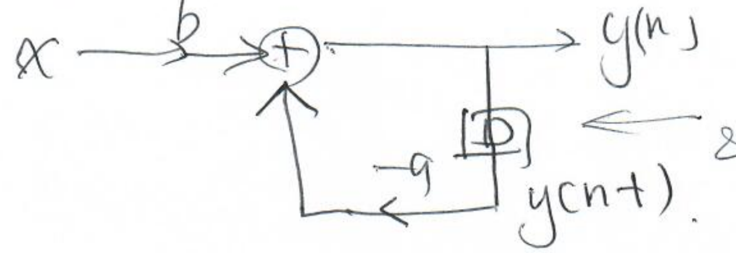
DT



$$x(n) \xrightarrow{a} a\,x(n)$$

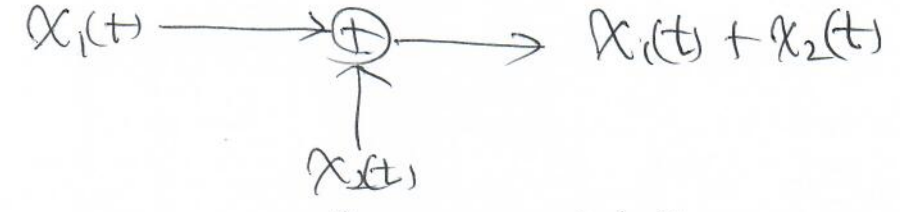
$$x(n) \rightarrow \boxed{D} \rightarrow x(n-1)$$

Ex) $y(n) + a\,y(n-1) = b\,x(n)$



← memory & initial ~~rest~~ value for memory.

CT

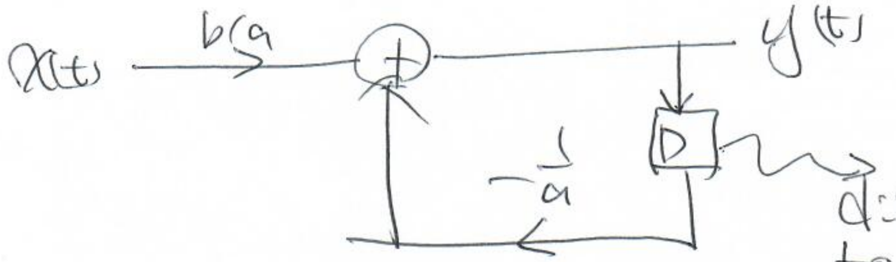


$$x(t) \xrightarrow{a} a\,x(t)$$

$$x(t) \rightarrow \boxed{D} \rightarrow \frac{dx(t)}{dt}$$

$$x(t) \rightarrow \boxed{\int} \rightarrow \int_{-\infty}^{t} x(\tau)\,d\tau$$

Ex) $\frac{dy(t)}{dt} + a\,y(t) = b\,x(t)$



difficult to implement in analog.

3/17 Done

"Break" before teaching periodicity.

- Periodicity property in DT complex exp.

~~In DT domain $e^{j2\pi fn}$ is not always periodic~~

~~A show example Fig 1.25 (page 24)~~

property 1

$$e^{j2\pi f_0 n} = e^{j2\pi f_0 n}\underbrace{e^{j2\pi k}}_{1} = e^{j2\pi f_0 n}\underbrace{e^{j2\pi kn}}_{1}$$

i.e $f_0 = f_0 + k$ k: integer.

or $\omega_0 = \omega_0 + 2\pi k$

This means f_0 is not distinct

and has an interval (1 in f_0, 2π in ω_0)

$\Rightarrow$ $0 \le f_0 < 1$, or $-1/2 \le f_0 < \frac{1}{2}$

$0 \le \omega_0 < 2\pi$, $-\pi \le \omega_0 < \pi$

$\Rightarrow$ $f_0 = 1/2$: highest frequency.

STOP 3/18

$$e^{j2\pi(1/2)n} = e^{j\pi n} = (-1)^n$$

~~very~~ fastest change.

Show Fig 1,27

property 2 periodicity.

To be periodic with N , $e^{j2\pi f_0(n+N)} = e^{j2\pi f_0 n}$

i.e. $e^{j2\pi f_0 N} = 1$

$f_0 N = m$ (N : Integer, m : Integer

$\therefore$ if $f_0 = \frac{m}{N}$ has to be a rational #

then $e^{j2\pi f_0 n}$ is periodic w period N

(ex) $f_0 = \frac{1}{12} \rightarrow e^{j2\pi f_0 n}$ periodic

$f = \frac{1}{12\pi} \rightarrow$ not periodic

Show Ex fn. Figure 1,25 (page 24)

$\boxed{\text{property 3}}$ Fundamental Period

From property 2 $f_0 = \frac{m}{N}$

$\therefore N = \frac{m}{f_0}$ assuming m&N have no common factor

→ This has m times longer period than $1/f_0$ which is fund. period for CT exp.

Example) $\cos\left(\frac{8\pi n}{31}\right)$ $f_0 = \frac{4}{31}$ $N = \frac{31}{4}\overset{\to 4}{m}$

$\therefore$ period $= 31$ $= 31$

$\cos\left(\frac{8\pi t}{31}\right)$ $f_0 = \frac{4}{31}$ $T_{\text{fund.}} = \frac{31}{4}$

show Fig. 1.25(b)

Fundamental frequency $f_{\text{fund}} = \frac{1}{N}$

Ex) $\left[\begin{array}{ll} \cos\left(2\pi\frac{1}{12}n\right) & N = 12 \\ \cos\left(2\pi\frac{1}{12}t\right) & t_0 = 12 \end{array}\right.$

$\left[\begin{array}{ll} \cos\left(8\pi n/31\right) & N = 31 \\ \cos\left(8\pi t/31\right) & T_0 = 31/4 \end{array}\right.$

$\left[\begin{array}{ll} \cos\left(n/6\right) & N = \text{NONE!} \\ \cos\left(t/6\right) & T_0 = 12\pi \end{array}\right.$

- Harmonically periodic exponentials

$\phi_k(n) = e^{jk(\underset{\downarrow \text{ fund. period}}{\frac{2\pi}{N}})n}$, k: Integer

$\phi_{k+N}(n) = e^{jk(\frac{2\pi}{N})n}e^{jN\frac{2\pi}{N}n} = e^{jk\frac{2\pi}{N}n} = \phi_k(n)$

→ only N distinct periodic exp.

$\phi_0(n) = 1$ $\phi_1(n) = e^{j2\pi\frac{n}{N}}$ . . . . $\phi_{N-1}(n) = e^{j2\pi\frac{(N-1)}{N}n}$

→ distinct (are they ~~auto~~ orthogonal?)

1.4. Unit impulse and unit step functions

- Discrete time

unit impulse

$\delta(n) = \begin{cases} 0 & n \neq 0 \\ 1 & n = 0 \end{cases}$ → Kronecker δ

unit step

$u(n) = \begin{cases} 0 & n < 0 \\ 1 & n \geq 0. \end{cases}$

$\delta(n) = u(n) - u(n-1)$

$u(n) = \sum_{m=-\infty}^{n} \delta(m) = \sum_{k=0}^{\infty} \delta(n-k)$

→ very interesting. Show it in Figure!

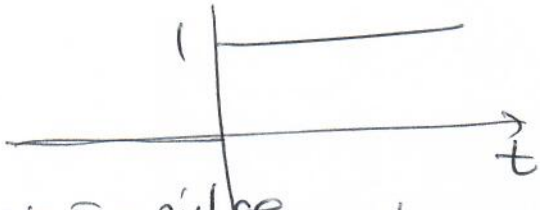
property $x(n)\delta(n) = x(0)\underbrace{\delta(n)}$

Q: why do we need this?

$x(n)\delta(n-n_0) = x(n_0)\delta(n-n_0)$

- Continuous time.

-unit step

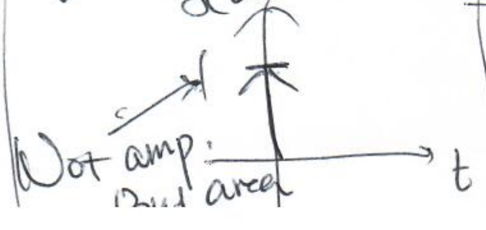


$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t > 0 \end{cases}$

-Unit impulse

$u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$

$\delta(t) = \frac{d\,u(t)}{dt}$ (one way to define it)



→ but $u(t)$ not defined @ $t=0$

$\delta(t) = \begin{cases} +\infty & t = 0 \\ 0 & t \neq 0 \end{cases}$ Dirac δ

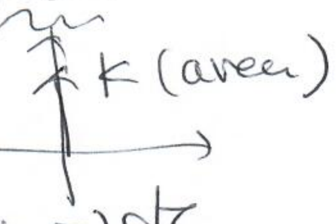
$\int_{-\infty}^{\infty} \delta(t) d\tau = 1$ (generalized

Read p33-35 for your reference

property of $\delta(t)$

- $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau$

$$k u(t) = \int_{-\infty}^{t} \underbrace{k \delta(\tau)} d\tau$$



- $u(t) = \int_{-\infty}^{t} \delta(\tau) d\tau = \underline{\int_{0}^{\infty} \delta(t-\tau) d\tau}$

illustrate in time domain

- $x(t)\delta(t) = x(0)\delta(t)$
- $x(t)\delta(t-t_0) = x(t_0)\delta(t-t_0)$

1.5 Continuous-time & Discrete-time Systems

a system: a process in which input signals are transformed by the system, resulting in other signals as output.

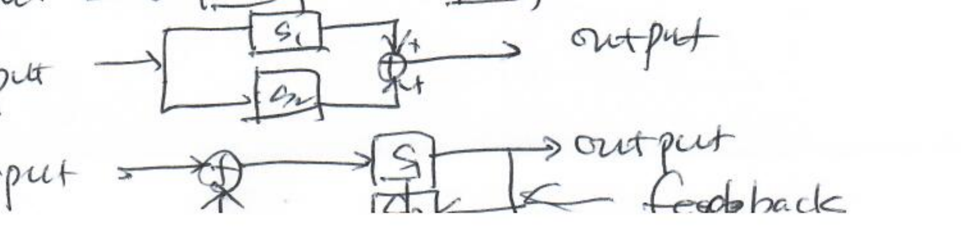
$$x(t) \rightarrow \boxed{\text{CT system}} \rightarrow y(t)$$

$$x(n) \rightarrow \boxed{\text{DT system}} \rightarrow y(n)$$

Example:

This course will focus on a ~~simple~~ class of system named "Linear Time Invariant sys."

- Interconnections of systemes:



1.6 Basic system properties

property 1 : memoryless system: output only depends on current input.

ex) $y(n) = 2x(n) + x^2(n)$

system w memory

ex) $y(n) = \sum_{k=-\infty}^{n} x(k)$, capacitor

$y(n) = x(n-1)$

* Include future input as well

ex) $y(n) = x(n+1)$

property 2 : Invertible system

$$x(n) \to \boxed{S1} \xrightarrow{y(n)} \boxed{S2} \to x(n)$$

↑ Invertible system.

ex) $y(n) = \sum_{k=-\infty}^{n} x(k)$ $\quad w(n) = y(n) - y(n-1)$

$y(t) = x^2(t)$ ⟸ Not Invertible.

encoding decoding.

property 3 : Causality.

Output only depends on input at the present time and the past (i.e. No future data needed)

ex) noncausal $y(n) = x(n+1)$

If data is already collected (e.g. picture) noncausal is not a big issue.

property 4 : Stability

~~Small~~ Bounded input ⇒ bounded output.

ex) stable ball dropped.

unstable: micro/speaker. positive feedback.

~~Property 5 & 6 Time Invariance & Linearity ⇒ Next chapter.~~

property 5 Time invariance

time shift in input results in an identical time shift in output

$$x(n) \xrightarrow{S} y(n) \qquad (n \text{ or } t)$$

$$x(n-n_0) \xrightarrow{S} y(n-n_0)$$

ex) Time variant sys.

<u>ask</u> → $y(n) = n\,x(n)$

property 6 Linearity

$$x(t) \xrightarrow{S} y(t)$$

- Scaling $\quad a x(t) \xrightarrow{S} a y(t)$

- Superposition $\quad x_1(t) \xrightarrow{S} y_1(t)$

$$x_2(t) \xrightarrow{S} y_2(t)$$

$$x_1(t)+x_2(t) \xrightarrow{S} \cancel{\neq} y_1(t)+y_2(t)$$

very important.

ex) Is $y(t) = x^2(t)$ linear?

$\boxed{\text{Stop!} \mid 3/10}$

Chapter 2

LTI

3月15日 Start

①

Chapter 2 LTI Systems

2.0 LTI ?

What is linear ?

What is TI ?

examples of LTI system ~~$y(t)=\delta(t)$~~ $y(t)=x(t)$

" of L but not TI $x(t^2)$

" nonlinear but TI $x^2(t)$

" nonlinear & not TI $x^2(t^2)$

in terms of { math →, electrical →, Life. → vending machine, neuron, girl friend / boy friend, ~~Lunch Coupon~~.

2.1. DT LTI system : ~~"Magic"~~ → Convolution Sum ← Magic

Show Fig 2.1

$$x(n) = \cdots + x(-3)\delta(n+3) + x(-2)\delta(n+2) + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + \cdots$$

$$= \sum_{k=-\infty}^{\infty} x(k)\delta(n-k)$$

Sifting property of discrete-time unit impulse
체를 거르다.

→ a discrete time signal can be represented by sum of shifted $\delta(n)$

$$x(n) \longrightarrow \boxed{\mathcal{H}_d} \longrightarrow y(n)$$

$$y(n) = \mathcal{H}_d[x(n)] = \mathcal{H}_d\left[\sum_{k=-\infty}^{\infty} x(k)\delta(n-k)\right]$$

If h(n) is linear.

Superposition → $= \sum_{k=-\infty}^{\infty} \mathcal{H}(x(k)\,\delta(n-k))$

Scaling → $= \sum_{k=-\infty}^{\infty} x(k)\,\mathcal{H}(\delta(n-k))$

$$= \sum_{k=-\infty}^{\infty} x(k)\,h_k(n)$$

If h(n) is linear & time invariant

$$= \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$

(where $\tilde{h}(n)$ is unit impulse response function of the system.

i.e. $\delta(n) \rightarrow \boxed{\mathcal{H}} \rightarrow h(n)$

$\delta(n-k) \rightarrow \boxed{\mathcal{H}} \rightarrow h(n-k)$ ∵ time-invariant

∴ in LTI systems

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)\,h(n-k)$$ ✓ very important.

So we have a name for this : Convolution.

$$y(n) \triangleq x(n) * h(n)$$

More importantly, an LTI system is completely characterized by an "impulse"

h(n): Impulse response function

for any known input, we can calculate the output of the LTI system if we know impulse response fn

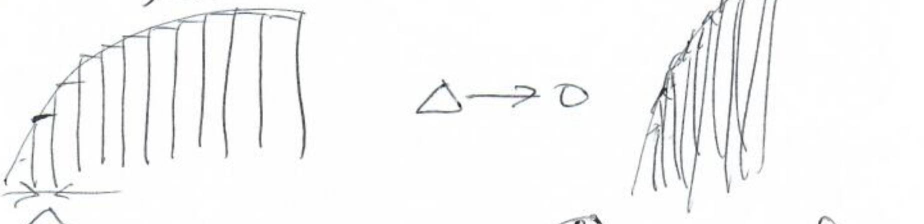
- Examples of convolution by hand by computer program,
  This is very important!

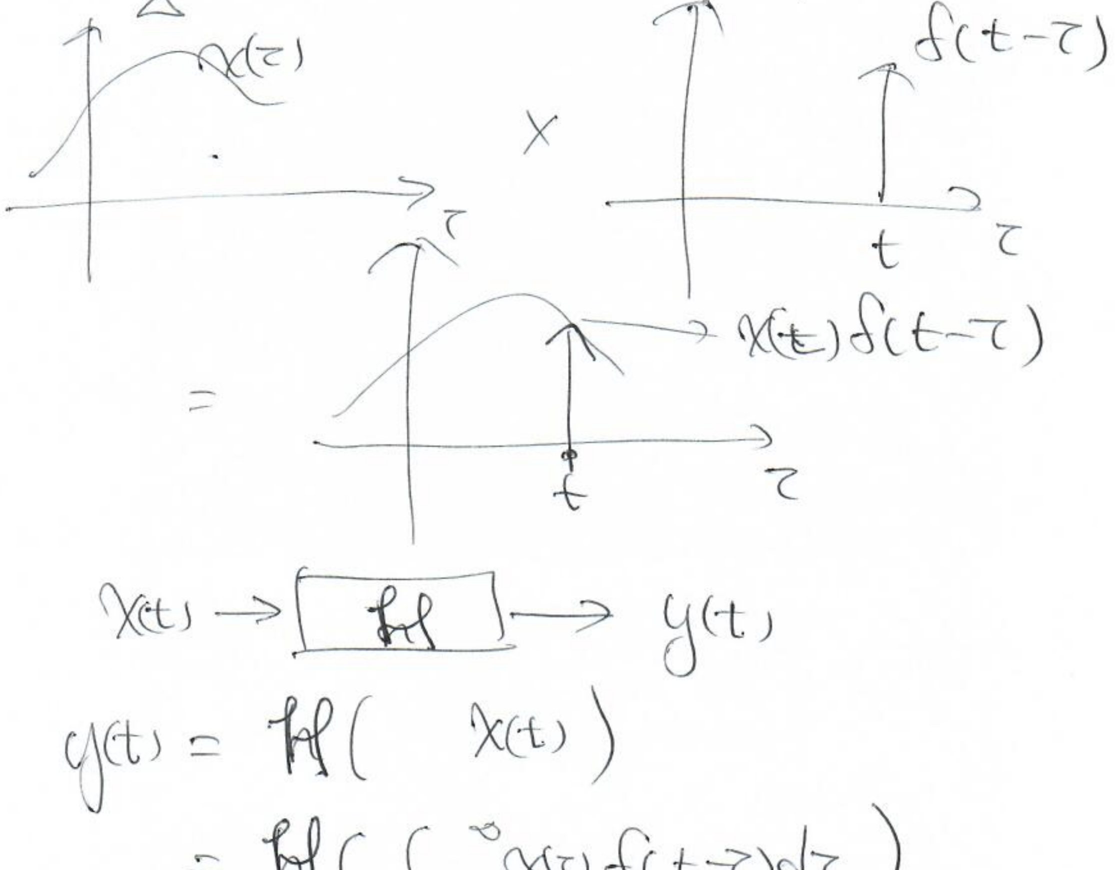
- 2.2. CT ~~⊖~~ LTI system : Convolution integral

Similarly to DT case

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau$$

why?





$$x(t) \rightarrow \boxed{\mathcal{H}} \rightarrow y(t)$$

$$y(t) = \mathcal{H}(\ \ x(t)\ )$$

$$= \mathcal{H}\left(\int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)d\tau \right)$$

$$\xrightarrow{\text{Superposition}} = \int_{-\infty}^{\infty} \mathcal{H}(x(\tau)\delta(t-\tau))\,d\tau$$

$$\xrightarrow{\text{Scaling}} = \int_{-\infty}^{\infty} x(\tau)\,\mathcal{H}(\delta(t-\tau))\,d\tau .$$

$$\xrightarrow{\text{Time Invariance}} = \int_{-\infty}^{\infty} x(\tau)h(t-\tau)d\tau \Big) \text{ Convolution Integral.}$$

$$\triangleq x(t) * h(t)$$

Show examples (Fig 2, 17, 19)

2.3. properties of LTI systems

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = x(n) * h(n)$$

$$y(t) = \int_{\infty}^{\infty} x(\tau)h(t-\tau)d\tau = x(t) * h(t)$$

* impulse response fully characterize the system. (only in LTI) ~~what if it is not linear or T.I~~

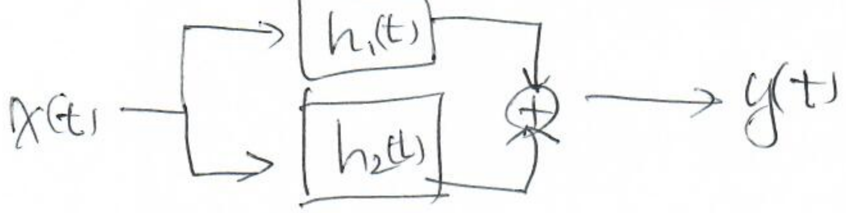
2.3.1 Commutative property

$$x(t) * h(t) = h(t) * x(t) = \int_{-\infty}^{\infty} h(\tau)x(t-\tau)d\tau$$

→ This means we can choose ~~what~~ which one to reverse/shifts in convolution.

2.3.2. Distributive property.

$$x(t) * (h_1(t) + h_2(t)) = x(t) * h_1(t) + x(t) * h_2(t)$$



$$x(t) \longrightarrow \boxed{h_1(t)+h_2(t)} \longrightarrow y(t).$$

2.3.3 Associative property.

$$(x(t) * (h_1(t) * h_2(t)) = (x(t) * h_1(t)) * h_2(t)$$

$$x(t) \longrightarrow \boxed{h_1(t)} \longrightarrow \boxed{h_2(t)} \longrightarrow y(t)$$

$$x(t) \longrightarrow \boxed{h_1(t) * h_2(t)} \longrightarrow y(t)$$

Commutative →

$$x(t) \longrightarrow \boxed{h_2(t)} \longrightarrow \boxed{h_1(t)} \longrightarrow y(t)$$

When ? in LTI   What if it is not LTI?

2.3.4. Memory

memoryless $h(n) = k\delta(n)$

$h(t) = k\delta(t)$

otherwise the system has memory

2.3.5 Invertibility

Identity system : $\delta(n)$ or $\delta(t)$

if invertible ~~$h(x)h(t)$~~ $h(t) * h(t) = \delta(t)$

Stop

Show ex 2.12 $\begin{cases} h(n) = u(n) \\ h_1(n) = \delta(n) - \delta(n-1) \end{cases}$

2.3.6. Causality : $y(n)$ must not depend on $x(k)$ $k>n$

$h(n) = 0$ for $n<0$

initial rest

in causal system $y(n) = \sum_{k=-\infty}^{n} x(k)h(n-k)$

$= \sum_{k=0}^{\infty} h(k)x(n-k)$

or in CT $h(t) = 0$ for $t<0$

$$y(t) = \int_{-\infty}^{t} x(\tau)h(t-\tau)d\tau = \int_0^{\infty} h(\tau)x(t-\tau)d\tau.$$

2.3.7 Stability

BIBO

$$|x(n)| \leq B \text{ for all } n$$

$$|y(n)| = \left| \sum_{k=-\infty}^{\infty} h(k)\, x(n-k) \right|$$

$$\leq \sum_{k=-\infty}^{\infty} |h(k)|\,|x(n-k)|$$

$$\leq B \sum_{k=-\infty}^{\infty} |h(k)| \quad ($$

$$\Rightarrow \text{ if } \sum_{k=-\infty}^{\infty} |h(k)| < \infty \rightarrow \text{ stable system}$$

(sufficient cond.)

(absolutely summable) & necessary cond.
(Prob2,49).

2.3.8 Unit step response

$$s(n) = u(n) * h(n)$$

$$= h(n) * u(n)$$

$n-k \geq 0$

$$= \sum_{k=-\infty}^{\infty} h(k) \cdot u(n-k)$$

$n \geq k$

$$= \sum_{k=-\infty}^{n} h(k)$$

h(n) can be recovered by

$$s(n) - s(n-1)$$

In CT

$$s(t) = u(t) * h(t)$$

$$= \int_{-\infty}^{t} h(\tau)\, d\tau$$

$$h(t) = \frac{ds(t)}{dt} = s'(t)$$

2.4. Causal LTI system by differential & difference equations

"here 'first'"

2.4.1 Linear Constant-Coefficient Differential equations.

a large # of system can be written as

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}$$

- a_k & b_k are constant eg.

$$i(t) = \frac{x(t) - y(t)}{R}$$

$$i(t) = C \frac{dy(t)}{dt}$$

$$RC \frac{dy(t)}{dt} + y(t) = x(t)$$

- a_k & b_k constant: R, C does not change over time.
- We can show this is a linear equation.

(not time invariant yet: eg.) Cap. may have initial charge.

- How do we solve this?

assume $RC = 1$

→ need auxillary condition.

$y(t) = y_p(t) + y_h(t)$

y_p → particular solution

y_h → natural resp. or homogeneous solution.

→ different auxillary condition. results in diffent solutions. (e.g. Cap initial values)

→ one option is "initial rest"

i.e.) $x(t) = 0$ for $t \leq t_0$
$y(t) = 0$ for $t \leq t_0$
→ meaning cap has no charge initially.

Then LCCDE will be time invariant & causal

Constraint: $y(t_0) = \frac{dy(t_0)}{dt} = \cdots = \frac{d^k y(t_0)}{dt^k} = 0$.

→ We will learn how to solve this
In chapter 4 & 9

2.4.2 linear constant coefficient difference Eq.

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

Again if initial rest (i.e. $x(n) = 0$ for $n < n_0$
then $y(n) = 0$ "

the system is LTI & causal.

