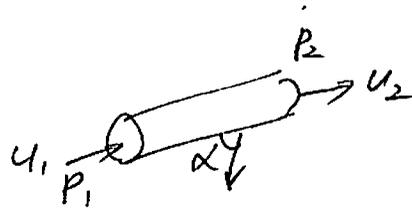
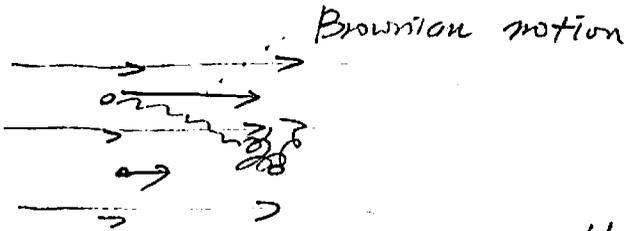


Laminar flow in pipe

previous in pipe, we use friction loss concept.

② we briefly discussed about the concept of viscosity

i.e. momentum transport across the streamline.

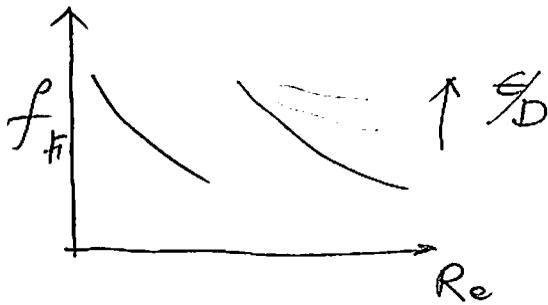


Wall shear stress

① f (friction loss) = $2u^2 f_{ff} (Re, \frac{\epsilon}{D}) \frac{L}{D} = \frac{\tau_w}{\frac{1}{2} \rho u^2}$

Compare E.B & M.B

f_{ff} (Fanning friction factor) = $\begin{cases} \frac{16}{Re} & \text{(laminar)} \\ 0.079 Re^{-0.25} & \text{(Turbulent)} \end{cases}$



$Re = \frac{\rho u D}{\mu}$

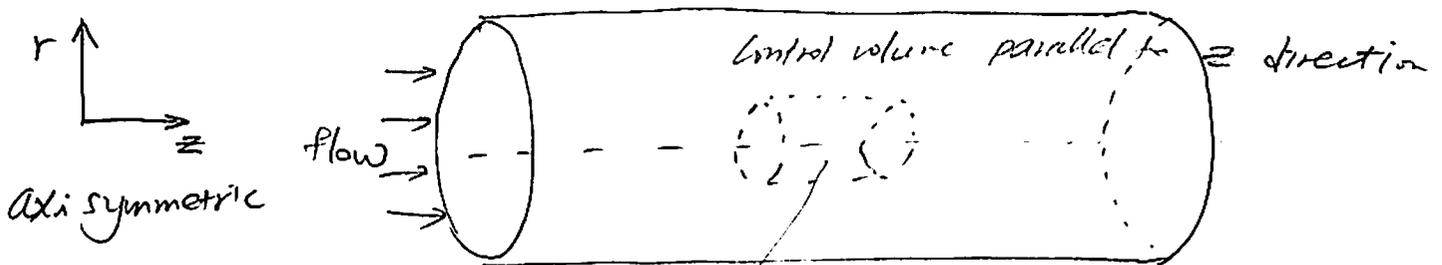
loss → from E.B
wall shear stress → from M.B } → Compare

So for τ_w is mysterious variable.

In this lecture, we will focus on

Lammar pipe flow and derive

The situation can be described as

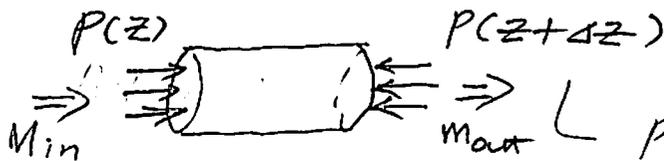


let's neglect gravity for simplicity.

so far our control volume was the whole pipe.

In this time, we will consider fixed small control volume inside pipe.

< Momentum balance for "small" CV >

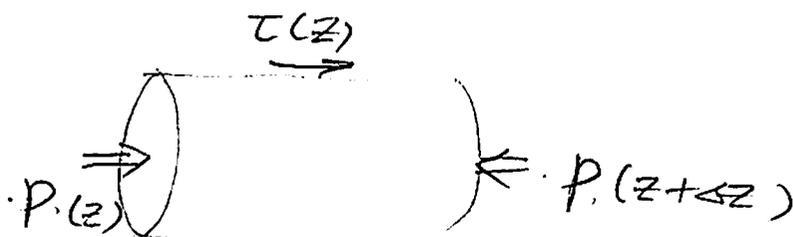


pressure is "isotropic"

which means it can exert the same amount of force in every directions.