Additional things in discrete-time:

$$y(n) = \frac{1}{a_0} \left\{ \sum_{k=0}^{M} b_k x(n-k) - \sum_{k=1}^{N} a_k y(n-k) \right\}$$

→ resursive equation.

when $N \geq 1$ causal LTI system has
an impulse response of infinite duration

→ infinite impulse response sys.
IIR

when $N = 0$
$$y(n) = \sum_{k=0}^{M} \left(\frac{b_k}{a_0} \right) x(n-k)$$

→ finite impulse response system
> cont for Chapter [illegible] FIR

2.4.3. Block diagram.

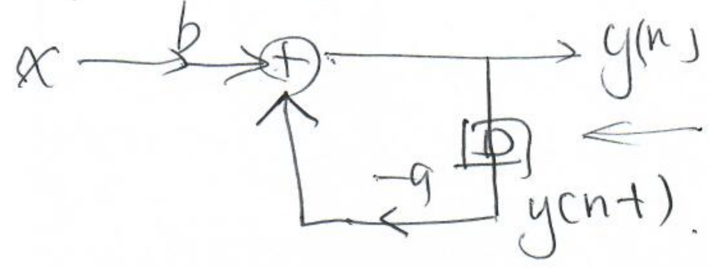
DT

$$\begin{tikzcd} & x_2(n) \arrow[d] & \\ x_1(n) \arrow[r] & + \arrow[r] & x_1(n)+x_2(n) \end{tikzcd}$$

$$x(n) \xrightarrow{a} a\,x(n)$$

$$x(n) \longrightarrow \boxed{D} \longrightarrow x(n-1)$$

Ex) $y(n) + a\,y(n-1) = b\,x(n)$



memory & initial ~~best~~ value for memory.

CT

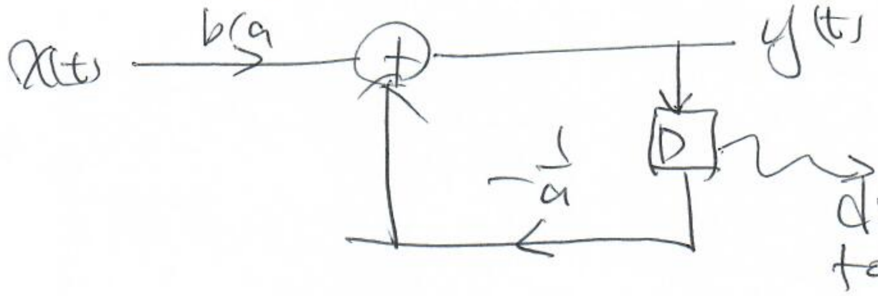
$$\begin{tikzcd} x_1(t) \arrow[r] & + \arrow[r] & x_1(t)+x_2(t) \\ & x_2(t) \arrow[u] & \end{tikzcd}$$

$$x(t) \xrightarrow{a} a\,x(t)$$

$$x(t) \longrightarrow \boxed{D} \longrightarrow \frac{dx(t)}{dt}$$

$$x(t) \longrightarrow \boxed{\int} \longrightarrow \int_{-\infty}^{t} x(\tau)d\tau .$$

Ex) $\frac{dy(t)}{dt} + a\,y(t) = b\,x(t)$



difficult to implement in analog.

3/17 Done

2.5 Singularity fn.

2.5.1

$$\delta(t) = \begin{cases} \infty & t = 0 \\ 0 & t \neq 0 \end{cases}$$

$$\int_{-\infty}^{\infty} \delta(\tau)\,d\tau = 1$$

$$x(t) = x(t) * \delta(t)$$

$$x(t) = \int_{-\infty}^{\infty} x(\tau)\delta(t-\tau)\,d\tau$$

$$\delta(t) = \delta(t) * \delta(t)$$

2.5.2 Unit impulse through Convolution.

- $x(t) * \delta(t) = x(t)$
- $1 = x(t) = x(t) * \delta(t) = \int_{-\infty}^{\infty} \delta(\tau)x(t-\tau)\,d\tau$

  $= \int_{-\infty}^{\infty} \delta(\tau)\,d\tau = 1$
- $g(-t) = g(-t) * \delta(\tau) = \int_{-\infty}^{\infty} g(\tau - t)\delta(\tau)\,d\tau$

for $t=0$ $\quad g(0) = \int_{-\infty}^{\infty} g(\tau)\delta(\tau)\,d\tau$

$\downarrow$ operational definition

~~$x(t) = g(0)$~~

- $f(t)\delta(t) = f(0)\delta(t)$
- $\int_{-\infty}^{\infty} f(\tau)\delta(\tau)\,d\tau = \int_{-\infty}^{\infty} f(0)\delta(\tau)\,d\tau$

  $= f(0)$
- $\delta(at) = \frac{1}{|a|}\delta(t)$.

2.5.3 Unit Doublet & ETC

A system : $y(t) = \frac{dx(t)}{dt}$

Impulse response fn "$\frac{d\delta(t)}{dt}$" : $u_1(t)$ unit doublet

$$\frac{dx(t)}{dt} = x(t) * u_1(t)$$

$$\frac{d^2x(t)}{dt^2} = x(t) * u_2(t) \quad \text{where } u_2(t) = u_1(t) * u_1(t)$$

$$u_k(t) = \underbrace{u_1(t) * u_1(t) * \cdots * u_1(t)}_{k \text{ times}}$$

Using operational definition.

for $x(t) = 1$

$$0 = \frac{dx(t)}{dt} = x(t) * u_1(t) = \int_{\infty}^{\infty} u_1(\tau)\, x(t-\tau)\, d\tau$$

$$= \underbrace{\int_{\infty}^{\infty} u_1(\tau)\, d\tau}_{\text{zero area}}$$

Unit step $y(t) = x(t) * u(t) = \int_{-\infty}^{\infty} x(\tau)\, u(t-\tau)\, d\tau$

$$= \int_{\infty}^{t} x(\tau)\, d\tau$$

$$u(t) = \int_{\infty}^{t} \delta(\tau)\, d\tau$$

operational definition $x(t) * u(t) = \int_{-\infty}^{t} x(\tau)\, d\tau$

$$u_{-2}(t) = u(t) * u(t) = \int_{-\infty}^{t} u(\tau)\, d\tau = \underbrace{t\, u(t)}_{\text{unit ramp}}$$

$$x(t) * u_{-2}(t) = x(t) * u(t) * u(t)$$

$$= \int_{\infty}^{t} \left(\int_{-\infty}^{\tau} x(\sigma)\, d\sigma \right) d\tau$$

$$u_{-k}(t) = u(t) * \cdots u(t) = \int_{\infty}^{t} u_{-(k-1)}(\tau)\, d\tau$$

$$= \frac{t^{k-1}}{(k-1)!}\, u(t)$$

better definitions

$\delta(t) = u_0(t)$ $\quad u(t) = u_{-1}(t)$.

Signals

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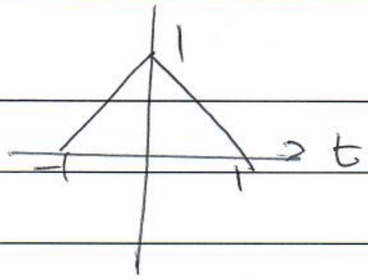
Time domain	~~Fourier domain~~
$\delta(t)$	
$\delta(at)$ $\frac{1}{\lvert a \rvert}\delta(\cancel{a}t)$	
$e^{i2\pi f_0 t}$	
$rect(t) \triangleq \begin{cases} 1, & \text{if } \lvert t\rvert \le 1/2 \\ 0, & \text{otherwise} \end{cases}$	
$\Lambda(t) = \underline{rect(t)} * \underline{rect(t)}$	
$sinc(t) \triangleq \sin(\pi t)/\pi t$	
$e^{-\pi t^2}$	
$sin2\pi f_0 t = \frac{\exp(+i2\pi f_0 t)-\exp(-i2\pi f_0 t)}{2i}$	
$cos2\pi f_0 t = \frac{\exp(+i2\pi f_0 t)+\exp(-i2\pi f_0 t)}{2}$	
$\begin{cases} e^{-at}, & \text{if } t \ge 0 \\ 0, & \text{otherwise} \end{cases}$ When $a > 0$,	
$1/a + j2\pi t$	
$\text{III}(t) \triangleq \sum_{k=-\infty}^{\infty} \delta(t-k)$	Shah Comb
$\frac{1}{T}\text{III}\left(\frac{t}{T}\right)$	
~~$\Lambda(t) =$~~	
$f(at)$	

Summary of Chapter 1&2

- CT & DT signal.
- Exponentials, $\delta(t)$, $u(t)$
- CT & DT system & system property.
  - memory
  - Invertibility
  - Causality
  - Stability
  - TI
  - Linearity
- LTI system. ① δ impulse response. function
  - ② convolution.
  - ③ Causality. Stability.
- Causal LTI system. : linear constant coeff diff. equation
- Singularity function.

Chapter 4. & Chapter 5.

How we find solutions for LTI system.

Chapter 4

Continuous Time. Fourier Transform.

3.2. Response of LTI system to complex exp.

- Complex exp. functions (e^{st} or z^n) are "magic functions" in LTI systems

Why?

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = h(t) * e^{st} = \int_{-\infty}^{\infty} h(\tau) e^{\cancel{+j}s(t-\tau)} d\tau$$

$$= \underbrace{\left[\int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau\right]}_{\text{independent of } t} e^{st}$$

$$= H(s) e^{st} \quad \text{where } H(s) = \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau$$

☆ output of complex exp. fn is the same e^{st} w modified mag/phase (by $H(s)$)

☆ $\underbrace{\text{This type of fn}}_{\text{magic fn}}$ is called "eigen function" and $H(s)$ is called "eigen value"

The same is true for z^n in DT.

$$y(n) = \left(\sum_{k=-\infty}^{\infty} h(k) z^{-k}\right) z^n = H(z) z^n$$

$$\text{where } H(z) = \sum_{k=-\infty}^{\infty} h(k) z^{-k}$$

So complex exp. are lovely.

What if our input is a combination of e^{st}?

i.e $x(t) = \sum_k a_k e^{s_k t}$

or $x(t) = \int^{\infty} X(s) e^{st} ds$

Then ~~the~~ output will be.

$$y(t) = \sum_{k} a_k H(s_k) e^{s_k t}$$

or

$$y(t) = \int_{-\infty}^{\infty} X(s) \cdot H(s) e^{st} ds$$

BTW "$\underbrace{s \& z}_{\text{Complex number.}}$" is too general to use it

~~here~~ For the next a few chapters, we will use $s = j2\pi f$ & $z = e^{j2\pi f}$ case. (i.e. pure Imaginary Complex exp. function).

☆ say something about evil ω empire & good f jedi.

· Let's define a transform.

$$\underline{X(f)} = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi f t} dt$$

→ aperiodic signal.

→ entire time.

↓ $X(j\omega)$ or $X(\omega)$ or $X(2\pi f)$

Then $x(t) = \int_{-\infty}^{\infty} X(f)\, e^{j2\pi f t} df$

if $X(f)$ exists.

Why?

$$\boxed{\text{Stop } 3/22}$$

In our last lecture

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow H(s)\,e^{st}$$

eigenfunction

$$z^{n} \longrightarrow \boxed{h(n)} \longrightarrow H(z)\,z^{n}$$

eigen value or "system function"

When $s=j2\pi f$, $z=e^{j2\pi f}$

i.e. evaluation of system function when amplitude is 1 ($|e^{j2\pi f}|=1$)

Then the system function is called "frequeny Response"

$$H(j2\pi f)=\int_{-\infty}^{\infty} h(t)\,e^{-j2\pi ft}\,dt$$

$$H(e^{j2\pi f})=\sum_{n=-\infty}^{\infty} h(n)\,e^{-j2\pi fn}$$

Since $e^{j2\pi ft}$ is eigenfunction, of an LTI system, we have a good motivation of writing $x(t)$ as a combination of $e^{j2\pi ft}$.

ex) if $x(t) = 3e^{j2\pi\cdot 5\cdot t}+5e^{j2\pi\cdot 10\,t}$

$$=\sum_{k} a_k e^{j2\pi f_k t}$$

or more generally

$$x(t)=\int_{-\infty}^{\infty} X(f)\,e^{j2\pi ft}\,df$$

then the output will be Complex coefficiant modified version of this input!

The question is if X(f) exist & if exist
how do we find it.

Thanks to Fourier, we already have a solution.

If
$$x(t) = \int_{\infty}^{\infty} X(f)\, e^{j2\pi ft} df$$
$$X(f) = \int_{-\infty}^{\infty} x(t)\, e^{-j2\pi ft} dt$$

Why?
$$x(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f\tau} d\tau\, e^{j2\pi ft} df$$
$$= \int_{\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)\, e^{-j2\pi f(\tau-t)} df\, d\tau$$
$$= \int_{\infty}^{\infty} x(\tau) \underbrace{\int_{-\infty}^{\infty} e^{-j2\pi f(\tau-t)} df}_{\delta(t-\tau)} d\tau$$
$$= \int_{\infty}^{\infty} x(\tau)\, \delta(\tau-t) d\tau = x(t) \;!$$

O.K. we still need to demonstrate
$$\int_{\infty}^{\infty} e^{-j2\pi f(\tau-t)} df = \delta(t-\tau).$$

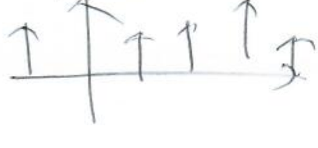
→ go to page XX

Let's rewrite the equations

"Fourier transform"
$$\underbrace{X(f)}_{\text{frequency spectrum}} = \int_{-\infty}^{\infty} x(t) \underbrace{e^{-j2\pi ft}}_{\text{basis function.}} dt$$
: analyze
decompose.

ex) $x(t) = e^{+j2\pi f_0 t}$
$$X(f) = \int_{-\infty}^{\infty} e^{-j2\pi(f-f_0)t} dt = \delta(f-f_0)$$

if $x(t) = \sum_{k=1}^{5} a_k e^{j2\pi f_k t}$ →



" Inverse Fourier Transform"

$$x(t) = \int_{-\infty}^{\infty} X(f)\, e^{+j2\pi ft}\, df \ : \ \text{Synthesis}$$

$$X(f) = \delta(f-f_0) \qquad x(t) = e^{j2\pi f_0 t}$$

4.1.2. Convergence.

When does F.T exist?

Condition 1 If $x(t)$ has finite energy $\left(\int_{-\infty}^{\infty} |x(t)|^2 dt < \infty \right)$

then $X(f)$ is finite & $\int_{-\infty}^{\infty} |e(t)|^2 dt = 0$

(i.e. $x(t)$ and $x(t)$ from F.T differs only @ individual values)

or

Condition 2

1. $\int_{-\infty}^{\infty} |x(t)| dt < \infty$
2. finite # of maxima & minima within finite interval
3. finite # of discontinuity within finite interval.

→ Condition 1 & 2 are both "sufficient" conditions

& some fn still have F.T not satisfying 1 or 2.

4.1.3 Examples of CT F.T.

Let's fill in your bucket list. (pull out your list)

① $x(t) = \delta(t)$

$$X(f) = \int_{-\infty}^{\infty} \delta(t) e^{-j2\pi ft} dt = 1$$

② $x(t) = \delta(at)$

$$X(f) = \int_{-\infty}^{\infty} \delta(at) e^{-j2\pi ft} dt = \frac{1}{|a|}\delta\left(\frac{f}{a}\right)$$

③ $x(t) = \delta(t-t_0)$

$$X(f) = \int_{-\infty}^{\infty} \delta(t-t_0) e^{-j2\pi f t} dt = e^{-j2\pi f t_0}$$

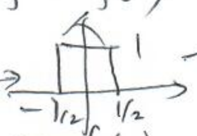
④ $x(t) = 1$

$$X(f) = \int_{-\infty}^{\infty} 1 e^{-j2\pi f t} dt = \delta(f)$$

⑤ $x(t) = e^{j2\pi f_0 t}$

$$X(f) = \delta(f-f_0)$$

⑥ $x(t) = \mathrm{rect}(t) \longrightarrow$



→ different def. than book.

$$X(f) = \int_{-1/2}^{1/2} e^{-j2\pi f t} dt = \frac{e^{-j\pi f} - e^{+j\pi f}}{-j2\pi f}$$

3/24

$$= \frac{e^{j\cancel{2}\pi f} - e^{-j\pi f}}{j2\pi f} = \frac{\sin \pi f}{\pi f} = \mathrm{sinc}\, f$$

⑦ $\Lambda(t) = \mathrm{rect}(t) * \mathrm{rect}(t)$ ← Do it later

$$X(f) = \mathrm{sinc}^2 f$$

⑧ $\mathrm{sinc}(t)$ ⇐ Do it later

⑨ $e^{-\pi t^2}$ ⇐ Do it later.

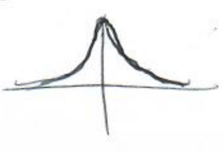
⑩ $\sin 2\pi f_0 t$ $\quad \frac{\delta(f-f_0) - \delta(f+f_0)}{2j}$

⑪ $\cos 2\pi f_0 t$ $\quad \frac{\delta(f-f_0) + \delta(f+f_0)}{2}$

⑫ $e^{-at} u(t) \quad a > 0$

$$\int_0^{\infty} e^{-at} e^{-j2\pi f t} dt = \frac{1}{a + j2\pi f}$$ → Lorentzian function

$$\left| \frac{1}{a+j2\pi f} \right| = \frac{1}{\sqrt{a^2 + (2\pi f)^2}}$$



⑬ Later

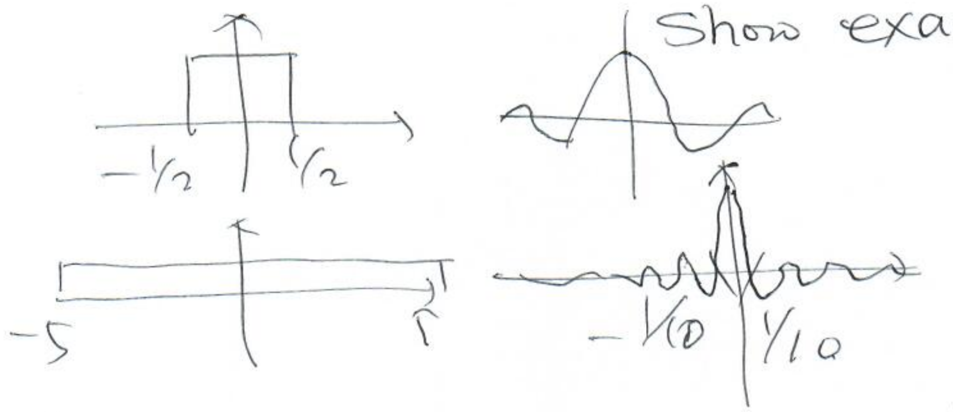
⑭ $\text{III}(t) = \sum_{k=-\infty}^{\infty} \delta(t-k) \xleftrightarrow{\mathcal{F}} \text{III}(f)$

⑮ $f(at) \xleftrightarrow{\mathcal{F}} \frac{1}{|a|} F\left(\frac{f}{a}\right)$

⑯ $\frac{1}{T}\text{III}\left(\frac{t}{T}\right) \xleftrightarrow{\mathcal{F}} \text{III}(Tf)$

⑰ $\text{rect}(at) \xleftrightarrow{\mathcal{F}} \frac{1}{a}\text{sinc}\left(\frac{f}{a}\right)$

Show example plots



Do example 4.5 & show how difficult it is.

Signals

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Time domain	Fourier domain
$\delta(t)$	1
$\delta(at)$	$1/\lvert a\rvert$
$\delta(t-t_0)$	$e^{-i2\pi f t_0}$
1	$\delta(f)$
$e^{i2\pi f_0 t}$	$\delta(f-f_0)$
$rect(t) \triangleq \begin{cases} 1, & \text{if } \lvert t\rvert \le 1/2 \\ 0, & \text{otherwise} \end{cases}$	$\dfrac{sin\pi f}{\pi f} \triangleq sinc(f)$
$\Lambda(t) = rect(t) * rect(t)$	$sinc^2(f)$
$sinc(t) \triangleq \sin(\pi t)/\pi t$	$rect(f)$
$e^{-\pi t^2}$	$e^{-\pi f^2}$
$sin2\pi f_0 t = \frac{\exp(+i2\pi f_0 t)-\exp(-i2\pi f_0 t)}{2i}$	$\dfrac{\delta(f-f_0)-\delta(f+f_0)}{2i}$
$cos2\pi f_0 t = \frac{\exp(+i2\pi f_0 t)+\exp(-i2\pi f_0 t)}{2}$	$\dfrac{\delta(f-f_0)+\delta(f+f_0)}{2}$
$\begin{cases} e^{-at}, & \text{if } t \ge 0 \\ 0, & \text{otherwise} \end{cases}$ When $a > 0$,	$1/(a+j2\pi f)$ $Magnitude\text{:} \dfrac{1}{\sqrt{a^2+(2\pi f)^2}}$ $Phase\text{:} -\tan^{-1}(\dfrac{2\pi f}{a})$
$1/a + j2\pi t$	
$\text{III}(t) \triangleq \sum_{k=-\infty}^{\infty} \delta(t-k)$	$\text{III}(f)$
$\frac{1}{\mathrm{T}}\text{III}\left(\frac{t}{\mathrm{T}}\right)$	$\text{III}(Tf)$
$f(at)$	$\frac{1}{a}F(\frac{f}{a})$

4.3 properties

- Linearity $a\,x(t) + b\,y(t) \overset{F_1}{\longleftrightarrow} a\,X(f) + b\,Y(f)$
- Time shift $x(t-t_0) \overset{F_1}{\longleftrightarrow} \underbrace{e^{-j2\pi f t_0}}_{} X(f)$

what is this?

magnitude $|X(f)|$ is the same

phase $\angle X(f)$ has "linear phase shift"

go to handout

- Conjugate & symmetry $x^*(t) \overset{F_1}{\longleftrightarrow} X^*(-f)$

  if $x(t)$ is real $X(-f) = \cancel{X}^*(f)$

  $\longrightarrow$ Hermition.

  " $\text{Even}\{x(t)\} \overset{F_1}{\longleftrightarrow} \text{Real}\{X(f)\}$

  " $\text{Odd}\{x(t)\} \overset{F_1}{\longleftrightarrow} j\,\text{Imag}\{X(f)\}$

- Differentiation & Integration

$$\frac{dx(t)}{dt} \overset{F_1}{\longleftrightarrow} j2\pi f X(f)$$

$$\int_{-\infty}^{t} x(\tau)d\tau \overset{F_1}{\longleftrightarrow} \frac{1}{j2\pi f}X(f) + \frac{1}{2}X(0)\delta(f)$$

- Time & frequency scaling

$$x(at) \overset{F_1}{\longleftrightarrow} \frac{1}{|a|}X\left(\frac{f}{a}\right)$$

$$x(-t) \overset{F_1}{\longleftrightarrow} X(-f)$$

- "Duality" very important

Show Fig 4.17 again

Properties of Symmetry

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A real function, f(t), is

"even function" if $f(t) = f(-t)$
"odd function" if $f(t) = -f(-t)$

A real function can be divided into even and odd parts of the function

$$f_{even}(t) = \{f(t) + f(-t)\}/2$$
$$f_{odd}(t) = \{f(t) - f(-t)\}/2$$

A function, f(x), is

"real function" if $f(t) = f^*(t)$
"imaginary function" if $f(t) = -f^*(t)$

A function, f(x), is

"Hermitian function" if $f^*(t) = f(-t)$
"Anti-hermitian function" if $f^*(t) = -f(-t)$

Hermitian means real part of the function is even and imaginary part is odd

$$f(t) = a(t) + ib(t)$$

where $a(t)$ and $b(t)$ are real functions

$$a(t) = a(-t)$$
$$b(t) = -b(-t)$$

Fourier transform of a real function, h(t), is Hermitian

$$H^*(f) = H(-f)$$

And

$$h(t) = h_{even}(t) + h_{odd}(t)$$
$$FT\{h_{even}(t)\} = Re\{H(f)\} = Re\{H(-f)\}$$
$$FT\{h_{odd}(t)\} = j\,Im\{H(f)\} = -j\,Im\{H(-f)\}$$

Using duality

$$-j2\pi t\, x(t) \longleftrightarrow \frac{dX(f)}{df}$$

$$e^{j2\pi f_0 t}x(t) \longleftrightarrow X(f-f_0)$$

$$-\frac{1}{j2\pi t}x(t) + \frac{1}{2}x(0)\delta(t) \longleftrightarrow \int_{-\infty}^{f} x(\tau)d\tau$$

- Parseval's Relation

$$\int_{\infty}^{\infty} |x(t)|^2 dt = \int_{-\infty}^{\infty} \underbrace{|X(f)|^2}_{} df$$

→ total energy is the same.

↓ enery density spectrum.

"Short Break" ⇐ here"

4.4. Convolution property.

$$y(t) = \int_{\infty}^{\infty} x(\tau)h(t-\tau)d\tau$$

$$Y(f) = \mathcal{F}\{y(t)\} = \int_{\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)h(t-\tau)\,d\tau\, e^{-j2\pi ft}dt$$

$$= \int_{\infty}^{\infty} x(\tau)\int_{-\infty}^{\infty} \underbrace{h(t-\tau)}_{\text{time shift}}\, e^{-j2\pi ft}dt\,d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\, H(f)e^{-j2\pi f\tau}d\tau$$

$$= H(f)\int_{\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}d\tau = H(f)X(f)$$

Awesome!

$y(t) = h(t) * x(t)$ ⇐ very complex

$Y(f) = H(f)\cdot X(f)$ ⇐ very simple.

☆ very very important.

3/28 stop →

Show matlab example.

* correction in handout. j in front of $\underline{\text{Im}\{H(f)\}}$

* correction in book 4.42, page 311

$$-\frac{1}{j2\pi t}x(t)+\frac{1}{2}X(0)\delta(t) \leftrightarrow \int_{-\infty}^{f} X(\eta)d\eta$$

– Duality

$$x(t) \longleftrightarrow X(f)$$

$$X(t) \longleftrightarrow ?$$

$X(t) = ?$

$$X(f) = \int_{-\infty}^{\infty} x(t)e^{-j2\pi ft}dt$$

$$= \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi f\tau}d\tau$$

$$X(t) = \int_{-\infty}^{\infty} x(\tau)e^{-j2\pi t\tau}d\tau$$

$$\int_{-\infty}^{\infty} X(t)e^{-j2\pi ft}dt = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi\tau t}e^{-j2\pi ft}d\tau dt$$

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x(\tau)e^{-j2\pi(\tau+f)t}dt\,d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\underline{\int_{-\infty}^{\infty} e^{-j2\pi(\tau+f)t}dt}\,d\tau$$

$$= \int_{-\infty}^{\infty} x(\tau)\,\delta(\tau+f)\,d\tau = \int_{-\infty}^{\infty} x(-f)\delta(\tau+f)d\tau$$

$$= x(-f)$$

→ go to your handout (next page)

Example 4.19 & 4.20 very important.

4.5 Multiplication or modulation property

$$s(t)\cdot p(t) = \int_{-\infty}^{\infty} S(\theta)P(f-\theta)d\theta$$

$$= S(f)*P(f)$$

→ used in AM podulation!

Do example 4.21

emphasize 4.22 is very important.

Signals

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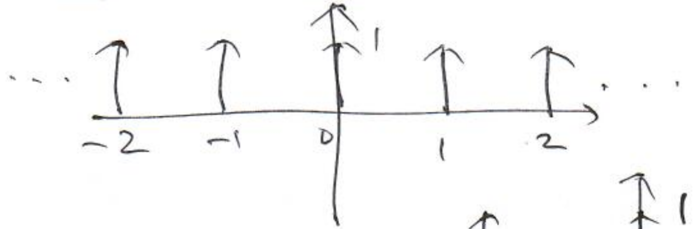
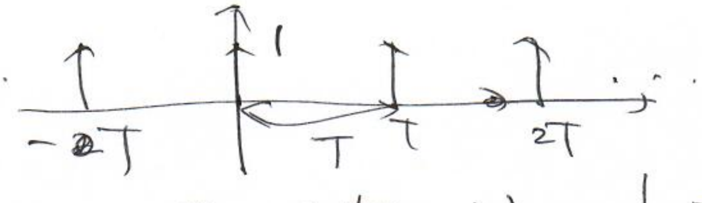
Time domain	Fourier domain	
$\delta(t)$	1	
$\delta(at)$	$1/\|a\|$	
$\delta(t-t_0)$	$e^{-i2\pi f t_0}$	
1	$\delta(f)$	← duality
$e^{i2\pi f_0 t}$	$\delta(f-f_0)$	
$rect(t) \triangleq \begin{cases} 1, & \text{if } \|t\| \le 1/2 \\ 0, & \text{otherwise} \end{cases}$	$\frac{\sin\pi f}{\pi f} \triangleq sinc(f)$	
$\wedge(t) = rect(t) * rect(t)$	$sinc^2(f)$	← convolution
$sinc(t) \triangleq \sin(\pi t)/\pi t$	$rect(f)$	← duality
$e^{-\pi t^2}$	$e^{-\pi f^2}$	← try it yourself
$sin2\pi f_0 t = \frac{\exp(+i2\pi f_0 t)-\exp(-i2\pi f_0 t)}{2i}$	$\frac{\delta(f-f_0)-\delta(f+f_0)}{2i}$	
$cos2\pi f_0 t = \frac{\exp(+i2\pi f_0 t)+\exp(-i2\pi f_0 t)}{2}$	$\frac{\delta(f-f_0)+\delta(f+f_0)}{2}$	
$\begin{cases} e^{-at}, & \text{if } t \ge 0 \\ 0, & \text{otherwise} \end{cases}$ When $a > 0$,	$1/(a+j2\pi f)$ $Magnitude: \frac{1}{\sqrt{a^2+(2\pi f)^2}}$ $Phase: -\tan^{-1}(\frac{2\pi f}{a})$	
$1/a + j2\pi t$		← duality
$\text{III}(t) \triangleq \sum_{k=-\infty}^{\infty} \delta(t-k)$	$\text{III}(f)$	
$\frac{1}{T}\text{III}\left(\frac{t}{T}\right)$	$\text{III}(Tf)$	
$f(at)$	$\frac{1}{\|a\|}F(\frac{f}{a})$	

4.2 FT for periodic signals

Let's review $\text{Ш}(t)$

$$\text{Ш}(t) = \sum \delta(t-n)$$

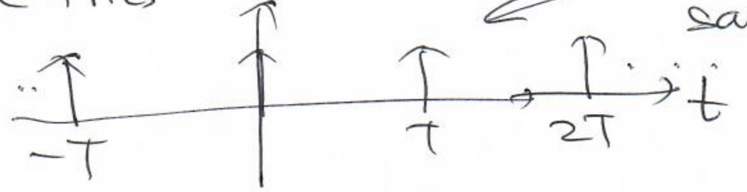


What about 

$$\sum\delta(t-nT) = \sum\delta\left(T\left(\frac{t}{T}-n\right)\right) = \frac{1}{T}\sum\delta\left(\frac{t}{T}-n\right)$$

$$= \frac{1}{T}\,\text{Ш}\left(\frac{t}{T}\right)$$

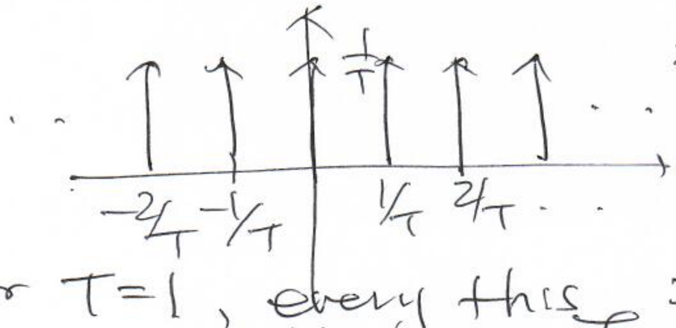
Let's memorize this ← also called sampling fn.

$\frac{1}{T}\text{Ш}\left(\frac{t}{T}\right)$ 

What is FT of $\frac{1}{T}\text{Ш}\left(\frac{t}{T}\right)$

$$\text{Ш}(Tf) = \sum\delta(Tf-n) = \sum\delta\left(T\left(f-\frac{n}{T}\right)\right)$$

$$= \frac{1}{T}\sum\delta\left(f-\frac{n}{T}\right)$$



for T=1, every this is good!

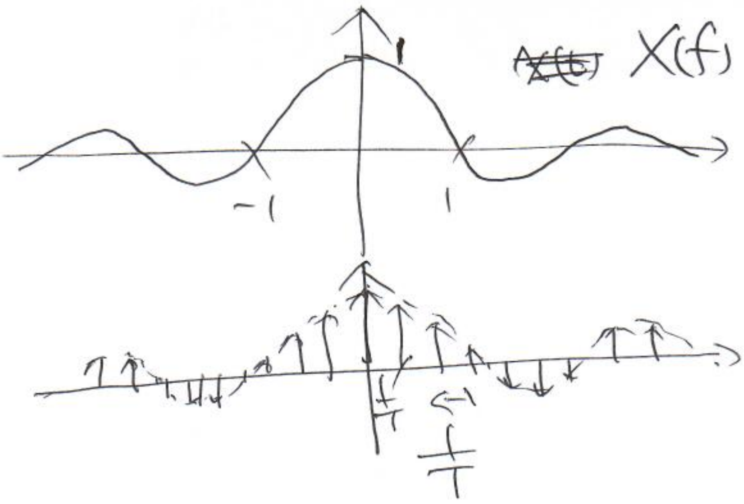
$$\text{Ш}(t) \xleftrightarrow{F_1} \text{Ш}(f) \quad (\text{will prove later})$$

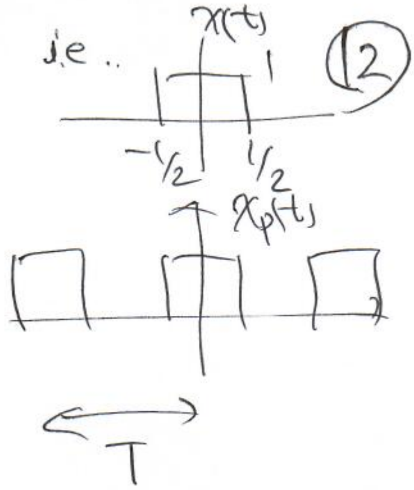
- FT of periodic signal.

$$x_p(t) = x(t) * \frac{1}{T}\text{Ш}\left(\frac{t}{T}\right)$$

$$F_1\{x_p(t)\} = F_1\left\{x(t) * \frac{1}{T}\text{Ш}\left(\frac{t}{T}\right)\right\}$$

$$\underset{\substack{\uparrow \\ \text{periodic fn}}}{X_p(f)} = \underbrace{X(f)\cdot\text{Ш}(Tf)}_{\text{sampled version of}}$$





T ~~short~~ vs T long.

$$X_p(f) = X(f)\,Ш(Tf)$$

$$= \int_{-\infty}^{\infty} x(t)\,e^{-j2\pi ft}\,dt \cdot Ш(Tf)$$

$$= \int_{-\cancel{\infty}T/2}^{\cancel{\infty}T/2} x(t)\,e^{-j2\pi ft}\,dt \cdot \frac{1}{T}\sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$$

$$= \underbrace{\frac{1}{T}\int_{-T/2}^{T/2} x(t)\,e^{-j2\pi ft}\,dt}_{a_k} \cdot \sum_{k=-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)$$

$$= \sum_{k=-\infty}^{\infty} a_k\,\delta\left(f-\frac{k}{T}\right) \quad \text{where } a_k = \frac{1}{T}\int_{-T/2}^{T/2} x(t)\,e^{-j2\pi\frac{k}{T}t}\,dt$$

$$x_p(t) = \int_{-\infty}^{\infty} X_p(f)\,e^{j2\pi ft}\,df$$

$$= \int_{-\infty}^{\infty}\sum_{k=-\infty}^{\infty} a_k\,\delta\left(f-\frac{k}{T}\right)e^{j2\pi ft}\,df$$

$$= \sum_{k=-\infty}^{\infty} a_k\int_{-\infty}^{\infty}\delta\left(f-\frac{k}{T}\right)e^{j2\pi ft}\,df$$

$$= \sum_{k=-\infty}^{\infty} a_k\,e^{j2\pi\frac{k}{T}t} = \sum_{k=-\infty}^{\infty} a_k\,e^{j2\pi kf_0t}$$

→ F.S pair

∴ F.S is a special case of F.T.

$\uparrow$ periodic signal $\qquad$ $\uparrow$ aperiodic signal

4.7. LTI system in differential equations

$$\sum_{k=0}^{N} a_k \frac{d^k y(t)}{dt^k} = \sum_{k=0}^{M} b_k \frac{d^k x(t)}{dt^k}.$$

what is this? LCCDE   initial rest → TI & Causal.

We are ready to solve!

FT for both sides

$$\sum_{k=0}^{N} a_k (j2\pi f)^k Y(f) = \sum_{k=0}^{M} b_k (j2\pi f)^k X(f)$$

$$H(f) = \frac{Y(f)}{X(f)} = \frac{\sum_{k=0}^{M} b_k (j2\pi f)^k}{\sum_{k=0}^{N} a_k (j2\pi f)^k}$$

→ rational fn.

Remember we want $y(t)$...

Solve 4.26.

stable LTI system.

$$\frac{d^2 y(t)}{dt^2} + 4\frac{dy(t)}{dt} + 3y(t) = \frac{dx(t)}{dt} + 2x(t)$$

$$x(t) = e^{-t}u(t).$$

What is $h(t)$? what is $y(t)$?

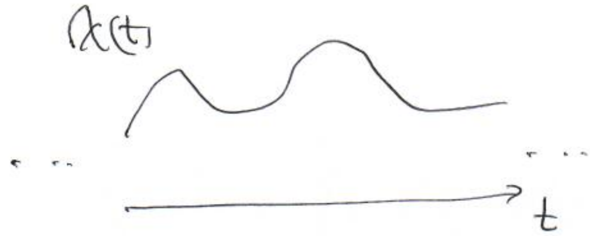
Chapter 5

Discrete-Time Fourier Transform

Chapter 5

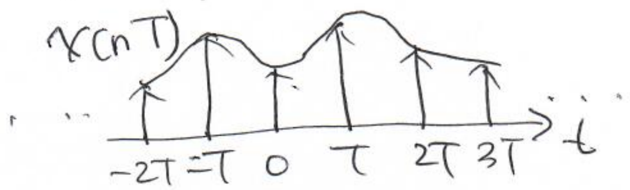
5.1 Discrete - Time Fourier Transform

Let's start from CT-FT

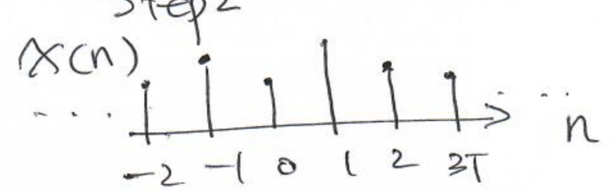


Discrete Time is

→ Step1



Step2



→ Do we loose a lot of information?

→ Chapter 7.

Step 1

$$x(nT) = x(t) \cdot \frac{1}{T} Ш\left(\frac{t}{T}\right)$$

$$= x(t) \sum_{k=-\infty}^{\infty} \delta(t-kT)$$

$$= \sum_{k=-\infty}^{\infty} x(kT)\,\delta(t-kT)$$

~~Then $x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(t-k)$ note $t \neq n$ but $t = \frac{n}{T}$~~

$$X(f) = \int_{-\infty}^{\infty} x(nT)\, e^{-j2\pi ft}\, dt$$

$$= \int_{-\infty}^{\infty} \sum_{k=-\infty}^{\infty} x(kT)\,\delta(t-kT)\, e^{-j2\pi ft}\, dt$$

$$= \sum_{k=-\infty}^{\infty} x(kT) \int_{-\infty}^{\infty} \delta(t-kT)\, e^{-j2\pi ft}\, dt$$

$$= \sum_{k=-\infty}^{\infty} x(kT)\, e^{-j2\pi fkT}$$

Step 2 replace kT to n . discrete time x(n)

DT-FT

$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x(n)\, e^{-j2\pi fn}$$

In many other books including DSP, $X(f)$ for both CT & DT

Let's compare it with CT-FT

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt.$$ pretty easy to remember!

- Inverse Fourier Transform of DT-FT.

$$x(n) = \int_1 X(e^{j2\pi f}) e^{j2\pi fn} df$$ uh-oh

Let's proove this

$$x(n) = \int_0^1 \sum_{k=-\infty}^{\infty} x(k) e^{-j2\pi fk} e^{j2\pi fn} df$$

$$= \sum_{k=-\infty}^{\infty} x(k) \underline{\int_0^1 e^{-j2\pi f(k-n)} df}$$

$= \delta(k-n)$ why? if $k=n$ ans $= 1$, if $k \neq n$ ans $= 0$.

$$= \sum_{k=-\infty}^{\infty} x(k)\delta(k-n) = x(n)$$

- DT-FT pair.

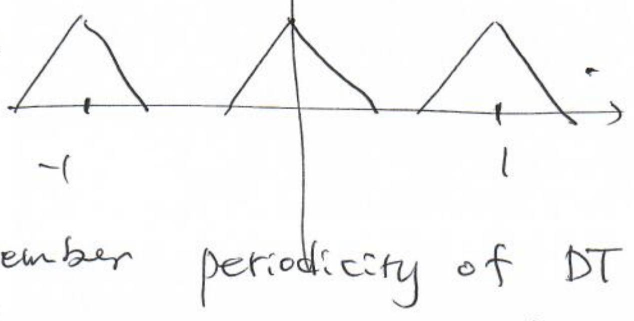
$$x(n) = \int_1 X(e^{j2\pi f}) e^{+j2\pi fn} df$$ → synthesis

$$X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi fn}$$ → analysis decompose.

frequency spectrum.

$X(e^{j2\pi f})$: periodic ∵ $2\pi f =$ ~~$4\pi f = 6\pi f$~~ .. $= 2\pi(f+1) = \cdots$ $2\pi(f+n)$

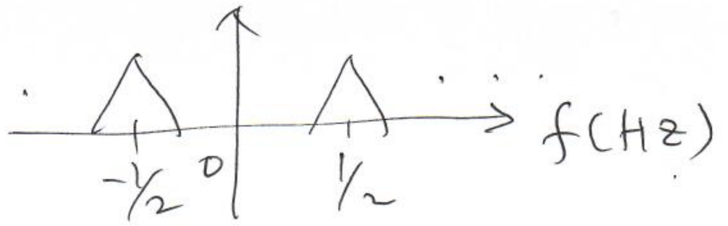
Meaning..



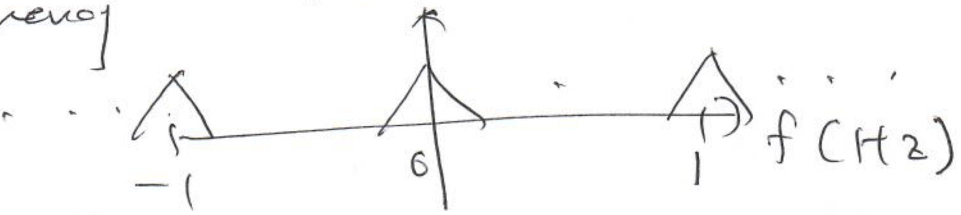
Do you remember periodicity of DT Complex exp.?

$X(e^{j2\pi f})$ is a linear Combination of DT. Complex exp

High frequency



Low frequency



☆ Very careful.

$e^{j2\pi fn}$ vs $e^{j2\pi ft}$.

- looks very similar but,
- $n = 1, 2, 3$ the same $\qquad$ t exist $-\infty$ to ∞.
- Do Example 5.1

5.1.3 Convergence

In DT-FT, we have infinite sum so for convergence

$$\sum_{n=-\infty}^{\infty} |x(n)| < \infty$$

Q: (Where are the conditions in CT case?)

or finite energy

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 < \infty$$

Synthesis eq. has no issue with convergence (finite interval)

~~5.2. FT of periodic signal~~

~~FT : aperiodic signal~~

~~FS : periodic signal ← special case of FT~~

~~if $x(n)$ is a periodic signal with period N~~

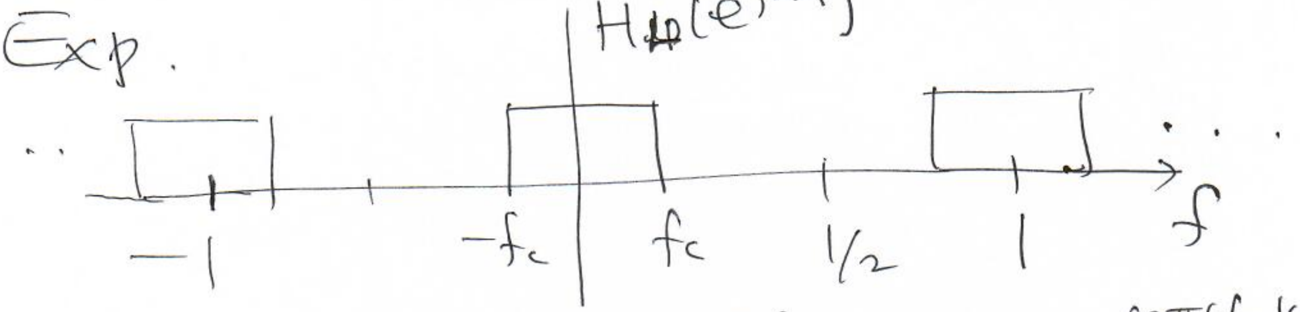
~~$$x(n) = \sum_{k=\langle N\rangle} a_k e^{jk\left(\frac{2\pi}{N}\right)n}$$~~

5.3 properties

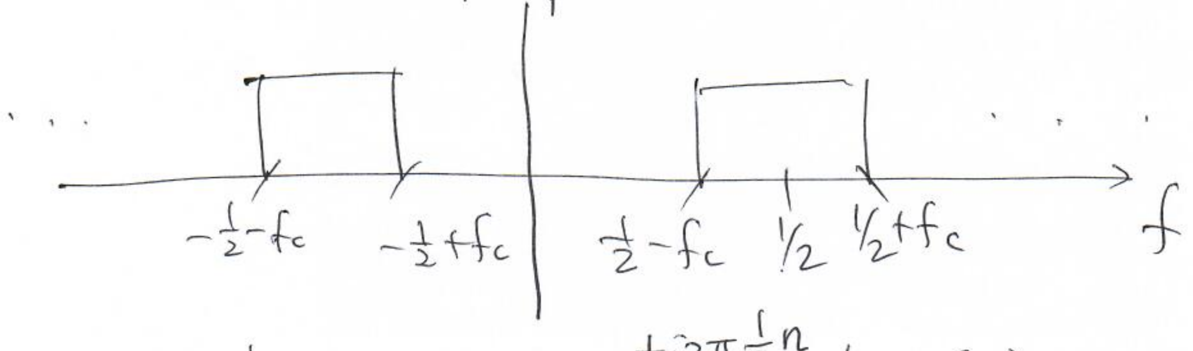
$$x(n) \xleftrightarrow{\mathcal{F}_1} X(e^{j2\pi f})$$

- periodicity $X(e^{j2\pi(f+1)}) = X(e^{j2\pi f})$
- linearity
- Time shift $x(n-n_0) \xleftrightarrow{\mathcal{F}_1} e^{-j2\pi f n_0} X(e^{j2\pi f})$
- Freq. shift $e^{j2\pi f_0 n} x(n) \xleftrightarrow{\mathcal{F}_1} X(e^{j2\pi(f-f_0)})$

Exp.



$$H_{HP}(e^{j2\pi f}) = H_{LP}(e^{j2\pi(f-1/2)})$$



$$h_{HP}(n) = e^{+j2\pi\frac{1}{2}n} h_{LP}(n)$$
$$= e^{j\pi n} h_{LP}(n) = (-1)^n h_{LP}(n)$$

- Conjugate & Conjugate symmetry

$$x^*(n) \xleftrightarrow{\mathcal{F}_1} X^*(e^{-j2\pi f})$$

if $x(n)$ real $X(e^{j2\pi f}) = X^*(e^{-j2\pi f})$

i.e. Hermitian

Then $\mathrm{Re}\{X(e^{j2\pi f})\}$: even fn

$\mathrm{Im}\{X(e^{j2\pi f})\}$: odd fn

- Difference

$$x(n) - x(n-1) \xleftrightarrow{FT} (1 - e^{-j2\pi f}) X(e^{j2\pi f})$$

- Accumulation

$$y(n) = \sum_{m=-\infty}^{n} x(m) \xleftrightarrow{FT} \frac{1}{1-e^{-j2\pi f}} X(e^{j2\pi f}) + \frac{1}{2} X(e^{j0}) \sum_{k=-\infty}^{\infty} \delta(f-k)$$

- Time Reversal

$$x(-n) \xleftrightarrow{FT} X(e^{-j2\pi f})$$

- Time expansion

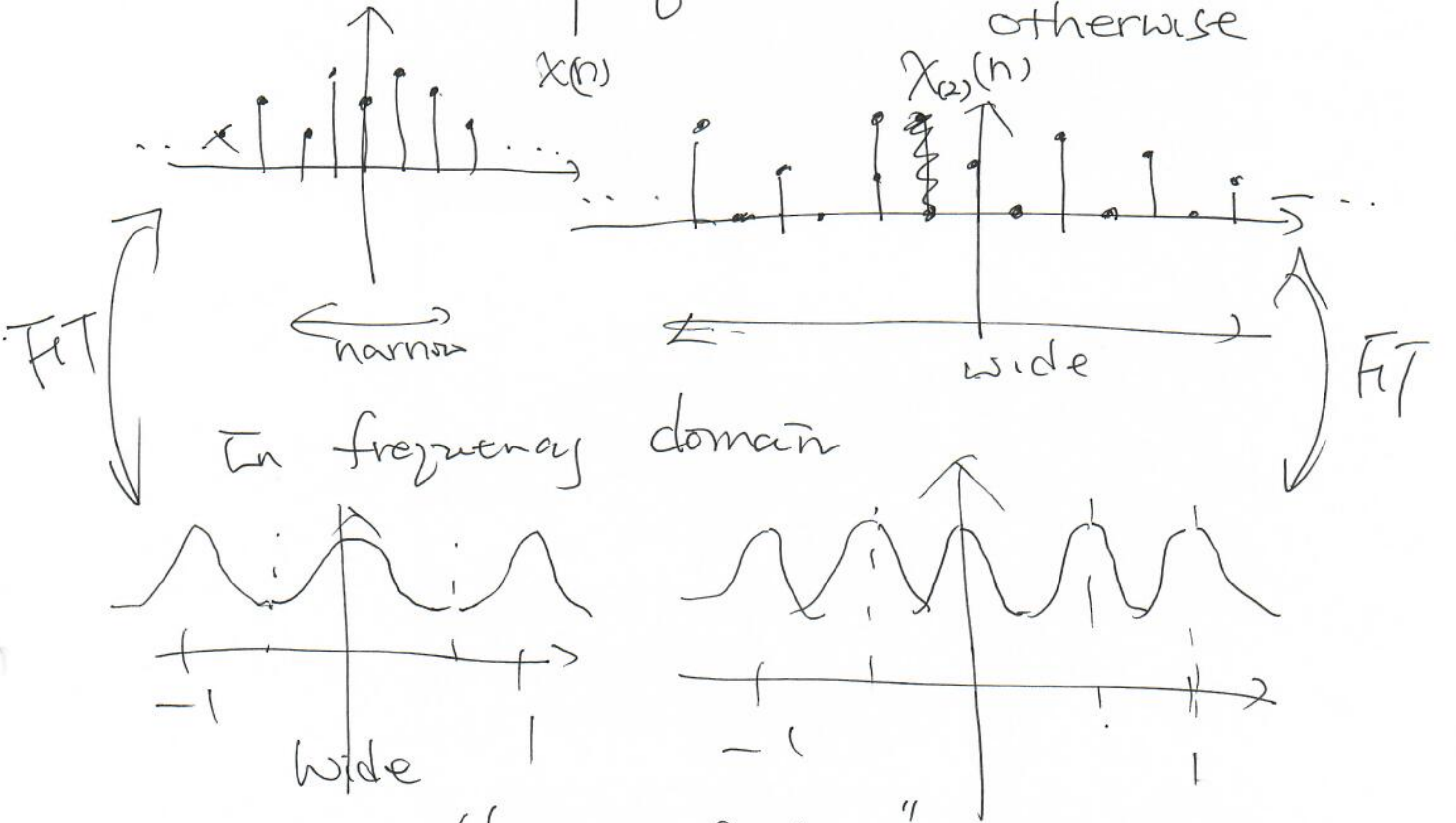
CT-FT : $x(at) \xleftrightarrow{FT} \frac{1}{|a|} X\left(\frac{f}{a}\right)$

DT-FT : $x_{(k)}(n) \xleftrightarrow{FT} X(e^{j2\pi f k})$

↑ zero filling.