→ pressure force always acting on normal to a given surface



Force balance in z -direction

Force = Area \times Pressure or Area \times stress

so

$$\underbrace{\pi r^2 P(z)}_{\text{inlet}} - \underbrace{\pi r^2 P(z+\Delta z)}_{\text{outlet}} + \underbrace{2\pi r \int_z^{z+\Delta z} \tau(z') dz'}_{\text{side}}$$

(Note that τ is fn of z)

$$+ \text{momentum in} - \text{momentum out} = 0 \dots \textcircled{1}$$

$$\left[\begin{array}{l} \rightarrow (\pi r^2 \rho U_z(z)) \quad \quad \quad \pi r^2 \rho U_z(z+\Delta z) \\ \downarrow \\ \text{you are attempt to write in this way,} \end{array} \right]$$

but at the end of the class $u = u(r, z)$
you will realize this is not true (velocity distribution)

① from mean-value

$$2\pi r \int_z^{z+\Delta z} \tau(z') dz' = 2\pi r \tau(z^*)$$

$$z < z^* < z + \Delta z$$

② Here we only consider "fully-developed" flow

U_z (velocity in z -direction) \neq fn (z).
does not vary along z -dir

\rightarrow the side of surface of C.V. is 
a stream tube (collection of streamlines)

Let's simplify the equation ①

$$P(z) - P(z + \Delta z) + \frac{2}{r} \tau(z^*) \Delta z = 0$$

(Note that momentum in = momentum out
cancelled out due to
fully developed condition)

$$\rightarrow \frac{P(z + \Delta z) - P(z)}{\Delta z} - \frac{2}{r} \tau(z^*) = 0$$

Let's shrink $\Delta z \rightarrow 0$

$$\lim_{\Delta z \rightarrow 0} \left[\frac{P(z + \Delta z) - P(z)}{\Delta z} - \frac{2}{r} \tau(z^*) \right] = 0$$

$$\rightarrow \frac{dP}{dz} - \frac{2}{r} \tau(z^*) = 0$$

$$z < z^* < z + \Delta z$$

$$\text{as } \Delta z \rightarrow 0 \quad \underline{z = z^*}$$

$$\rightarrow \tau(z) = \frac{r}{2} \frac{dP}{dz} \quad \dots \quad \textcircled{2}$$

Note $\tau < 0$ (negative)

because P decrease along the pipe.

In other words, shear stress is balanced
by pressure gradient in
fully developed flow.

← It is from force balance.

So how can we get τ ?

We can imagine that some kind of "motion" of fluid will invoke the τ shear stress.

That requires a fundamental law that is specific to a material (fluid).

The constitute equation is a description of this fundamental law that "connect" motion of fluid to stress.

e.g.) Ideal gas law $PV = nRT$

Newtonian fluid is a simplest fluid that follow the constitute eqn below.

$$\tau = \mu \left(\frac{du}{dr} \right)$$

shear stress

"velocity gradient"
rate of velocity change along radial direction.
motion (rate of strain)

We have two eqn

① Conservation eqn: $\tau(z) = \frac{r}{2} \frac{dp}{dz}$

② Constitutive eqn: $\tau(z) = \mu \frac{du}{dr}$

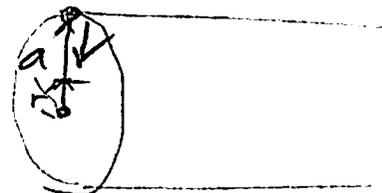
Combine ① & ②.

$$\mu \frac{du}{dr} = \frac{r}{2} \frac{dp}{dz}$$

$$\rightarrow \frac{du}{dr} = \frac{r}{2\mu} \frac{dp}{dz}$$

\rightarrow Integrate this eqn from a to r

$$\int_a^r \frac{du}{dr} dr = \frac{1}{2\mu} \frac{dp}{dz} \int_a^r r' dr' \dots \textcircled{3}$$



Note that $r = a$, $u = 0$

It is called no-slip condition.