~~Show Fig 5.13 8~~.

$$x_{(k)}(n) = \begin{cases} x(n/k) & \text{if } n \text{ is a multiple of } k \\ 0 & \text{otherwise} \end{cases}$$



"Zero filling"

- Differentiation in frequency

$$n\,x(n) \longleftrightarrow \frac{j}{2\pi}\frac{dX(e^{j2\pi f})}{df}$$

- Parseval's Relation

$$\sum_{k=-\infty}^{\infty} |x(n)|^2 = \int_1 |X(e^{j2\pi f})|^2 df$$

5.4 Convolution property

$$y(n) = x(n) * h(n)$$
$$Y(e^{j2\pi f}) = X(e^{j2\pi f}) \cdot H(e^{j2\pi f})$$

5.5 Multiplication property

$$y(n) = x_1(n) \cdot x_2(n)$$
$$Y(e^{j2\pi f}) = \int_0^1 X_1(e^{j2\pi F_1}) X_2(e^{j2\pi(f-F_1)}) dF_1$$

5.6. Table 5.2

① $\delta(n) \overset{\mathcal{F}}{\longleftrightarrow} 1$

② $\delta(n-n_0) \overset{\mathcal{F}}{\longleftrightarrow} e^{-j2\pi f n_0}$

③ $1 \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \delta(f-k)$

④ $e^{j2\pi f_0 n} \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} \delta(f-f_0-k)$

⑤ $\sum_{k=\langle N\rangle} a_k e^{j2\pi(\frac{k}{N})n} \overset{\mathcal{F}}{\longleftrightarrow} \sum_{k=-\infty}^{\infty} a_k \delta(f-\frac{k}{N})$

⑥ $\sin 2\pi f_0 n \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2j}\sum_{k=-\infty}^{\infty}\{\delta(f-f_0-k) - \delta(f+f_0-k)\}$

⑦ $\cos 2\pi f_0 n \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}\sum_{k=-\infty}^{\infty}\{\delta(f-f_0-k) + \delta(f+f_0-k)\}$

⑧ $\frac{1}{N}\text{Ш}(\frac{n}{N}) \overset{\mathcal{F}}{\longleftrightarrow} \text{Ш}(Nf)$

⑨ $a^n u(n) \quad |a|<1 \xleftrightarrow{\mathcal{F}_1} \dfrac{1}{1-ae^{-j2\pi f}}$

⑩ $\dfrac{(n+r-1)!}{n!(r-1)!} a^n u(n) \quad |a|<1 \xleftrightarrow{\mathcal{F}_1} \dfrac{1}{(1-ae^{-j2\pi f})^r}$

⑪ $\mathrm{rect}\left(\dfrac{n}{2N_1}\right) \xleftrightarrow{\mathcal{F}_1} \dfrac{\sin(2\pi f(N_1+\frac{1}{2}))}{\sin(\pi f)}$

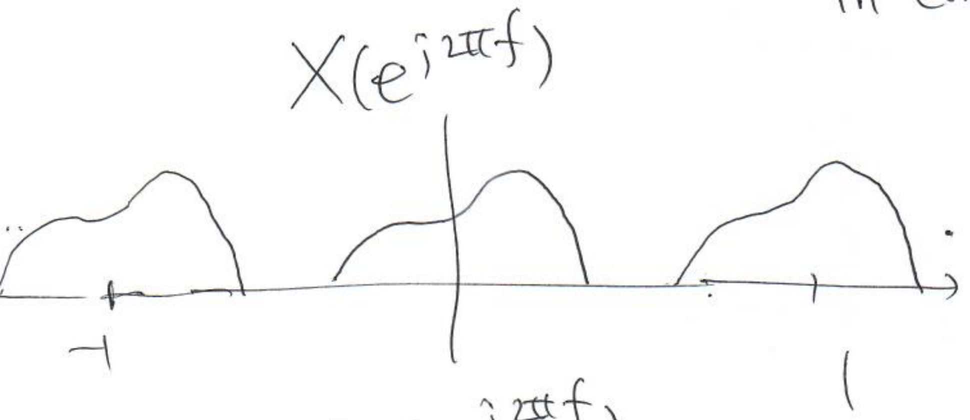
⑫ $2f_0 \mathrm{sinc}(2f_0 n) \xleftrightarrow{\mathcal{F}_1} \mathrm{rect}\left(\dfrac{f}{2f_0}\right)$ with period of "1"

5.2 DT-FiT for periodic signal

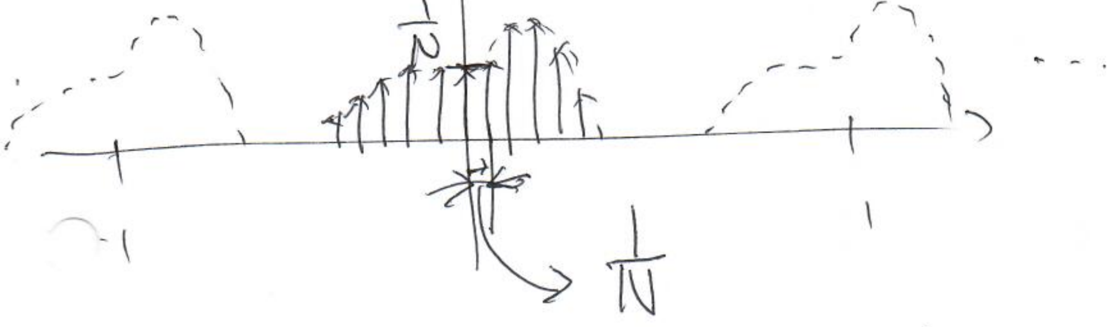
$$x_p(n) = \underbrace{x(n)}_{\text{finite duration}} * \frac{1}{N} ⧢\left(\frac{n}{N}\right)$$

$$\underbrace{\mathcal{F}_1\{x_p(n)\}}_{\text{DT-FiT of periodic fn}} \underset{}{=} X_p(e^{j2\pi f}) \qquad = X(\cancel{f}\,e^{j2\pi f}) \cdot \underbrace{⧢(Nf)}_{\text{multiplication of } ⧢(Nf) \text{ in continuous f domain.}}$$

$X(e^{j2\pi f})$



$X_p(e^{j2\pi f})$



※ DT-FiT of periodic signal (or DT-FiS) has ~~discrete point~~ [illegible] points of distinct spectrum in frequency.

$$X_p(e^{j2\pi f}) = X(e^{j2\pi f}) \cdot Ш(Nf)$$

$$= \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi f n} \cdot \frac{1}{N} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{N})$$

$$= \frac{1}{N} \sum_{n=<N>} x(n) \sum_{k=-\infty}^{\infty} e^{-j2\pi f n} \delta(f - \frac{k}{N})$$

$$= \underbrace{\frac{1}{N} \sum_{n=<N>} x(n) e^{-j2\pi(\frac{k}{N})n}}_{a_k} \sum_{k=-\infty}^{\infty} \delta(f - \frac{k}{N})$$

$$= \sum_{k=-\infty}^{\infty} a_k \delta(f - \frac{k}{N})$$

$$\therefore a_k = \frac{1}{N} \sum_{n=<N>} x(n) e^{-j2\pi(\frac{k}{N})n}$$

or $X(k)$

$$x_p(n) = \int_1 X_p(e^{j2\pi f}) e^{j2\pi f n} df$$

$$= \int_1 \sum_{k=<N>} a_k \delta(f - \frac{k}{N}) e^{j2\pi f n} df$$

$$= \sum_{k=<N>} a_k \int_1 \delta(f - \frac{k}{N}) e^{j2\pi f n} df$$

$$= \sum_{k=<N>} a_k e^{j2\pi \frac{k}{N} n}$$

//

so we have a new pair for periodic DT signal

$$a_k = \frac{1}{N} \sum_{n=<N>} x(n) e^{-j2\pi(\frac{k}{N})n}$$

or $X(k)$

$$x(n) = \sum_{k=<N>} a_k e^{j2\pi \frac{k}{N} n}$$

both $x(n)$ & a_k (or $X(k)$) are

distinctive finite duration !

5.8 LCC DE

$$\sum_{k=0}^{N} a_k y(n-k) = \sum_{k=0}^{M} b_k x(n-k)$$

$$H(e^{j2\pi f}) = \frac{Y(e^{j2\pi f})}{X(e^{j2\pi f})}$$

$$\sum_{k=0}^{N} a_k e^{-j2\pi fk} Y(e^{j2\pi f}) = \sum_{k=0}^{M} b_k e^{-j2\pi fk} X(e^{j2\pi f})$$

$$H(e^{j2\pi f}) = \frac{\sum_{k=0}^{M} b_k e^{-j2\pi fk}}{\sum_{k=0}^{N} a_k e^{-j2\pi fk}}$$

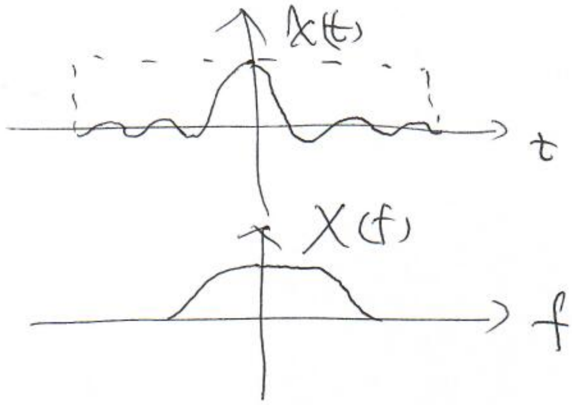
Do Ex. 5.19 if time allows.

- Summary of FT & FS

CT-FT

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi ft} dt$$

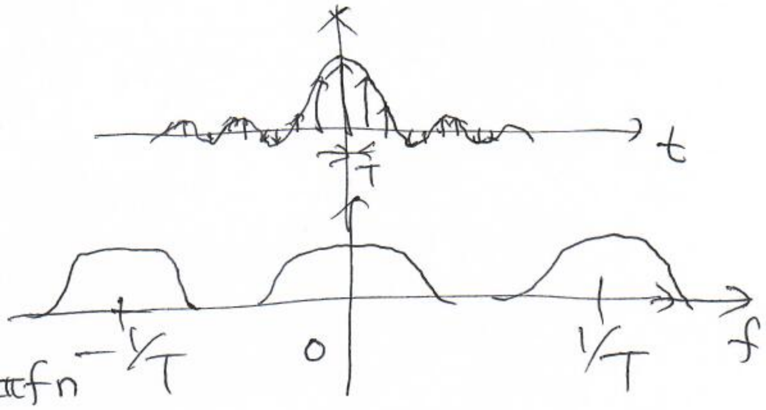
$$x(t) = \int_{-\infty}^{\infty} X(f) e^{+j2\pi ft} df$$



DT-FT

$$x(t) \cdot \frac{1}{T}\text{Ш}\left(\frac{t}{T}\right)$$

$$X(f) * \text{Ш}(Tf)$$



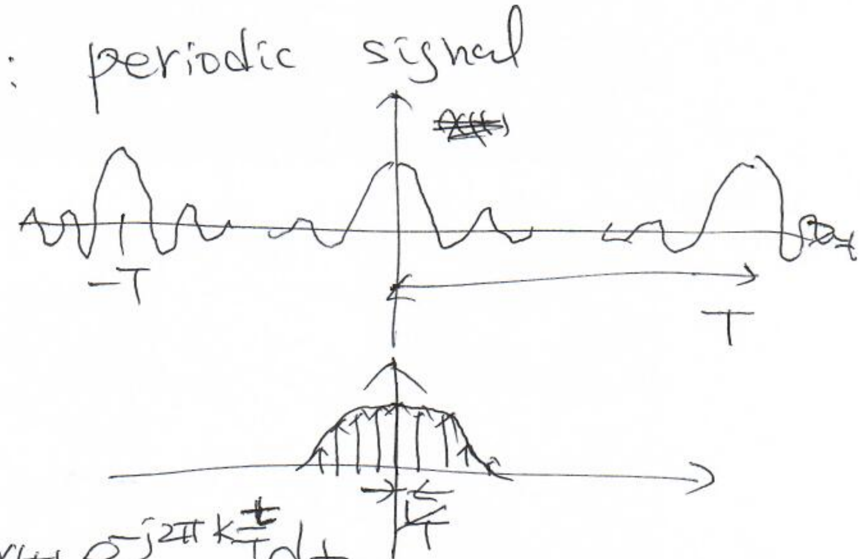
$$X(f) = X(e^{j2\pi f}) = \sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi fn}$$

$$x(n) = \int_{1} X(e^{j2\pi f}) e^{+j2\pi fn} df$$

~~DT~~ CT-FS : periodic signal

$$x(t) * \frac{1}{T}\text{Ш}\left(\frac{t}{T}\right)$$

$$X(f) \cdot \text{Ш}(Tf)$$



$$X(k) = a_k$$

$$= \frac{1}{T}\int_{T} x(t) e^{-j2\pi k\frac{t}{T}} dt$$

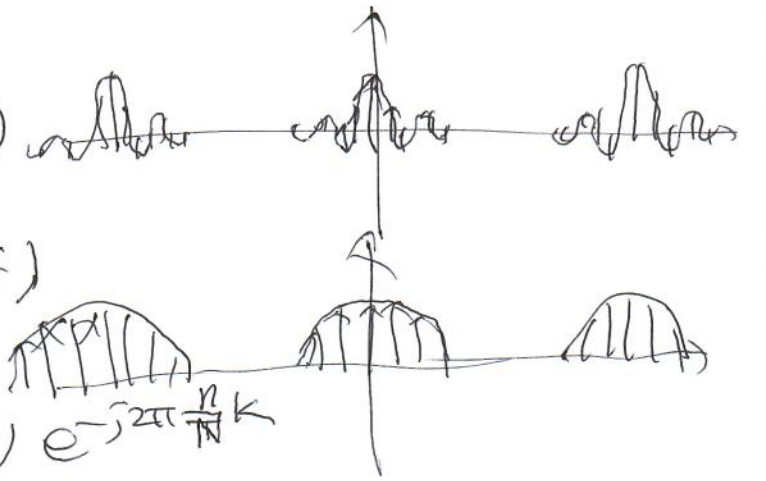
$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k\frac{t}{T}} \qquad \frac{1}{T} = f_0$$

DT-FS : periodic signal

① from CT-FS :

$$\{x(t) * \tfrac{1}{T_1}\text{Ш}(\tfrac{t}{T_1})\}\tfrac{1}{T_2}\text{Ш}(\tfrac{t}{T_2})$$

$$\{X(f)\cdot\text{Ш}(T_1 f)\} * \text{Ш}(T_2 f)$$



$$X(k) = a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x(n)\, e^{-j2\pi\frac{n}{N}k}$$

$$x(n) = \sum_{k=\langle N\rangle} a_k\, e^{j2\pi\frac{k}{N}n}$$

② from DT-FS

$$\{x(t)\,\tfrac{1}{T_1}\text{Ш}(\tfrac{t}{T_1})\} * \tfrac{1}{T_2}\text{Ш}(\tfrac{t}{T_2})$$

$$\{X(f) * \text{Ш}(T_1 f)\}\cdot\text{Ш}(T_2 f)$$

Chapter 3

Flourier Series

e^{st}

- Our magic function : $e^{j2\pi ft}$

Why? LTI system. $\downarrow$ eigenfunction.

if $x(t) = \sum_k a_k e^{j2\pi f_k t}$. → good for LTI

we did it for ~~FT~~ aperiodic signal (i.e FT)

What about periodic signal (YES we already did it!)

3.3. Fourier Series representation of CT periodic signal

- $e^{j2\pi f_0 t}$ is a periodic signal.
  → can we use this fn to represent a periodic signal?
- Lets consider a periodic signal

$$x(t) = x(t+T)$$

$$f_0 = \frac{1}{T} \text{ (fundamental frequency)}$$

Then $e^{j2\pi kf_0 t}$ or $e^{j2\pi\frac{k}{T}t}$ becomes harmonically related complex exp.

→ we suspect an $x(t)$ w fund. freq of f_0 can be written as

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi k\frac{1}{T}t}$$

When $k=0$ constant

$k=\pm 1$ fund freq. or first harmonic

$k=\pm N$ Nth harmonic.

Read page 188-189 when x(t) is real

- Q1: How many fn can be represented ...

  → a lot

Q2: What will be a_k?

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi \frac{k}{T} t}$$

periodic

$$\int_0^T x(t) e^{-j2\pi \frac{n}{T} t} dt = \int_0^T \sum_{k=-\infty}^{\infty} a_k e^{j2\pi \frac{(k-n)}{T} t} dt$$

$$= \sum_{k=-\infty}^{\infty} a_k \left[\underbrace{\int_0^T e^{j2\pi \frac{(k-n)}{T} t} dt}_{\delta} \right]$$

$$= \begin{cases} T & \text{when } k=n \\ 0 & \text{when } k \neq n \end{cases}$$

$$= T a_n$$

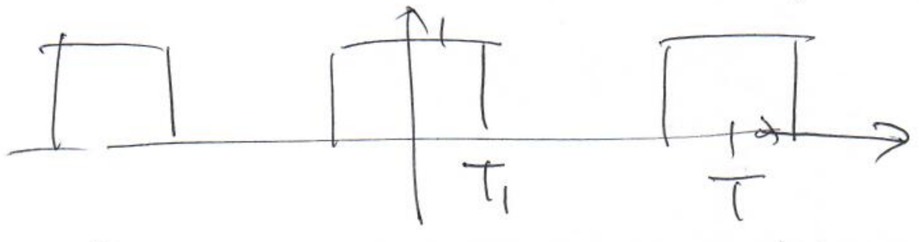
$$\therefore a_k = \frac{1}{T} \int_0^T x(t) e^{-j2\pi \frac{k}{T} t} dt$$

$$x(t) = \sum_{k=-\infty}^{\infty} a_k e^{j2\pi \frac{k}{T} t}.$$

$a_0 \rightarrow$ DC.

- OK! We already know this from FT!

Let's do an example



from FT $= \text{rect}(\frac{t}{2T_1}) * \frac{1}{T} ш(\frac{t}{T})$

$X(f) = 2T_1 \,\mathrm{sinc}(2T_1 f) \cdot Ш(Tf)$ // done!

$a_k = ?$

$$a_k = \frac{1}{T}\int_{-T_1}^{T_1} 1 \, e^{-j2\pi\frac{k}{T}t} dt$$

$$= \frac{1}{T} \frac{e^{-j2\pi\frac{k}{T}t}}{-j2\pi\frac{k}{T}} \Bigg|_{-T_1}^{T_1}$$

$$= \frac{e^{-j2\pi\frac{k}{T}T_1} - e^{+j2\pi\frac{k}{T}T_1}}{T(-j2\pi\frac{k}{T})} = \frac{2T_1}{T}\frac{\sin 2\pi\frac{k}{T}T_1}{2\pi\frac{k}{T}T_1}$$

$$= \frac{2T_1}{T}\,\mathrm{sinc}\left(2\frac{k}{T}T_1\right)$$

$$a_k = \frac{1}{T}X(f)\Big|_{\frac{k}{T}} \quad //$$

show p.195 figure 3.7.

3.4. Convergence.

$$x_N(t) = \sum_{k=-N}^{N} a_k e^{j2\pi\frac{k}{T}t}.$$

$$e_N(t) = x(t) - x_N(t)$$

$$\underbrace{E_N}_{\text{energy}} = \int_T |e_N(t)|^2 dt$$

- If x(t) has F.S. $E_N \to 0$ as $N \to \infty$

← Every continuous periodic signal has F.S. & a lot of discontinuous signals

— ~~we~~ inverse F.S signal & orignal signal may ~~are~~ not be the same!

- We say no energy difference between the two ⑨

Convergence condition A

- $\int_T |x(t)|^2 dt < \infty$: finite energy over one period

Convergence condition B

- dirichelet conditions (1, 2, 3) in page197-198
- discontinuity value is mean of two edges.

- Gibbs ringing : overshoot 9% page ~~199~~ 201
  still converge
  area → 0 as $n \to \infty$

3.5 properties of CT-FS

$$x(t) \xleftrightarrow{FS} a_k$$

- linearity
- Time shift $x(t-t_0) \longleftrightarrow e^{-j2\pi\frac{k}{T}t_0} a_k =$
- Time reversal $x(-t) \longleftrightarrow a_{-k}$
- Time scaling $x(at) \longleftrightarrow \underline{a_k}$ different meaning since $x(at) = \sum a_k e^{j[illegible]}$

convolution

- $\frac{1}{T}\int_T x(\tau)y(t-\tau)d\tau \longleftrightarrow a_k b_k$
- Multiplication.

$$x(t)\,y(t) \xleftrightarrow{FS} \sum_{l=-\infty}^{\infty} a_l b_{k-l}.$$

- Conjugate symmetry

$$x^*(t) \xleftrightarrow{FS} a^*_{-k}$$

if $x(t)$ real $a_{-k} = a^*_k$ (Hermitian)

- Parseval's Relation

$$\frac{1}{T}\int_T |x(t)|^2 dt = \sum_{k=-\infty}^{\infty} |a_k|^2.$$

3.6 FS of DT periodic signal

$$x(n) = x(n+N) \quad \text{with} \quad \underset{\text{fundamental freq.}}{f_0} = \frac{1}{N}$$

Harmonic functions $\phi_k(n) = e^{j2\pi\frac{k}{N}n}$

* Different from continous time case k is not infinite because $2\pi k\frac{1}{N} = 2\pi(k+N)\frac{1}{N}$

This means we have N distinct k,

i.e. $\phi_0(n), \phi_1(n), \cdots, \phi_{N-1}(n)$.

$$x(n) = \sum_{k=0}^{N-1} a_k e^{j2\pi k\frac{1}{N}n}$$

$$= \sum_{k=\langle N\rangle} a_k e^{j2\pi\frac{k}{N}n}$$

What would be a_k?

$$\sum_{n=\langle N\rangle} x(n) e^{-j2\pi\frac{n}{N}k} = \sum_{n=\langle N\rangle}\sum_{k=\langle N\rangle} \cancel{x(n)}\, a_k e^{-j2\pi\frac{(r-n)}{N}k}$$

$$= \sum_r \cancel{x(r)}\, a_k \sum_k e^{-j2\pi\frac{(r-n)k}{N}}$$

$$= N a_k$$

No Gibbs ringing No issue w Convergence.

$$\therefore a_k = \frac{1}{N}\sum_{r=\langle N\rangle} x(r) e^{-j2\pi\frac{r}{N}k}$$

$$a_k = \frac{1}{N}\sum_{n=\langle N\rangle} x(n) e^{-j2\pi\frac{n}{N}k}.$$

Why?

$$x(0) = \sum_{k=\langle N\rangle} a_k$$

$$x(1) = \sum_{k=\langle N\rangle} a_k e^{j2\pi k/N}$$

$$\vdots \qquad\qquad \vdots$$

$$x(N-1) = \sum_{k=\langle N\rangle} a_k e^{+j2\pi k(N-1)/N}.$$

→ N equations & N unknown.

$$\begin{bmatrix} x(0) \\ x(1) \\ \vdots \\ \vdots \\ x(N-1) \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 \cancel{e^{j\frac{2\pi}{N}}} & e^{j\frac{2\pi\cdot 2}{N}} & \cdots & & e^{j\frac{2\pi(N-1)}{N}} \\ 1 & & & & \\ \vdots & & & & \\ 1 & & \cdots & & e^{j\frac{2\pi(N-1)^2}{N}} \end{bmatrix}} \begin{bmatrix} a_0 \\ a_1 \\ \vdots \\ \vdots \\ a_{N-1} \end{bmatrix}$$

is this orthogonal ~~ø~~
full rank? why?

so x(n) and a_k exist in all cases.

3.7. properties.

· ~~mult~~ Almost the same as CT-FS
Check page 221 / 222

3.8 FS and LTI systems

→ Better to solve in FT ~~&~~T of periodic signals.

$$y(t) = \sum_{k=-\infty}^{\infty} a_k \underbrace{H\left(\frac{k}{T}\right)}_{FT} e^{j2\pi\frac{k}{T}t}$$

$$y(n) = \sum_{k=\langle N\rangle} a_k H\left(e^{j2\pi\frac{k}{N}}\right) e^{j\frac{2\pi k n}{N}}$$

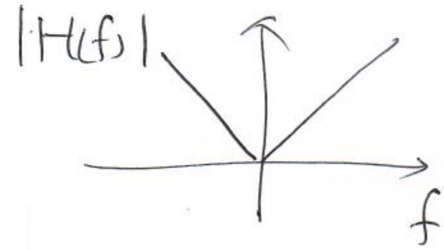
3.9 Filtering.

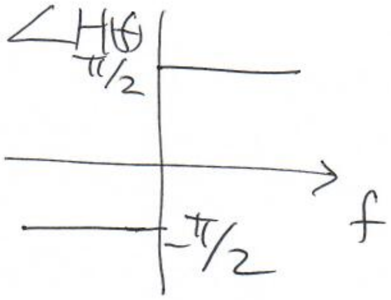
- Frequency-shaping filter: change shape of spectrum
  e.g.) equalizer

~~Frequency-selective filter: pass some attenuate/eliminate others spectrual components.~~

Example 1 ~~of a filter~~:

$y(t) = \frac{dx(t)}{dt}$ $H(f) = j2\pi f.$





→ enhance high frequency (i.e rapid variations)

→ enhance edges in picture. (Fig. 3.24)

* First explain an image

$m(t_1, t_2)$ or $m(x, y)$

$m(x, y) = e^{j2\pi f_1 x + j2\pi f_2 y}$

→ Computer PPT file.

→ digital (or discrete-time (or space) domain

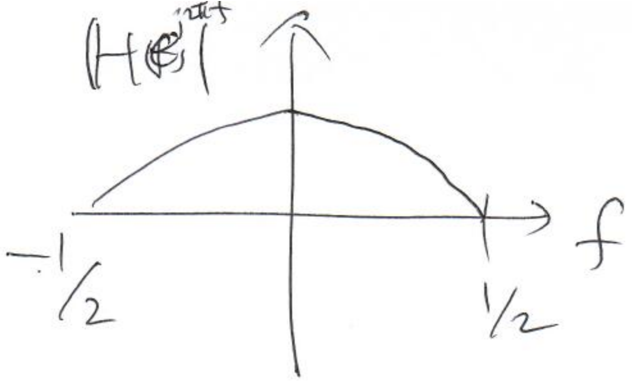
Difference operation $x(n) - x(n-1)$

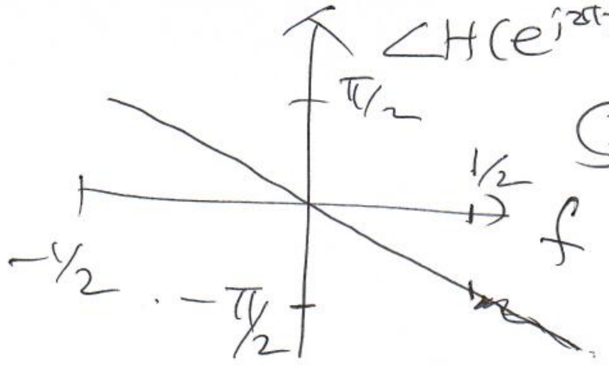
Example 2

$$y(n) = \frac{1}{2}(x(n) + x(n-1))$$

$$H(e^{j2\pi f}) = \mathcal{F}\{\frac{1}{2}(\delta(n) + \delta(n-1))\}$$

$$= \frac{1}{2}(1 + e^{-j2\pi f}) = e^{-j\pi f}\cos(\pi f)$$



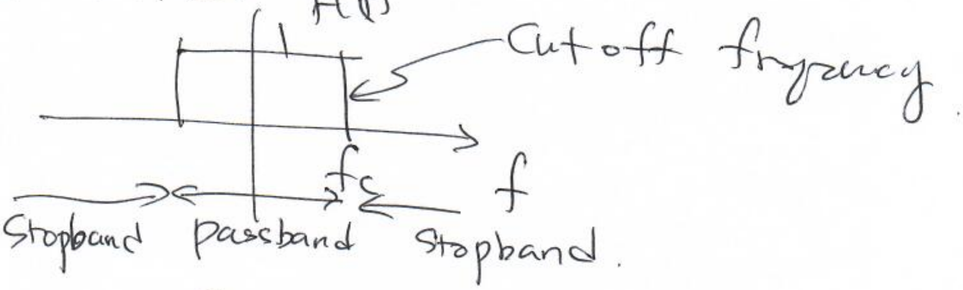


low frequency: OK
high frequency: suppressed.

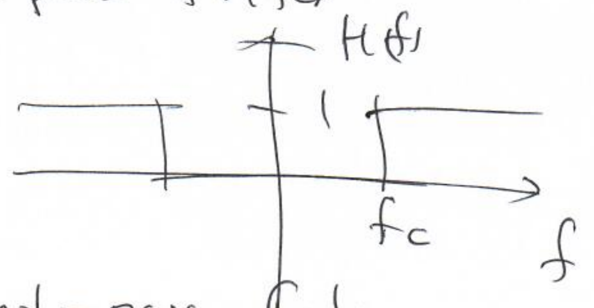
- Frequency-selective filter: select some frequency bands and reject others

Ex) male voice vs violin.
Radio channel selection.

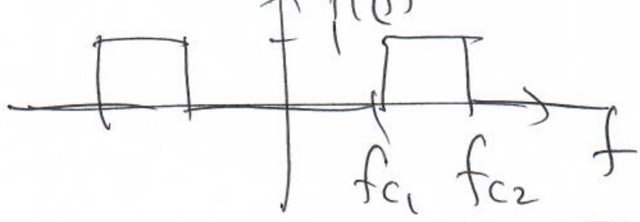
Ideal Low pass filter



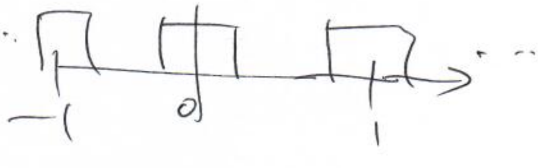
Ideal high pass filter



Ideal band-pass filter



In DT filter repeats



3.10 Examples.

RC Lowpass filter.

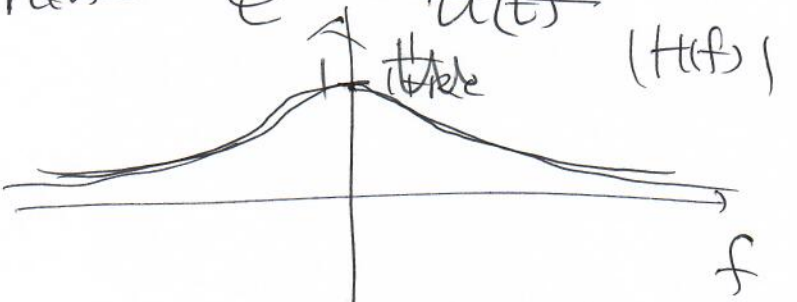
$$RC\frac{d v_c(t)}{dt} + v_c(t) = v_s(t)$$

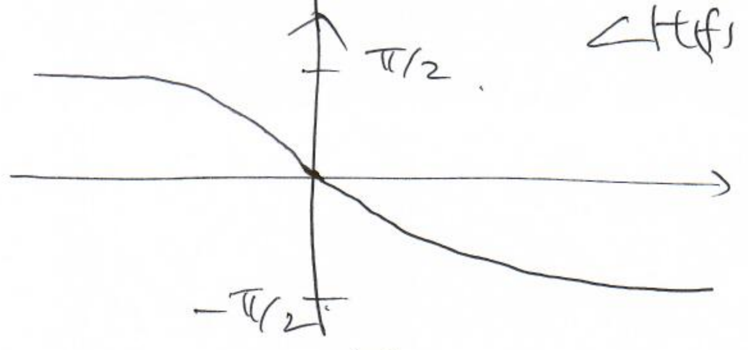
Assuming initial rest. $\underset{\overset{\downarrow}{y(t)}}{v_c(t)}$? LTI system.
$v_s(t) \rightarrow x(t)$.

h(t) ? $(1 + RCj2\pi f)V_c(f) = V_s(f)$

$$H(f) = \frac{V_c(f)}{V_s(f)} = \frac{1}{1 + RCj2\pi f}$$

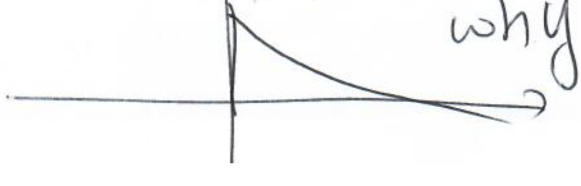
~~$h(t) = e^{-t/RC} u(t)$~~





$$h(t) = \frac{1}{RC}e^{-t/RC} \cdot u(t).$$

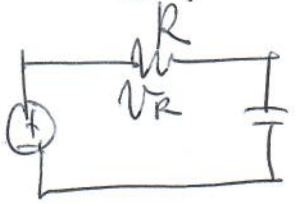
Let's talk about RC large or small.



why is this LPF
in time domain?

... h(t) ?

RC highpass filter.



$$RC\frac{dv_r(t)}{dt} + v_r(t) = RC\frac{dv_s(t)}{dt}.$$

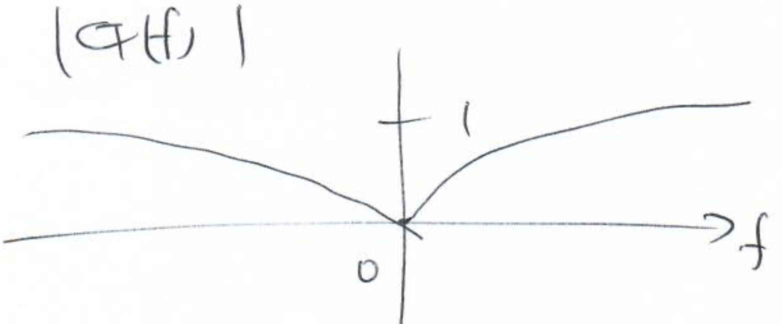
$$G(f) = \frac{j2\pi fRC}{1+j2\pi fRC}$$

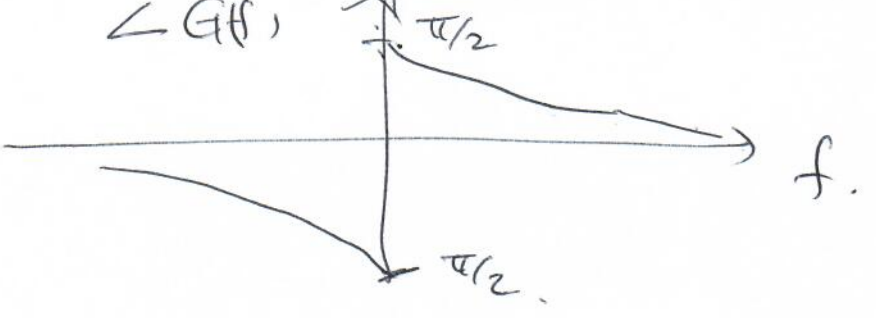
틀림.

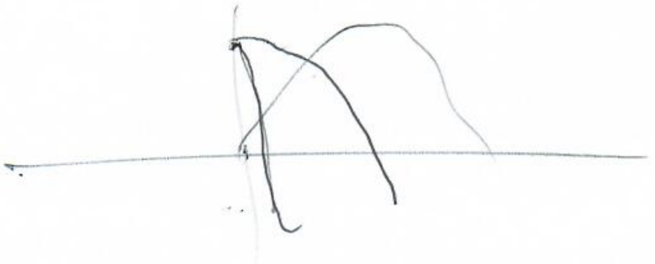
~~g(t) =~~

$$g(t) = e^{-\frac{t}{RC}}u(t)$$

$$v_r(t) = g(t) * ~~h(t)~~ v_s(t)$$







3.11. Discrete-Time filters by difference Eq. ⑤

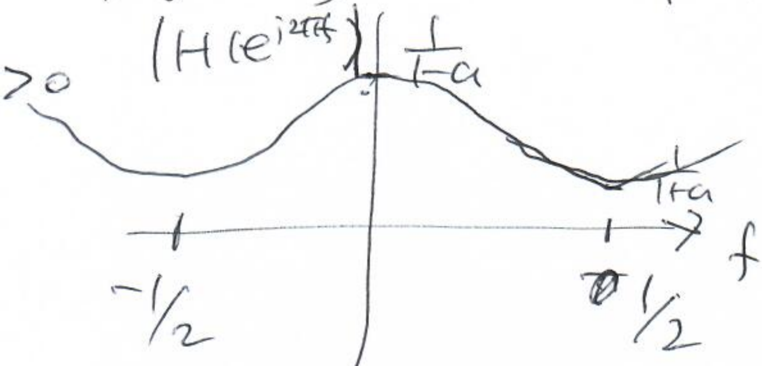
Recursive DT filter. (IIR)

$$y(n) - ay(n-1) = x(n)$$

$$Y(e^{j2\pi f}) - aY(e^{j2\pi f})e^{-j2\pi f} = X(e^{j2\pi f})$$

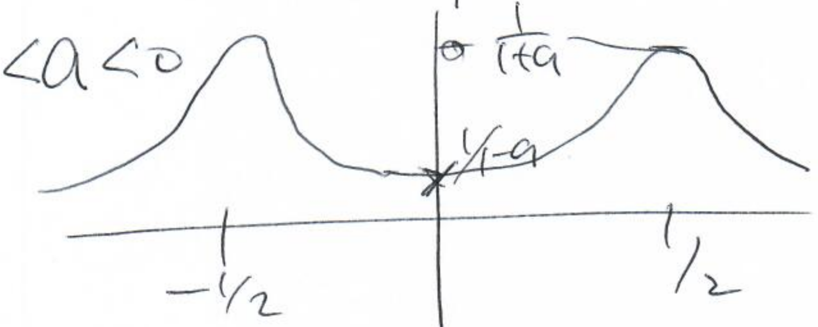
$$H(e^{j2\pi f}) = \frac{Y(e^{j2\pi f})}{X(e^{j2\pi f})} = \frac{1}{1-ae^{-j2\pi f}}$$

$1 > a > 0$



$e^{-j\pi} = -1$

$-1 < a < 0$



$h(n) = a^n u(n)$
↑
IIR

$a = +0.5$
vs -0.5.

Nonrecursive DT filter. (FIR)

$$y(n) = \sum_{k=-N}^{M} b_k x(n-k)$$

$$= \frac{1}{3}\left(x(n-1) + x(n) + x(n+1)\right)$$

$$h(n) = \frac{1}{3}[\delta(n+1) + \delta(n) + \delta(n-1)]$$

↑ FIR

$$H(e^{j2\pi f}) = \frac{1}{3}[e^{j2\pi f} + 1 + e^{-j2\pi f}] = \frac{1}{3}(1 + 2\cos 2\pi f)$$

Chapter 7.

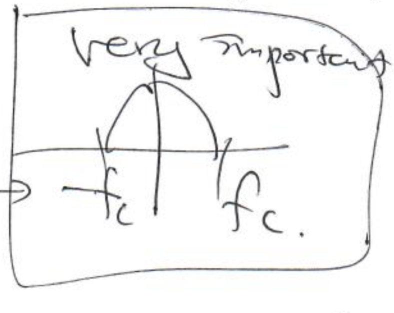
Chapter 7. Sampling.

Is an Image continous?

we need digitaized signal for computers,

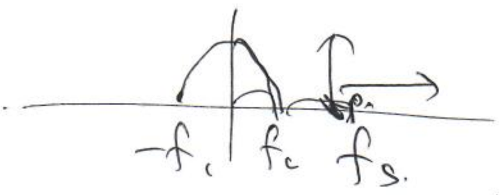
7.1. Sampling theorem

If a signal is band limited,



we can perfectly reconstruct the original signal from the samples of the signal.

To do so, we need to sample the signal more than twice the freq. of the maximum frequency of this signl.



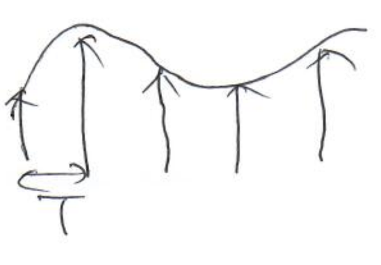
⇒ Nyquist sampling theorem (will be your guide)

7.1.1 Sampling function

— Remember Ш?

To have unit amplitude with spacing of T

sampling function becomes $\frac{1}{T}Ш(\frac{t}{T})$



$$x_p(t) = \sum^{\infty} x(nT)\,\delta(t-nT)$$

In frequency domain

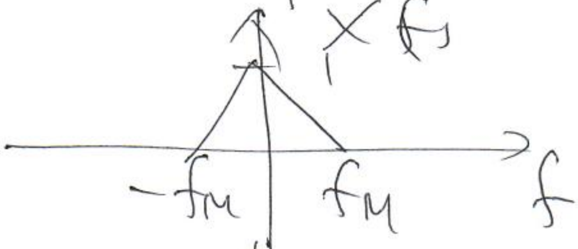
$$X_p(f) = \mathcal{F}\{x(t)\} * \mathcal{F}\{\tfrac{1}{T}Ш(\tfrac{t}{T})\}$$

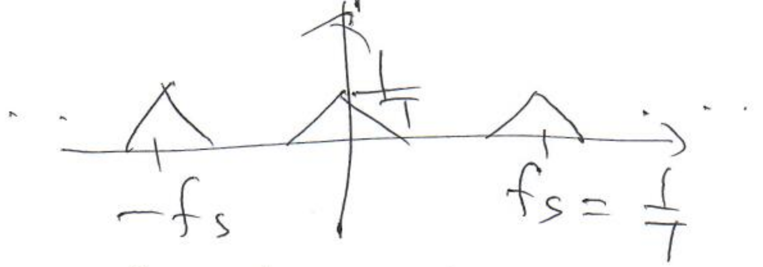
$$= X(f) * Ш(Tf)$$

$$= X(f) * \tfrac{1}{T}\sum \delta(f - \tfrac{n}{T})$$

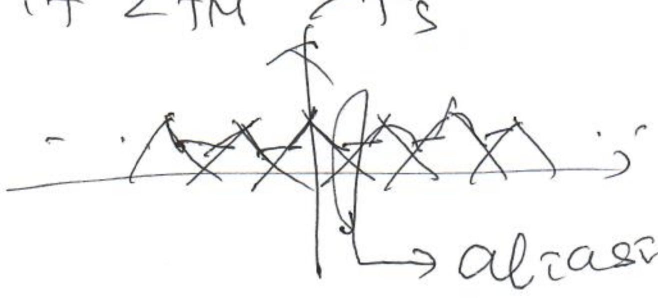
$$= \tfrac{1}{T}\sum X(f - \tfrac{n}{T})$$

$$= \tfrac{1}{T}\sum X(f - nf_s) \qquad f_s = \tfrac{1}{T} \text{ : sampling freq.}$$



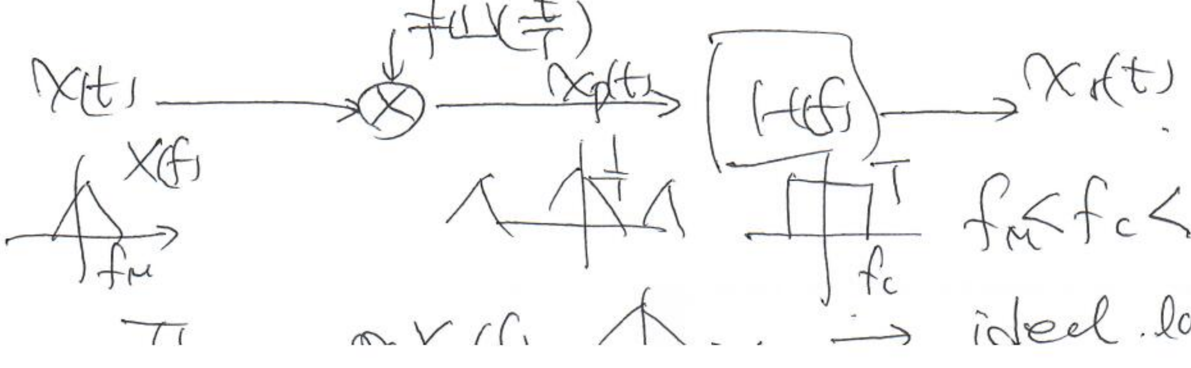


if $2f_M > f_s$



$2f_M$: Nyquist rate (need more than this)

- Reconstruction



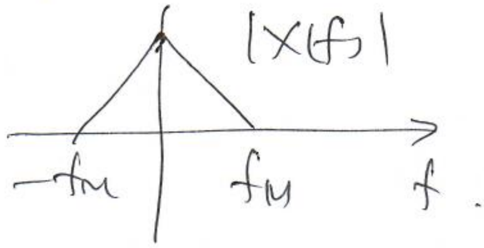
$f_M < f_c < f_s - f_M$

ideal low pass

A few things:

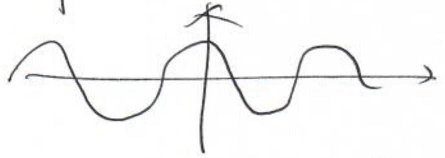
① Why negative frequency?

Voice is real signal. So the spectrum is hermitian. i.e $X(-f) = X^*(f)$

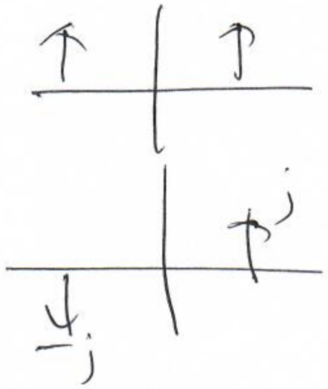


$$|X(-f)| = |X^*(f)|$$
$$= |X(f)|$$

example $\cos 2\pi f t$



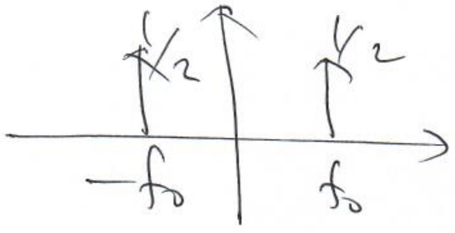
$\sin 2\pi f t$



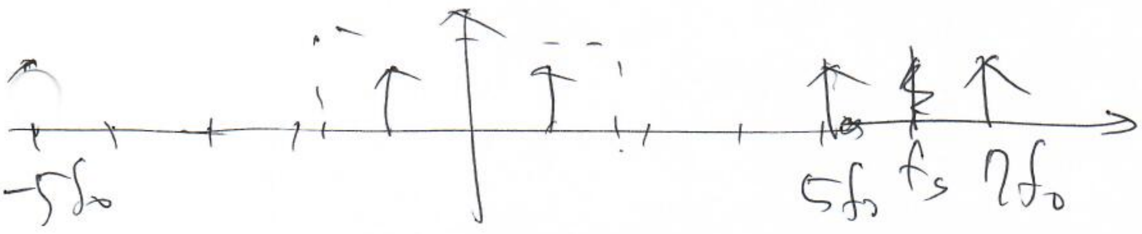
② What is aliasing?

If $f_s < 2f_M$, we cannot reconstruct the original signal.