" In old day people think that fluid slip at the wall but

In 19 century $\left\{ \begin{array}{l} Navier \& Poisson \\ Coulomb \& Stokes \end{array} \right.$

\rightarrow slip \checkmark

\rightarrow no slip

Proved by experiments
Poiseuille 1841
Whetham 1842

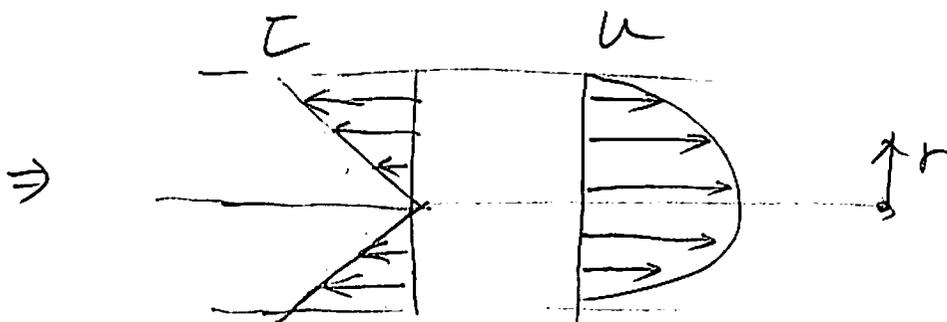
So Eq (3) becomes

$$\begin{aligned}
 u(r) - \underset{0}{\cancel{u(a)}} &= \frac{1}{2\mu} \frac{dp}{dz} \frac{1}{2} (r^2 - a^2) \\
 &= \frac{1}{4\mu} \left(\underset{\substack{\uparrow \\ \text{make positive}}}{-\frac{dp}{dz}} \right) (a^2 - r^2)
 \end{aligned}$$

$u(r)$ is not constant & parabola shape!

$$u(r) = \text{const} \times (a^2 - r^2)$$

$$\tau(r) = \ominus \text{const} \times r$$



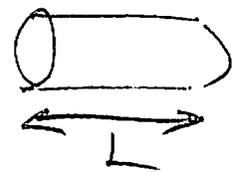
For fully developed flow

u does not depend on z

$\Rightarrow \frac{dP}{dz}$ is also independent of z !

Therefore $P(z)$ is linear and

$$\frac{dP}{dz} = \frac{P_2 - P_1}{L} = \frac{\Delta P}{L}$$



finite length

for example
pipe length

$$\therefore u = -\frac{\Delta P}{4\mu L} (a^2 - r^2)$$

The flow rate is

$$Q = \int_A u dA = \int_0^{2\pi} \int_0^a u(r) \frac{r dr d\theta}{dA}$$

$$= -\frac{2\pi \Delta P}{4\mu L} \int_0^a (a^2 - r^2) r dr$$

$$= -\frac{\pi \Delta P}{2\mu L} \left[\frac{a^2 r^2}{2} - \frac{r^4}{4} \right]_0^a$$

$$= -\frac{\pi}{8} \frac{\Delta P}{\mu L} a^4 = \frac{\pi a^4}{8\mu} \left(-\frac{\Delta P}{L} \right)$$

Hagen-Poiseuille law

Poiseuille showed that

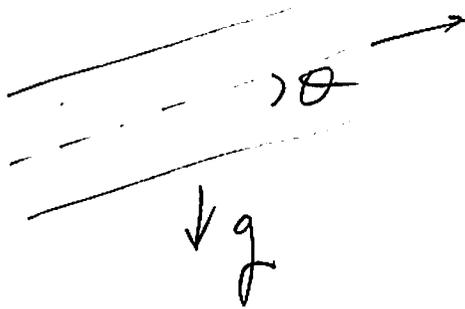
$$Q \propto \frac{\Delta P}{L} a^{\frac{4}{\uparrow}} \leftarrow \text{experimentally}$$

If fluid slip at wall $Q \propto \frac{\Delta P}{L} a^{\frac{3}{\downarrow}}$

If the pipe is inclined (include gravity),

we can apply the same analysis

but substitute $-\frac{dP}{dz} \rightarrow -\frac{dP}{dz} + \rho g \sin \theta$



θ is the angle btw
pipe and horizontal
direction.

Now we can compute friction loss for
laminar Newtonian fluid

from Bernoulli eqn

$$\Delta \left(\frac{u^2}{2} \right) + g \Delta z + \frac{\Delta P}{\rho} + u f + f_i = 0$$

"average"
velocity

friction loss

representative
velocity for
kinetic E

$$-\Delta P = \frac{8\mu Q L}{\pi a^4}$$

from Hagen-Poiseuille
flow

$$f = -\frac{\Delta P}{\rho} = \frac{8\mu Q L}{\pi a^4 \rho} = \frac{8\mu \cdot u_m L}{\rho a^2}$$

Here u_m (mean velocity) = $\frac{\text{flow rate}}{\text{area}}$

$$u_m = \frac{Q}{\pi a^2} = \frac{a^2}{8\mu} \left(-\frac{\Delta P}{L} \right)$$

$$\therefore f = \frac{16\mu}{\rho u_m (2a)} \cdot \frac{2u_m^2}{2} \cdot \frac{L}{2a} = \left(\frac{16}{Re} \right) \frac{1}{2} \frac{u^2 L}{2a}$$

Note $\boxed{2a = D}$

↳ Fanning friction factor