Ex $\cos 2\pi f_0 t$

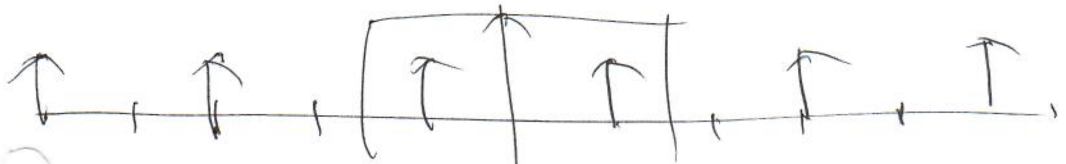


If $f_s = 6f_0$



→ No aliasing

If $f_s = 3f_0$



Still No aliasing

If $f_s = 3/2 f_0$

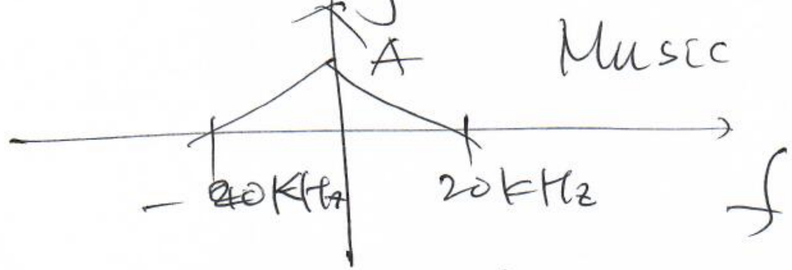
two ~~sin~~ cosine functions!

Show Fig 7.16. for Time domain.

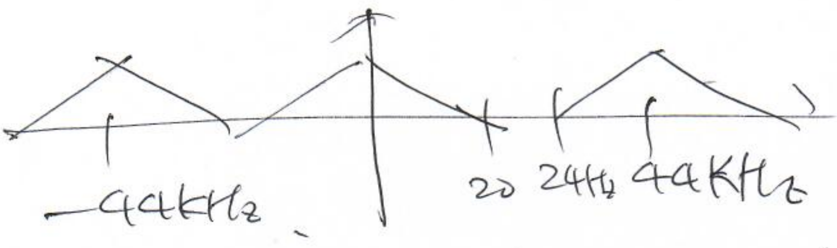
Show wheel.

③ Anti-aliasing filter.

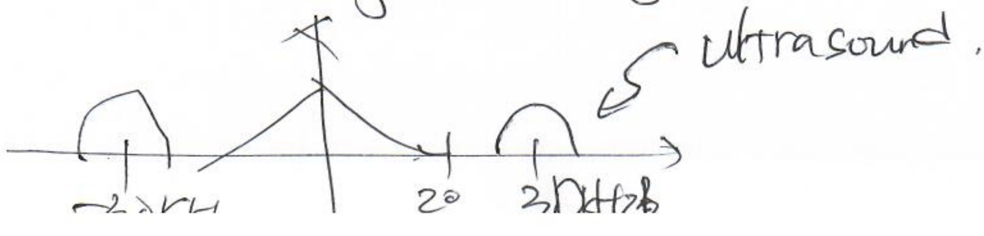
Music signal



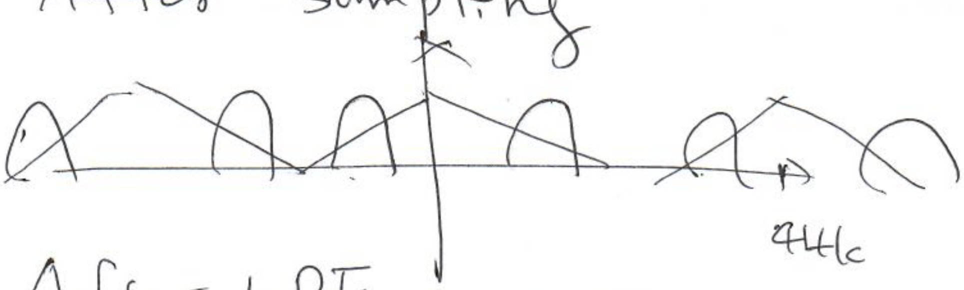
× $\frac{1}{T}Ш(\frac{t}{T})$ @ 44kHz.



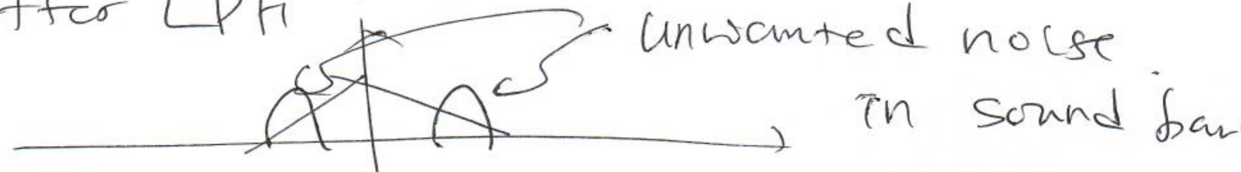
What happens if we had ultra-sound noise @ 30kHz. While you are recording a song.



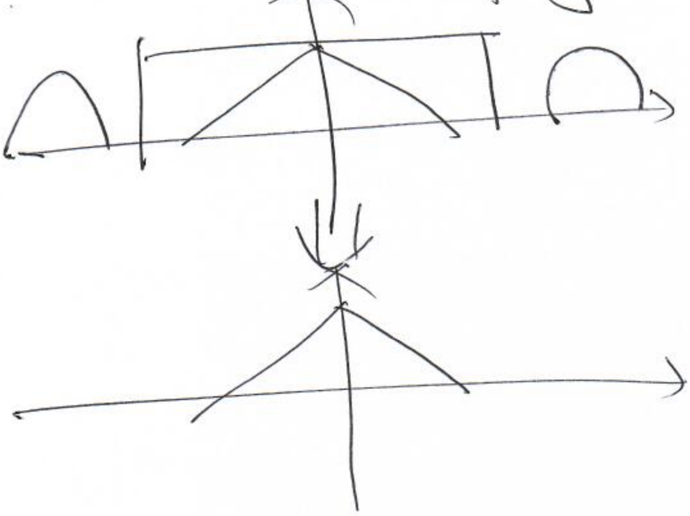
After sampling



After LPF



Hence we need to filter out ultrasound noise before sampling! Anti-aliasing filter



then sample

7.2 Reconstruction Using Interpolation

$$x_r(t) = x_p(t) * h_r(t)$$

$$= \left\{ \sum_{n=-\infty}^{\infty} x(nT)\,\delta(t-nT) \right\} * h_r(t)$$

$$= \sum_{n=-\infty}^{\infty} x(nT)\, h_r(t-nT)$$

- Ideal interpolator

$$H_r(f) = T\,\mathrm{rect}\left(\frac{f}{2f_c}\right)$$

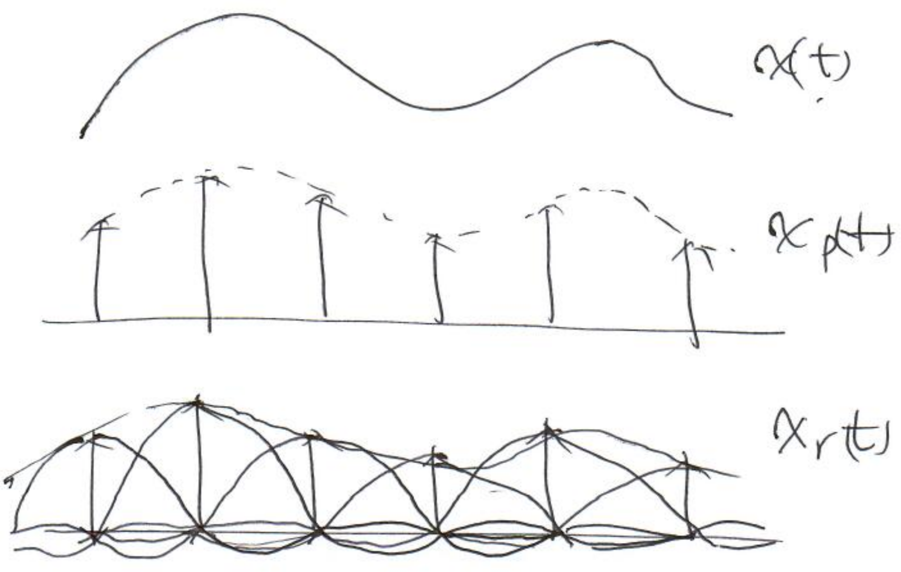
$$h_r(t) = T2f_c\,\mathrm{sinc}(2f_c t)$$

$$\therefore\ x_r(t) = T2f_c \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}(2f_c(t-nT))$$

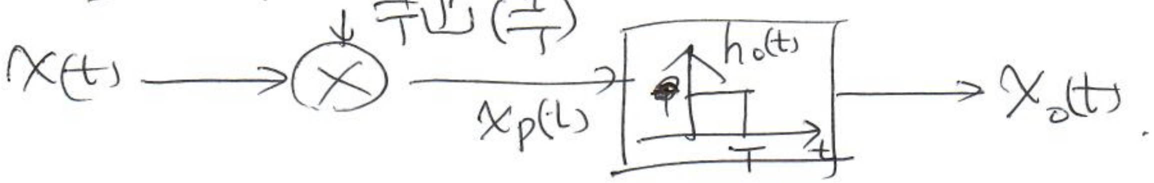
if $T = \frac{1}{2f_c}$ (What is this condition?)

$$T2f_c = 1$$

Then $x_r(t) = \sum_{n=-\infty}^{\infty} x(nT)\,\mathrm{sinc}(2f_c(t-nT))$



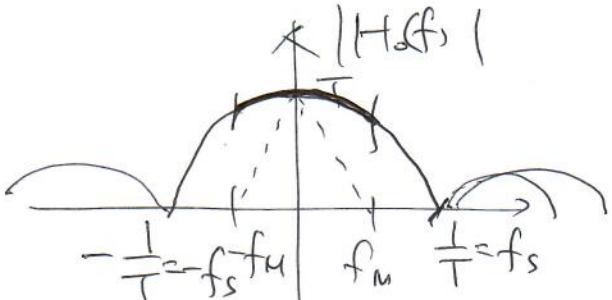
- Zero order hold Interpolator.



$$x_0(t) = \left\{ x(t) \cdot \frac{1}{T}Ш\left(\frac{t}{T}\right) \right\} * \mathrm{rect}\left(\frac{t}{T}-\frac{1}{2}\right)$$

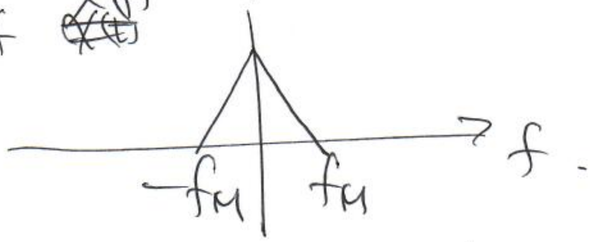
$$h_0(t) = \mathrm{rect}\left(\frac{t}{T}-\frac{1}{2}\right) = \mathrm{rect}\left(\frac{t-\frac{1}{2}T}{T}\right)$$

$$H_0(f) = T e^{-j\pi Tf} \mathrm{sinc}(Tf)$$

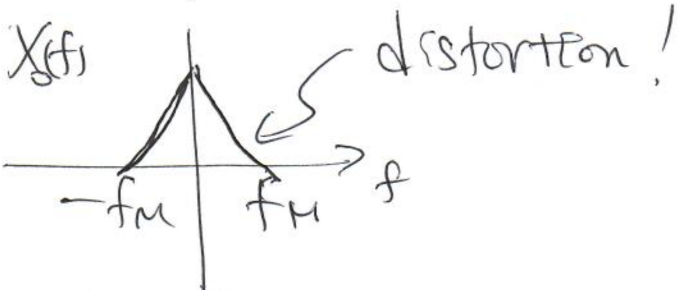


$\frac{1}{T} = f_s$

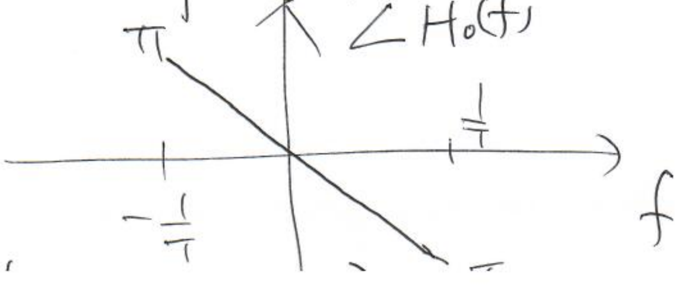
If $X(f)$ ~~$x(t)$~~



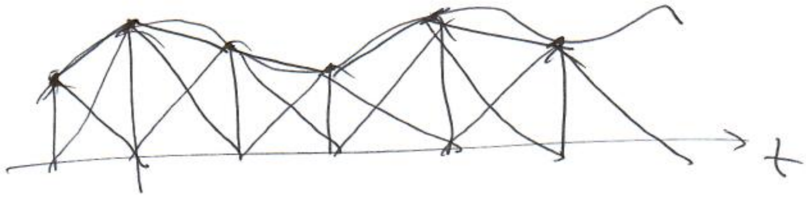
Then $X_0(f)$



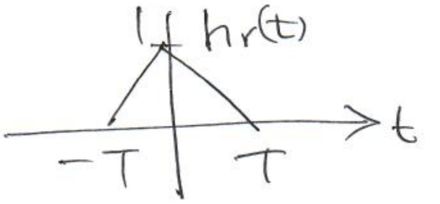
What about phase?



- Linear Interpolator

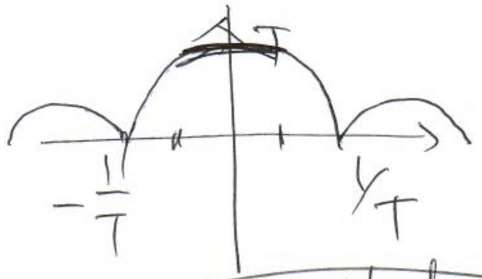


What is the interpolator?



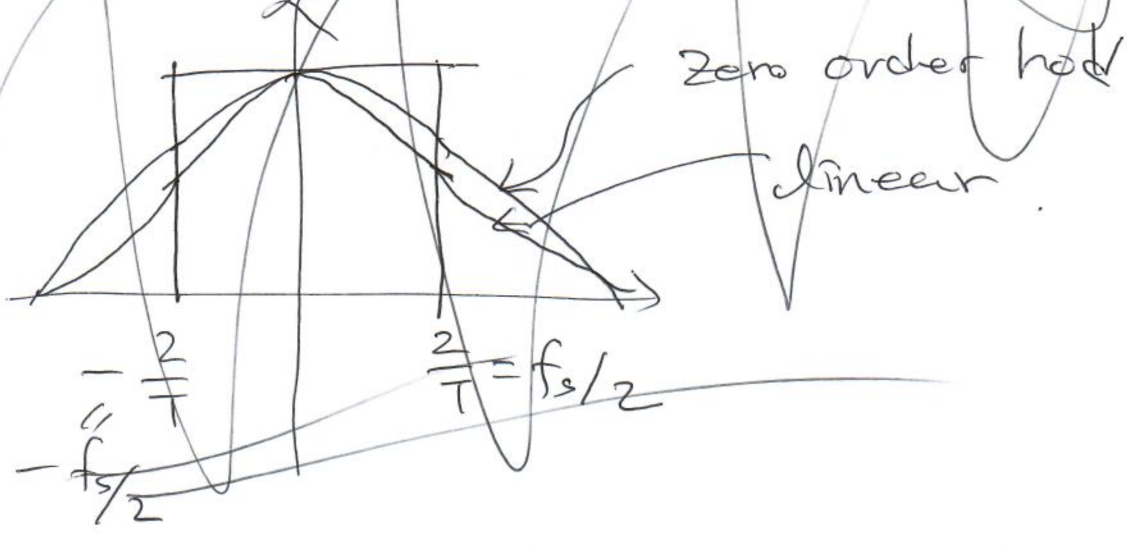
$\Lambda\left(\frac{t}{T}\right) = \frac{1}{T}\text{rect}\left(\frac{t}{T}\right) * \text{rect}\left(\frac{t}{T}\right)$

$H_r(f)$? $T\text{sinc}^2(Tf)$



° Reconstruction with ideal LPF, ZOH, linear interpolators ~~interpolators~~

Let's compare the three ~~reconstruction~~.



Show Figs 7.12 & 7.13 & 6.2.

+ Distortion correction

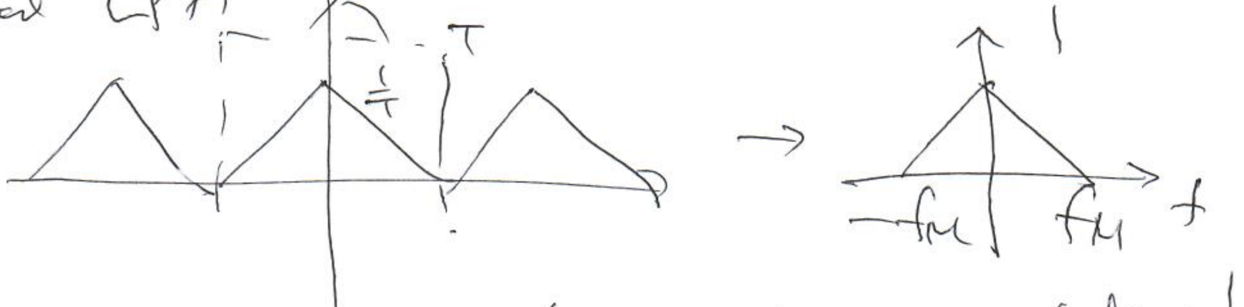
For ZOH

Looks like ZOH is better! Really?

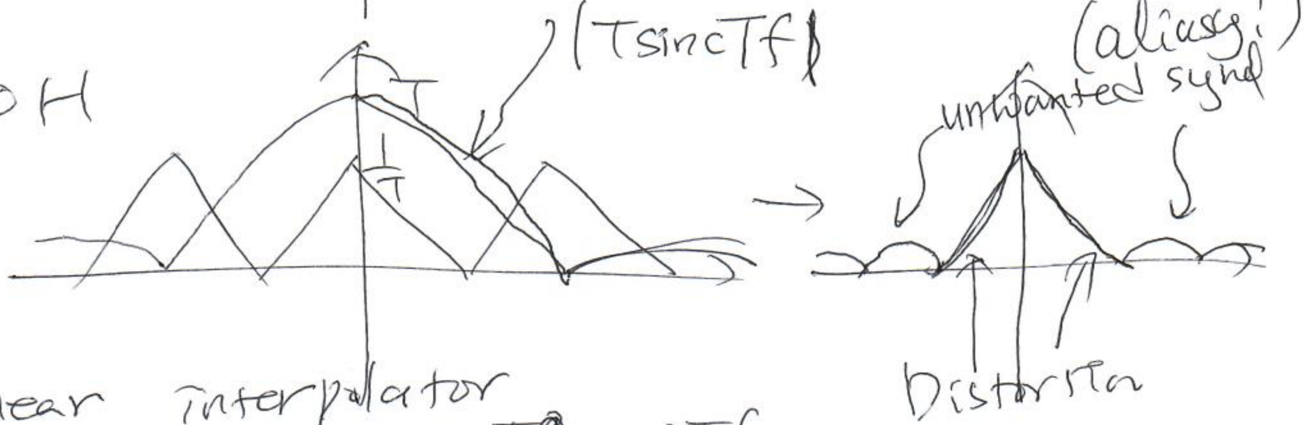
① Sampling w $\frac{1}{T}Ш(\frac{t}{T})$

② ~~Recon~~ ~~to~~ Sampling @ Nyquist rate.

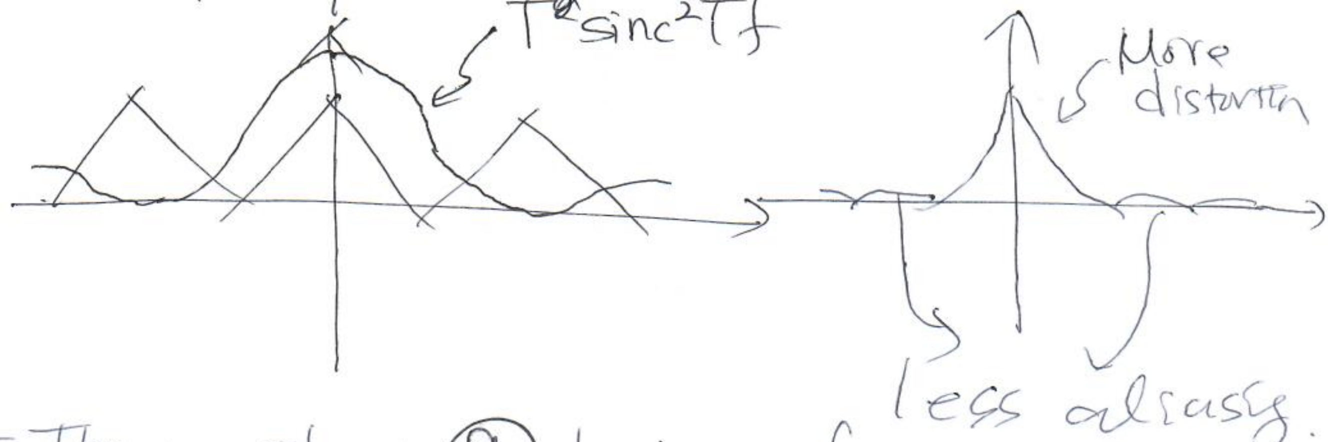
Ideal LPF



ZOH



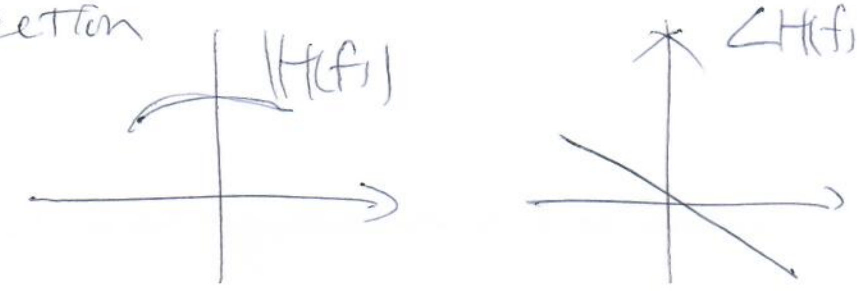
Linear interpolator



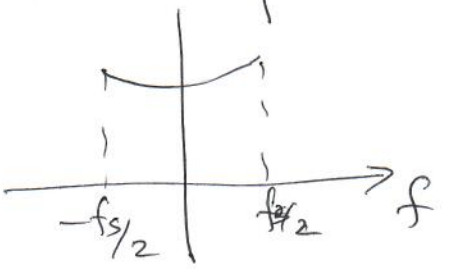
- Things chang @ higher fs.

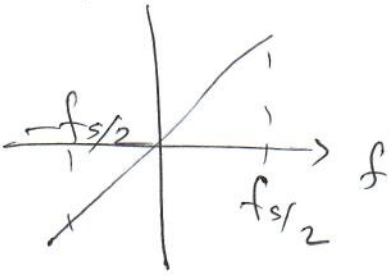
Distortion correction

For FOH



This can be compensated by





i.e $H_{comp}(f) = \dfrac{e^{j\pi Tf}}{\mathrm{sinc}(Tf)}$

For linear interpolaton

$$H_{comp}(f) = \frac{1}{\mathrm{sinc}^2(Tf)}$$

☆ Note ZOH is what happens in ADC.

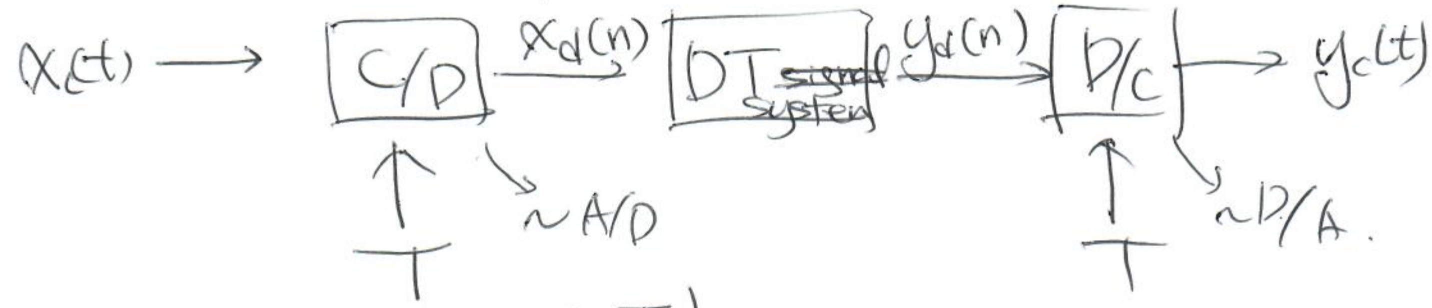
ADC ┬ sampling
└ quantization → explain this again

8Bit Quantization.

Dynamic Range.

often ADC works as "sampling with a zero order hold.

7.4 Discrete-time processing of CT-signals



$$x_d(n) = x_c(nT)$$

$$y_d(n) = y_c(nT)$$

- C/D ~~Cover~~ Conversion in time & freq. domain

$$x_p(t) = x_c(t)\cdot\frac{1}{T}\text{Ш}\left(\frac{t}{T}\right)$$

$$X_p(f) = X_c(f) * \text{Ш}(Tf)$$

or alternatively

$$x_p(t) = x_c(t)\cdot\sum_{n=-\infty}^{\infty}\delta(t-nT)$$

$$= \sum_{n=-\infty}^{\infty} x_c(nT)\,\delta(t-nT)$$

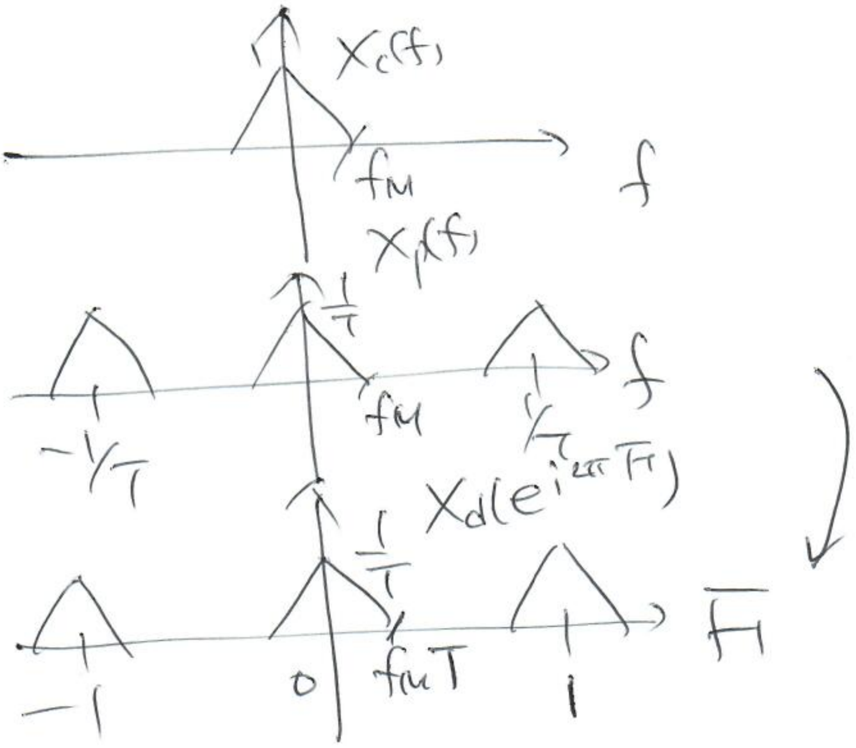
$$X_p(f) = \mathcal{F}\{x_p(t)\} = \underbrace{\sum_{n=-\infty}^{\infty} x_c(nT)\,e^{-j2\pi fnT}}_{\text{looks pretty familiar!}}$$

$$X_d(e^{j2\pi F}) = \sum_{n=-\infty}^{\infty} x_d(n)\,e^{-j2\pi Fn}$$

Compare ~~th~~ these $\underbrace{X_d(e^{j2\pi F})}_{DT} = \underbrace{X_p\left(\frac{F}{T}\right)}_{CT}$

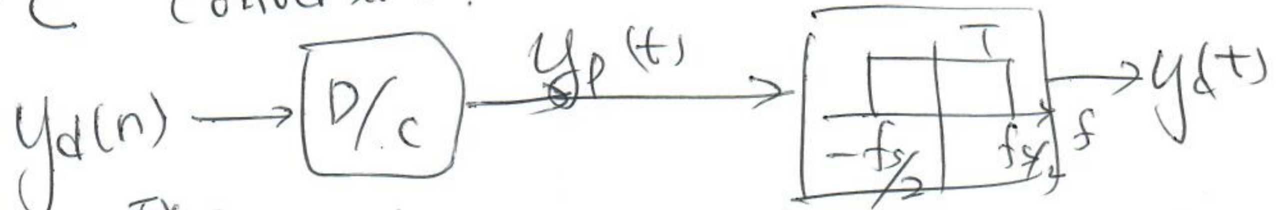
$$= X_c\left(\frac{F}{T}\right) * \text{Ш}(F)$$

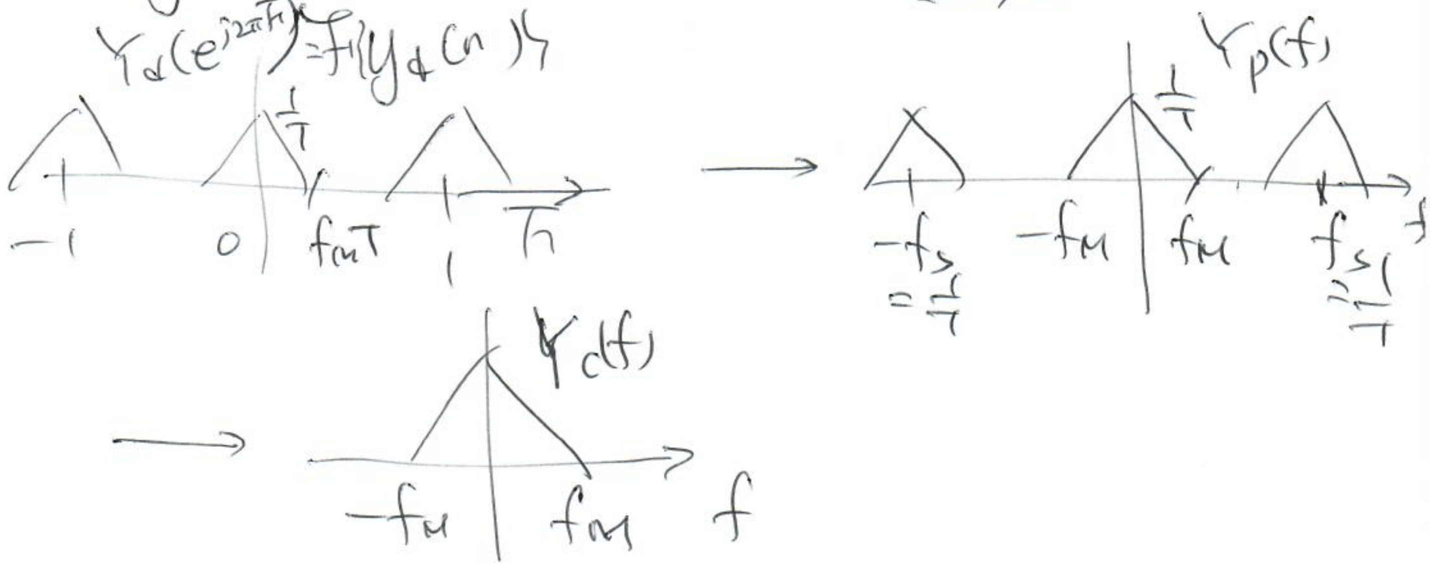
$$= \frac{1}{T}\sum_{n=-\infty}^{\infty} X_c\left(\frac{F-n}{T}\right)$$



Scaled in f axis
$f = F_1/T$

- D/C conversion.





Show Fig 7.24 & 7.25.

If Input is BAND-Limited & Sampling meets Nyquist rate. Fig 7.24 is equivalent to CT LTI system

$$H_c(f) = \begin{cases} H_d(e^{j2\pi fT}) & |f| < f_s/2 \\ 0 & |f| > f_s/2 \end{cases}$$

Chapter
✩ 7.4.1 & 7.4.2 Ex 7.2 & Ex 7.3

7.5 Sampling of DT signal

7.5.1 Impulse-Train sampling.

$$x_p(n) = \begin{cases} x(n) & \text{if } n = \text{integer multiple of } N \\ 0 & \text{otherwise.} \end{cases}$$

$$x_p(n) = x(n)p(n) = \sum_{k=-\infty}^{\infty} x(kN)\,\delta(n-kN)$$

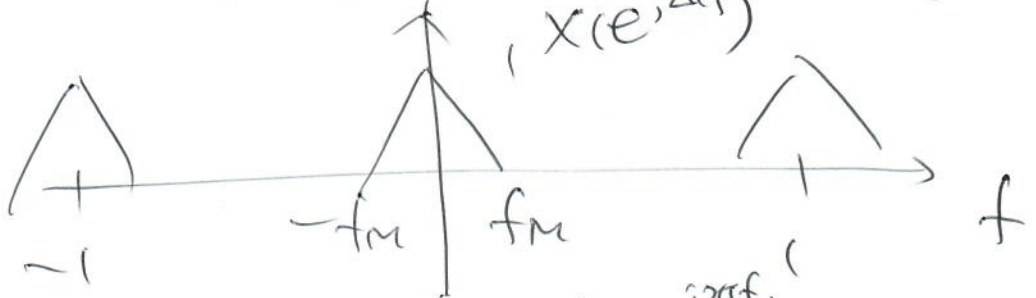
In freq domain,

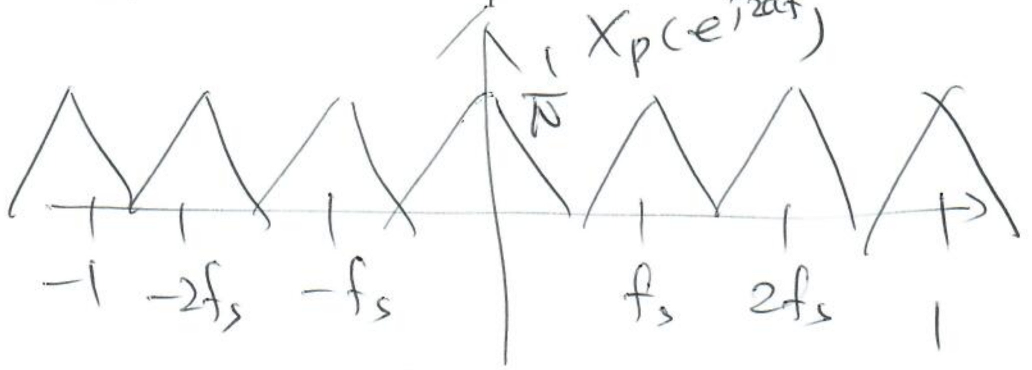
$$X_p(e^{j2\pi f}) = \int_1 P(e^{j2\pi F})X(e^{j2\pi(f-F)})df$$

$$P(e^{j2\pi f}) = \frac{1}{N}\sum_{k=-\infty}^{\infty}\delta(f-kf_s) \qquad f_s \overset{\Delta}{=} 1/N$$

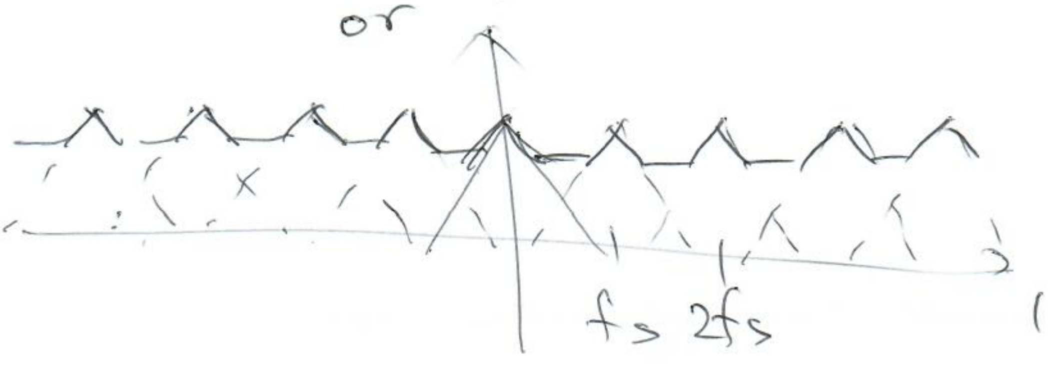
$$\therefore X_p(e^{j2\pi f}) = \frac{1}{N}\sum_{k=0}^{N-1} X(e^{j2\pi(f-kf_s)})$$

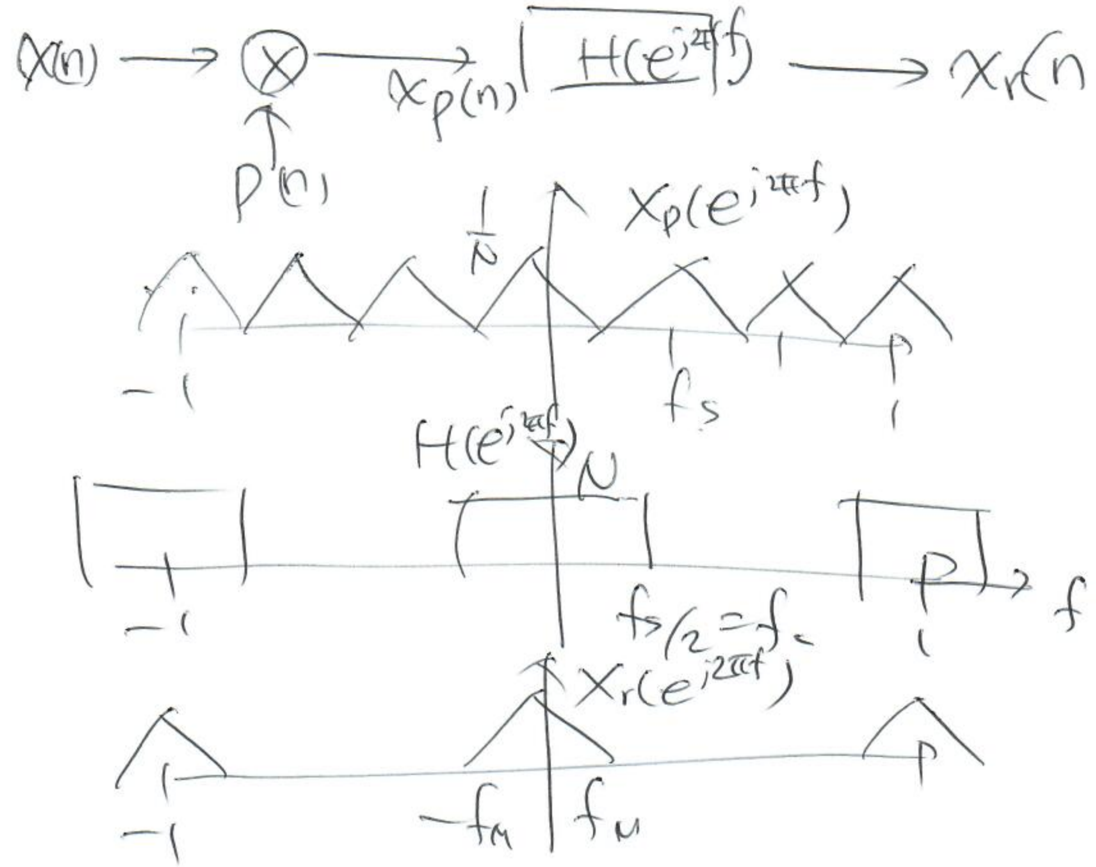
If $f_s > 2f_M$ → no aliasing





or





In time domain

$$H(e^{j2\pi f}) = N\,\mathrm{rect}\left(\frac{f}{f_s}\right) = N\,\mathrm{rect}\left(\frac{f}{2f_c}\right)$$

$$h(n) = 2Nf_c\,\mathrm{sinc}(2f_c n)$$

$$x_r(n) = x_p(n) * h(n)$$

$$x_r(n) = \sum_{k=-\infty}^{\infty} x(kN)\,2Nf_c\,\mathrm{sinc}(2f_c(n-kN))$$

7.5.2. Decimation & Interpolation

Decimation : $\underset{\sim}{x_b}(n) = x(nN)$

Ex) Data suppression

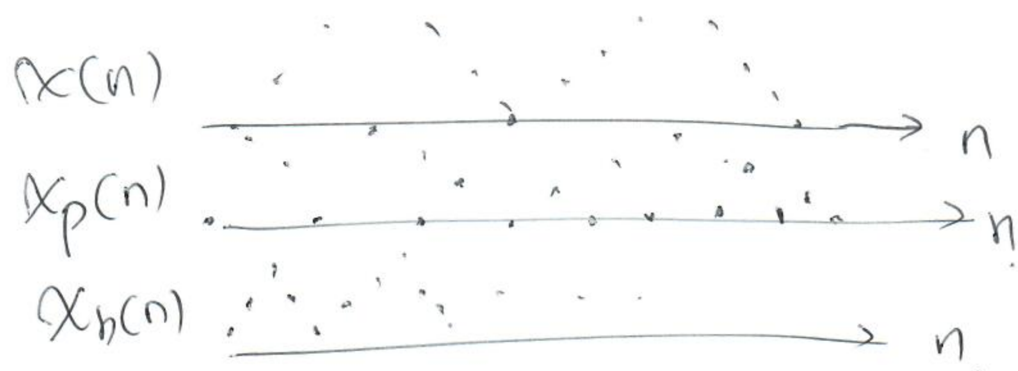
$$X_b(e^{j2\pi f}) = \sum_{k=-\infty}^{\infty} x_b(k)e^{-j2\pi fk}$$

$$= \sum_{k=-\infty}^{\infty} x_p(\underset{n}{kN})\, e^{-j2\pi fk}$$

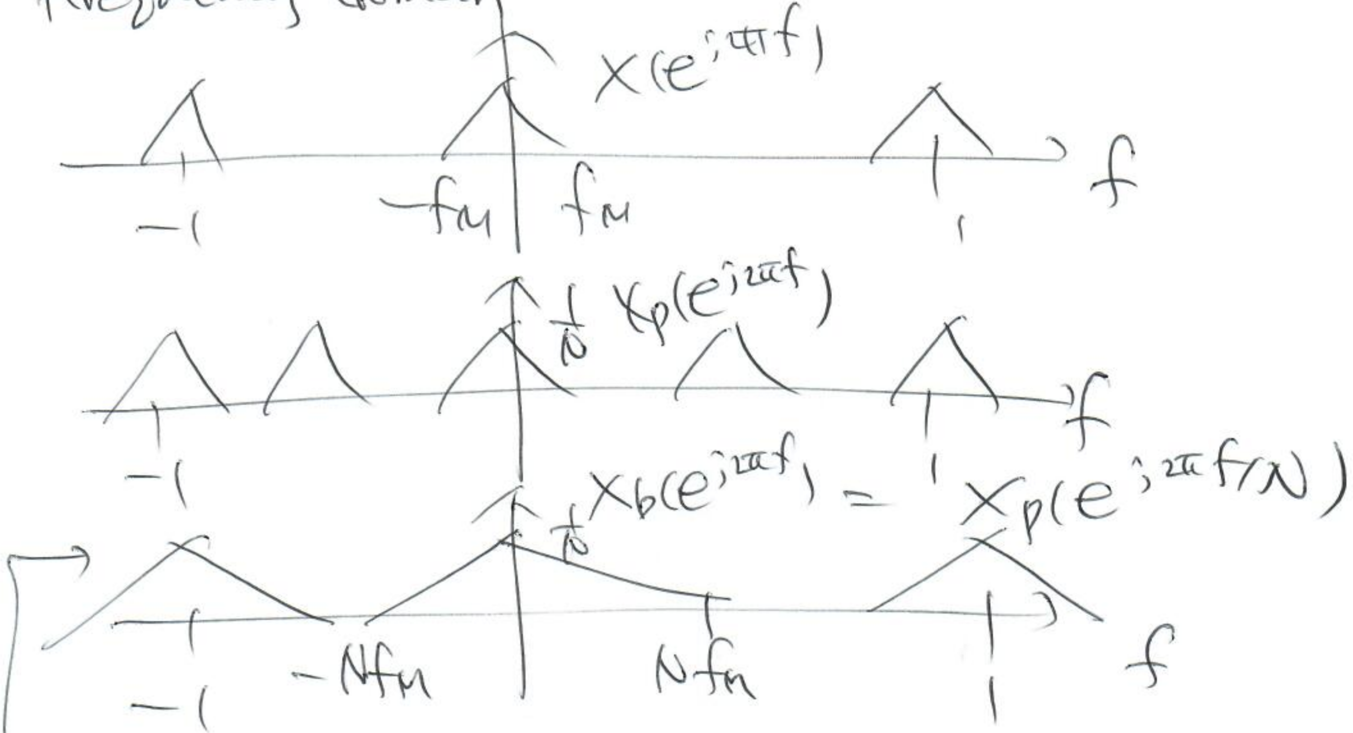
$$= \sum_{n=-\infty}^{\infty} x_p(n)\,e^{-j2\pi fn/N}$$

$$= X_p(e^{j2\pi f/N})$$

Time domain

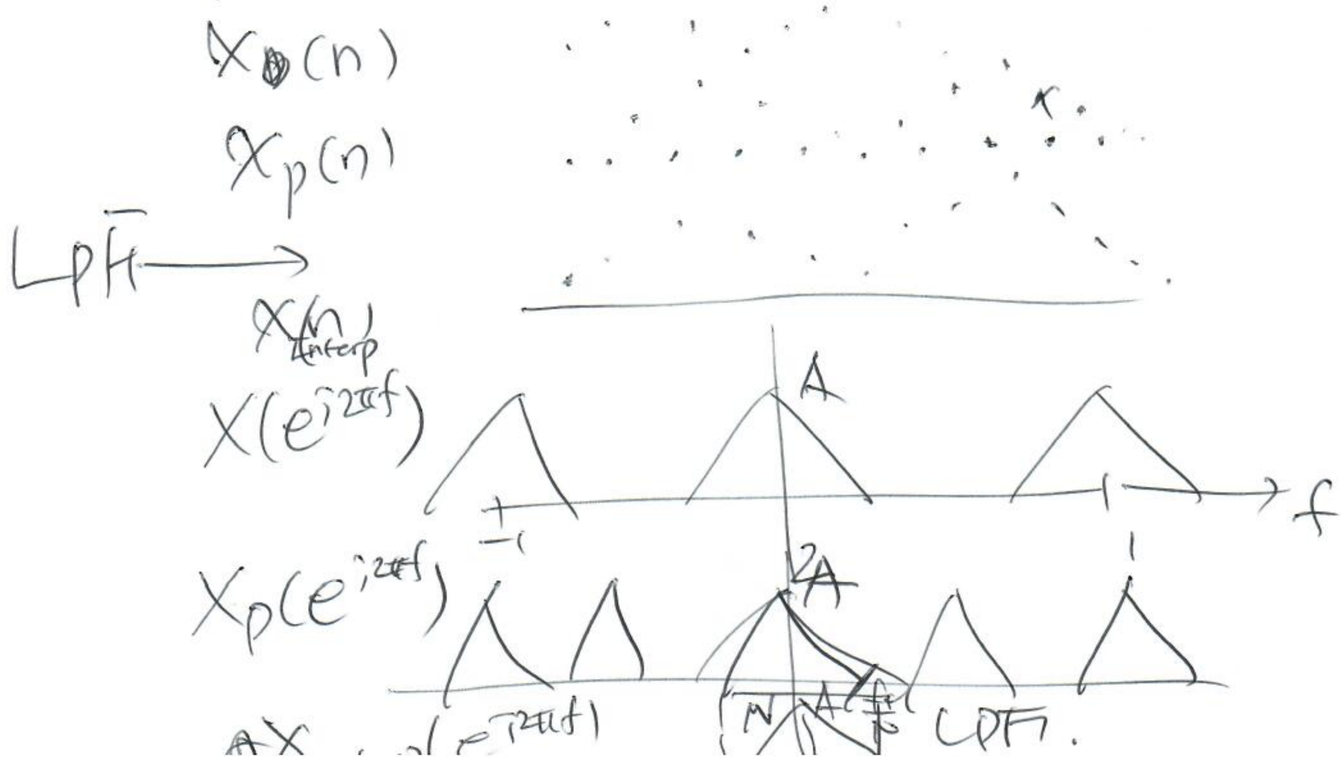


Frequency domain



Need redundancy. Decimation: Down sampling.

Interpolation (up sampling)



Chapter 6

Time & Frequency.

6.1. Mag & phase in FT

$$X(f) = \underbrace{|X(f)|}_{\text{mag}} e^{j \underbrace{\angle X(f)}_{\text{phase}}}$$

- magnitude mag

$|X(f)|^2$: energy density function.

$|X(f)|^2 \Delta f$: energy @ f to $f+\Delta f$.

- phase

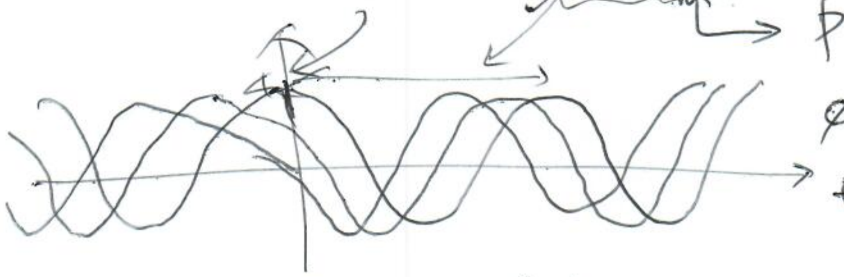
$\angle X(f)$??? why does it mean? does it matter.

Ex) ~~$4\cos(2\pi f_c t + \phi)$~~ $4e^{j(2\pi f_c t + \phi)}$

amplitude ~~frequency~~ phase.

$$Re\{4e^{j(2\pi f_c t + \phi)}\}$$

$$= 4\cos(2\pi f_c t + \underbrace{\phi}_{})$$ → phase offset.



ϕ determines ~~when~~ crosssection on y.

Show Figure 6.1 & Figure 6.2 for examples.

Ex) $\underbrace{x(-t)}_{\text{reverse play of audio}} \xleftrightarrow{F_1} X(f) e^{-j \cancel{2\pi} \underbrace{\angle X(f)}_{\text{phase reversed}}}$

6.2 Mag & Phase of frq. Resp. of LTI sys. ②

In LTI

$$Y(f) = H(f) X(f)$$

$$|Y(f)| e^{j\angle Y(f)} = |H(f)| |X(f)| e^{j\angle H(f)} e^{j\angle X(f)}$$

$$\therefore |Y(f)| = |H(f)| \cdot |X(f)| \longrightarrow \text{gain}$$

$$\angle Y(f) = \angle H(f) + \angle X(f) \longrightarrow \text{phase shift.}$$

both of them fn of f.

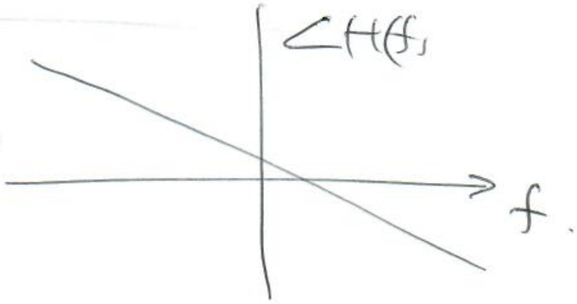
6.2.1. Linear & nonlinear phase

linear phase: $\angle H(f) = -2\pi fa$

↳ why minus here?

$$H(f) = |H(f)| e^{-j2\pi fa}$$

or



If $|H(f)| = 1$

$$Y(f) = H(f) X(f) = e^{-j2\pi fa} X(f)$$

$$\therefore y(t) = x(t-a) \ !$$

⇒ linear phase means time delay.

benign in most cases

Show Fig 6.3

Note: $|H(f)| = 1$ is called all pass filter.

still has significant effects on signals.

6.2.2. Group delay.

Approximate delay for asmall band of freq.

$$\angle H(f) \simeq -\phi - 2\pi f\alpha .$$

$$\cancel{\phi}\; Y(f) = X(f)\,|H(f)|\, \underset{\substack{\downarrow \\ \text{constant phase}}}{e^{-j\phi}}\; \underset{\substack{\downarrow \\ \text{linearphase}}}{e^{-j2\pi f\alpha}}$$

Group delay:

$$\tau(f) = \frac{1}{2\pi}\frac{d}{df}\{\angle H(f)\}$$

Ex) what does $\tau(f) = 5,\ 10,\ 15$ mean?
@100Hz @1kHz @10kHz ?

6.2.3. Log Mag / Bode plot

$$\log|Y(f)| = \log|H(f)| + \log|X(f)|$$

magitude are additive!

Amplitude: $20\log_{10}$ (dB)

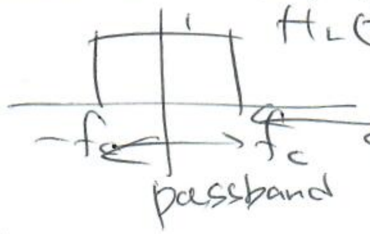
0dB : ×1
-20dB : ×1/10
20dB : ×10
6dB : ×2

power : $10\log 10$

3dB: ×2.

6.3. Time-domain properties of ideal freq.-selective filters.

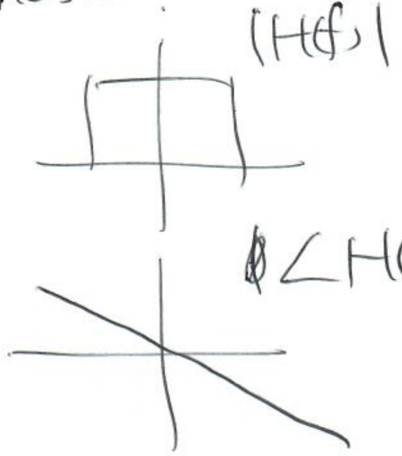
- Ideal LPF



$$H_L(f) = \begin{cases} 1 & |f| \le f_c \\ 0 & \text{otherwise} \end{cases}$$

- What about phase?

If linear



$\angle H(f) = -2\pi f \alpha$

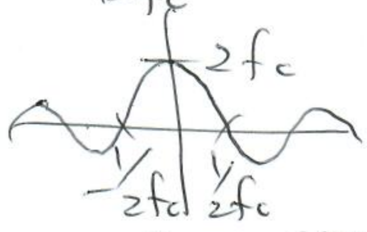
→ signal after filter will have time delay.

If nonlinear

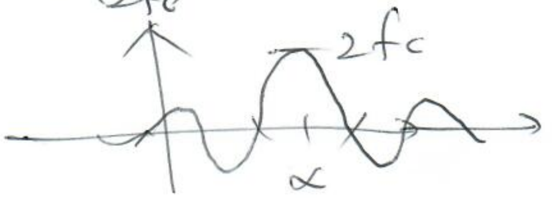
→ output will severely dependent on phase of filter.

- Impulse response of ideal filter.

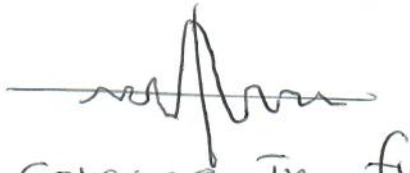
$\mathrm{rect}\left(\frac{f}{2f_c}\right) \rightarrow 2f_c \,\mathrm{sinc}(2f_c t)$



$\mathrm{rect}\left(\frac{f}{2f_c}\right) e^{-j2\pi f a} \rightarrow 2f_c \,\mathrm{sinc}(2f_c(t-a))$



f_c: large → "wide" coverage in frq

in time domain 

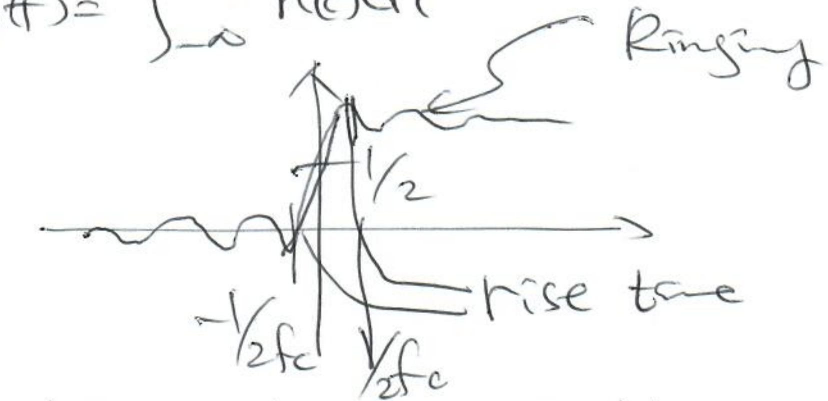
f_c: small → "narrow" coverage in frq

in time domain 

* Remember sinc has infinite duration
  → Ideal selective filter has infinite duration in time
  → Bad for implementation.

- Step response of ideal LP filter

$s(t) = \int_{-\infty}^{t} h(\tau) d\tau$



Q: Why this is a problem.

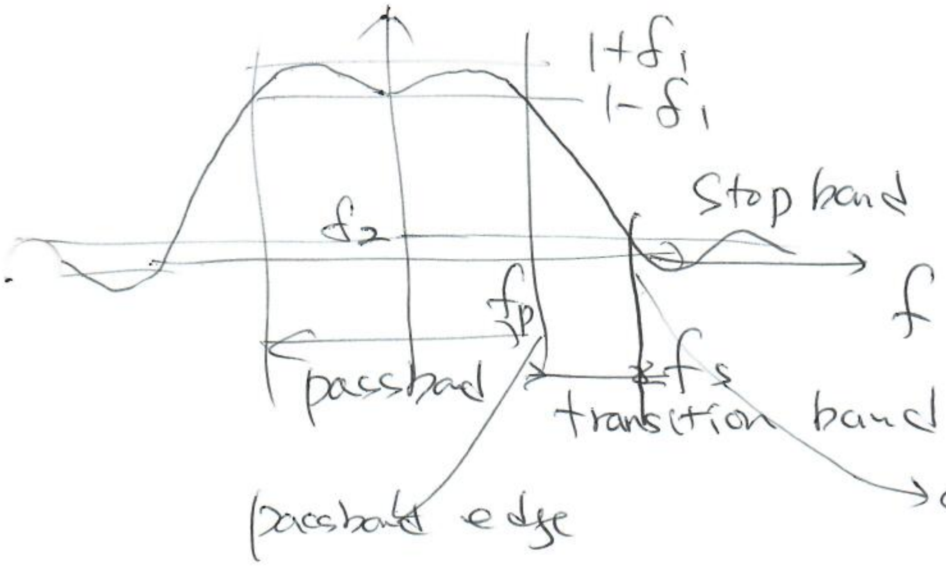
→ non causal

→ not applicable for real time processing

→ expensive to approximate.

6.4. Time/freq domain aspects of nonideal filter.

- Relax constraints in ideal filter to design practical filter

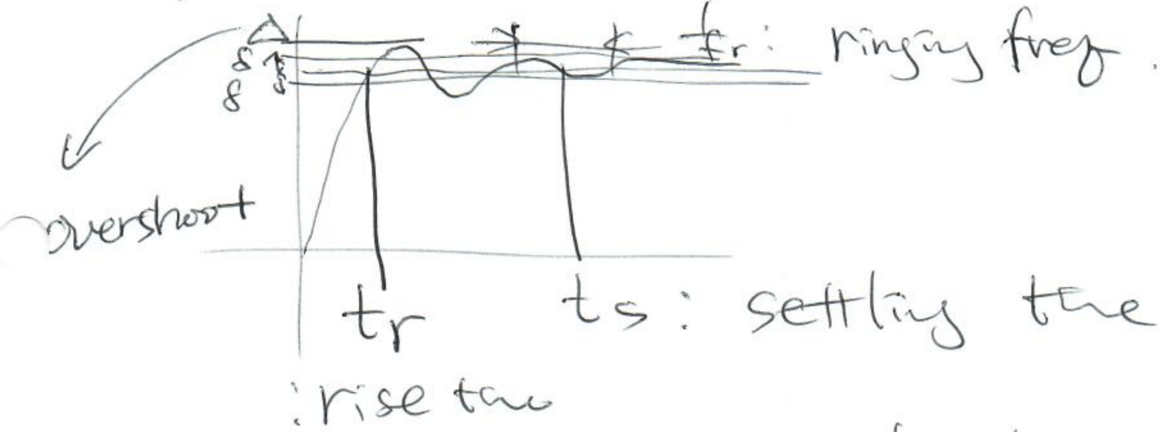


δ_1: passband ripple

δ_2: stopband ripple

- phase can be linear or nearly linear over passband

• Time domain behavior ~~cause~~



width of transition band $\propto \frac{1}{\text{settling time}}$ of step fn.

Chapter 9

Laplace Transform

9.1. Laplace Transform

- e^{st} : Complex exp.
  s : Complex number

In LTI system e^{st} : eigen function

$$e^{st} \longrightarrow \boxed{h(t)} \longrightarrow y(t)$$

$$y(t) = H(s)e^{st}$$

why:

$$\begin{aligned} y(t) &= h(t) * e^{st} \\ &= \int_{-\infty}^{\infty} h(\tau) e^{s(t-\tau)} d\tau \\ &= e^{st} \int_{-\infty}^{\infty} h(\tau) e^{-s\tau} d\tau \\ &= \underbrace{H(s)}_{\text{system function}} e^{st} \end{aligned}$$

- In CT-FT $s = j2\pi f$.
  (i.e. $y(t) = H(j2\pi f) e^{j2\pi f}$
  $= H(f) e^{j2\pi f}$)

  But not all signals have FT.
  (i.e. convergence issue)

- Laplace transform is a generalization of CT-FT by allowing $s = \underset{\text{non-zero}}{\underbrace{\sigma}} + j2\pi f$

LT : $$X(s) \triangleq \int_{-\infty}^{\infty} x(t) e^{-st} dt$$

So CT-FT is a special case of LT

when $s = j2\pi f$

i.e $X(s)\Big|_{s=j2\pi f} = \mathcal{F}_t\{x(t)\}$.

In otherwords, LT can be viewed as

$$X(\sigma + j2\pi f) = \int_{-\infty}^{\infty} (x(t)e^{-\sigma t})e^{-j2\pi f t}dt$$

i.e. FT of $x(t)\underbrace{e^{-\sigma t}}_{\text{decaying or growing}}$

you can see it covers more signals than FT.

Ex 9.1 $a<0$ LT exist but no FT

Ex 9.2. LT the same as 9.1 except ROC.

- ROC : Range of values of s LT converges
  Region of convergence

  Ex 9.1 & 9.2 have different ROC

  Show Figure 9.1

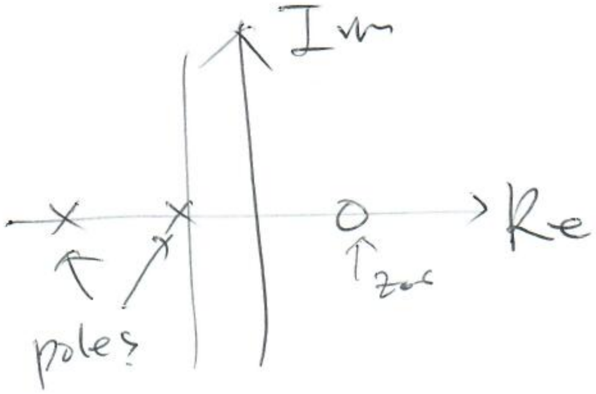
  Ex 9.3.

- If LT is a rational fn

  i.e. $X(s) = \frac{N(s)}{D(s)}$

  (e.g. LCCDE)

  a $\underbrace{\text{pole}}_{\text{Root of } D(s)}$ - $\underbrace{\text{zero}}_{\text{root of } N(s)}$ plot can be plotted.



- A rational LT is completely specified to within a scaling factor, by pole-zero plot + ROC
- If order of the rational LT function is not even, we assume poles/zeros @ infinity
- If ROC includes $s=j2\pi f$. then FT exists.

9.2. Region of Convergence

LT: need $X(s)$ + ROC

property1: ROC consists of strips parallel to $j2\pi f$-axis in s-plane

∵ In LT, convergence is $x(t)e^{-\sigma t}$

i.e. $\int_{-\infty}^{\infty}|x(t)|e^{-\sigma t}dt<\infty$

property2: For rational LT, ROC excludes poles

∵ poles make LT → ∞. (i.e. not converge

Property 3: $x(t)$ is of finite duration & absolutely integrable → ROC is entire s-plane. ④

∵ If $\int_{T_1}^{T_2} |x(t)| dt < \infty$ then $\int_{T_1}^{T_2} |x(t)| e^{-\sigma t} dt < \infty$.

Property 4&5: If $x(t)$ is right sided (or left-sided) & if $Re\{s\} = \sigma_0$ is in ROC then all values of s for $Re\{s\} > \sigma_0$ (or $Re\{s\} < \sigma_0$) will be in ROC

→ $x(t)$: right-sided → ROC is right-half plane

$x(t)$: left-sided → ROC is left-half plane

Property 6: $x(t)$ is two sided & $Re\{s\} = \sigma_0$ is in ROC → ROC will consist of a strip that includes $Re\{s\} = \sigma_0$

Solve Ex 9.7.

Property 7: $X(s)$ is rational → ROC is banded by poles or extends to ∞

Property 8: $X(s)$ is rational & $x(t)$ is right-sided (or left sided) → ROC is s-plane to right (or left) of the rightmost (leftmost) pole.

Solve Ex 9.8.

Laplace transform

2nd part missing.

Covered
① 9.3 Inverse Laplace Transform
② 9.4 Geometry evalution
③ 9.5 property (only Initial & final Value theorem)
④ 9.7. LT & LTI Causality
Stability

Chapter 10

z - transform

10.1 z-transform.

- generalization of DT-FT
- In LTI system

$y(n) = H(z) z^n$ where $H(z) = \sum_{n=-\infty}^{\infty} h(n) z^{-n}$

If $\underbrace{z = e^{+j2\pi f}}_{\text{unit circle}}$ then $H(z)$ becomes DT-FT

In general z is an arbitrary complex value $(re^{j2\pi f})$

so ~~[illegible]~~ covers wider range than DT-FT
z-transform

$$X(re^{j2\pi f}) = \sum_{n=-\infty}^{\infty} (x(n) r^{-n}) e^{-j2\pi f n}$$

$$= \mathcal{F}\{x(n) r^{-n}\}$$

- Convergence: $x(n) r^{-n}$ in DT-FT

i.e. depend not only $x(n)$
but also r

Example 1 & 2

10, 2 ROC in z-transform

property 1 ROC consits of a ring
centered about the orgin.
→ $x(n)r^{-n}$ is absolutely summable

property 2 ~~ROC~~ No pole @ ROC

property 3. $x(n)$ finite duration → ROC : entire
z plane except $z=0$ or ∞
∵ $X(z)=\sum_{n=N_1}^{N_2} x(n)z^{-n}$ Converge.
z^{-m} can exist $z=0$ ∞,

property 4. $x(n)$ right sidede & $|z|=r_0$ in ROC
→ $|z|>r_0$ in ROC
Show Figure 10.7.

property 5
$x(n)$ left sided & $|z|=r_0$ in ROC
→ $0<|z|<r_0$ in ROC.

Property 6 $x(n)$ two-sided & $|z|=r_0$ in ROC
→ ROC is a ring Including $|z|=r_0$
→ Ex 10.7

property 7 $X(z)$ rational, ROC is bounded
by poles or extends to infinity

property 8 $X(z)$ rational & right-sided
→ ROC : outside outermost pole
& If causal, ROC includes $z=\infty$

Property 9 $X(z)$ rational & left-sided.
→ ROC : inside innermost pole

10.3 Inverse z-transform

$$x(n) = \frac{1}{2\pi j} \underset{\sim}{\oint} X(z) z^{n-1} dz$$

Counter clockwise closed circular contour.

Why?

$$X(re^{j2\pi f}) = \mathcal{F}\{x(n) r^{-n}\}$$

$$x(n) r^{-n} = \mathcal{F}^{-1}\{X(re^{j2\pi f})\}$$

$$x(n) = r^n \mathcal{F}^{-1}\{X(re^{j2\pi f})\}$$

$$= r^n \int X(re^{j2\pi f}) e^{j2\pi fn} df$$

$$= \int X(re^{j2\pi f}) (re^{j2\pi f})^n df$$

$$z = re^{j2\pi f}, \quad dz = j2\pi r e^{j2\pi f} df$$

$$\therefore x(n) = \frac{1}{j2\pi} \oint X(z) z^{n-1} dz$$

f : $-\frac{1}{2}$ to $\frac{1}{2}$ or 0 to 1

z : circle.

Still difficult to evaluate.

For rational z-transform

$$\left(\text{e.x} \right) \sum_{i=1}^{M} \frac{A_i}{1 - a_i z^{-1}} \quad \text{or so} \Big)$$

Use partial-fraction expansion solutions.

e.g. $A_i a_i^n u(n)$ if ROC outside of outermost pole.

$-A_i a_i^n u(-n-1)$ if ROC inside of innermost pole.

Ex 10.9 Z10, 11

10.4. Geometric evaluation

④

FT : use pole-zero plot and evaluate on the contour $|z| = 1$

Ex) $h(n) = a^n u(n)$

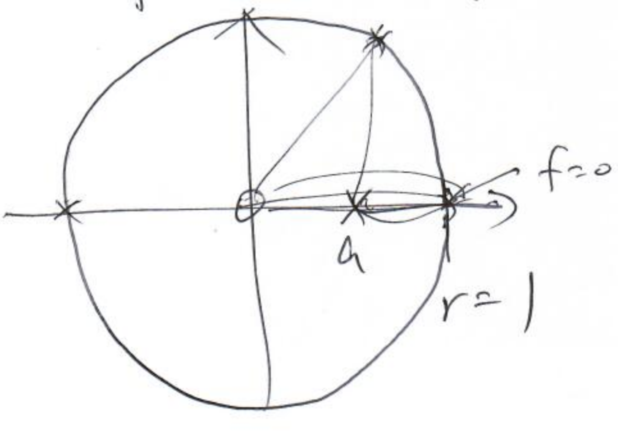
$$H(z) = \frac{1}{1 - az^{-1}} \qquad |z| > |a|$$

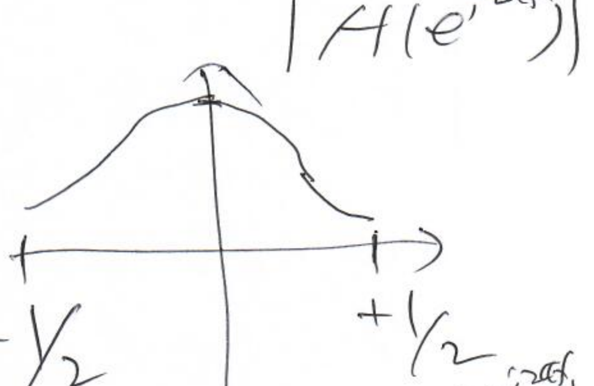
if $|a| < 1$, ROC includes $|z| = 1$

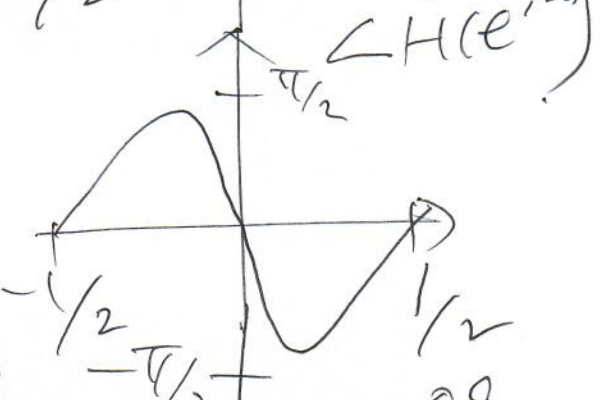
Then FT exists

$$H(e^{j2\pi f}) = \frac{1}{1 - ae^{-j2\pi f}}$$

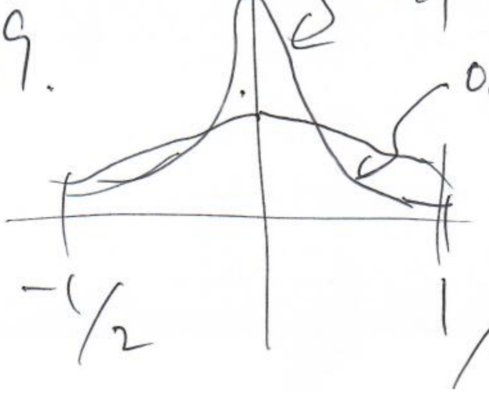
From pole-zero plot

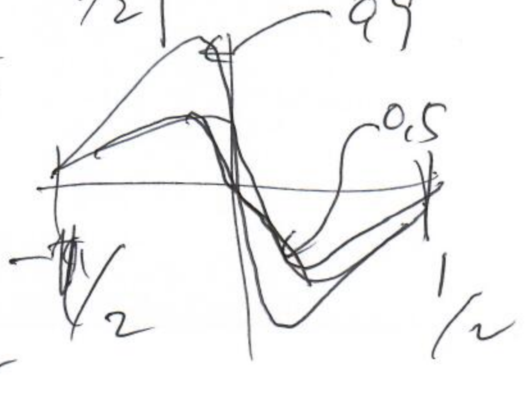






$a = 0.5$
vs
$a = 0.9.$





10,5 properties

- linearity   ROC: $R_1 \cap R_2$
- Time shifting $x(n-n_0) \longleftrightarrow z^{-n_0}X(z)$

  ROC: R except addition or deletion
  @ infinity or zeros
- Scaling in z

  $z_0^n \cancel{x}\, x(n) \longleftrightarrow X\left(\frac{z}{z_0}\right)$ ROC: $|z_0|R$
- Time reversal

  $x(-n) \Longleftrightarrow X\left(\frac{1}{z}\right)$ ROC: $\frac{1}{R}$
- Time expansion

  $$x_{(k)}(n) = \begin{cases} x(n/k) & \text{if } n \text{ is multiples of } k \\ 0 & \text{otherwise} \end{cases}$$

  $x_{(k)}(n) \Longleftrightarrow X(z^k)$ ROC $= R^{1/k}$
- Conjugate

  $x^*(n) \longleftrightarrow X^*(z^*)$
- Convolution

  $x_1(n) * x_2(n) \longleftrightarrow X_1(z)X_2(z)$ $R_1 \cap R_2$
- differentiation in z

  $nx(n) \longleftrightarrow -z\frac{dX(z)}{dz}$
- Initial Value theorem

  If $x(n)=0$ for $n<0$

  $$x(0) = \lim_{z \to \infty} X(z)$$

10.7 LTI systems & ~~using~~ z-transform

In DT LTI,

$$Y(z) = \underline{H(z)}X(z)$$

system function or
transfer function

becomes "frequency response" if evaluated
for $z = e^{j2\pi f}$

- Causality

  $h(n) = 0$ for $n < 0$ ⟺ right sided
  ⟺ ROC exterior of a circle includg ∞

- Stability

  - An LTI system is stable iff ROC of $H(z)$ includes $|z| = 1$
  - A causal LTI with rational ~~system~~ fn is stable iff all poles of $H(z)$ lies inside the unit circle.

- LTI w LCCDE

Ex 10.25 "delay"

Ex 10.26 & 29 important.