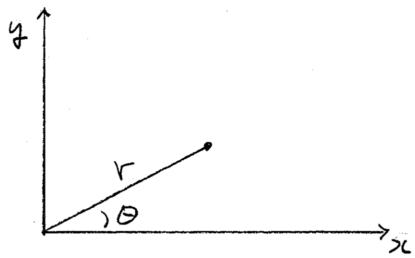
Polar coordinate

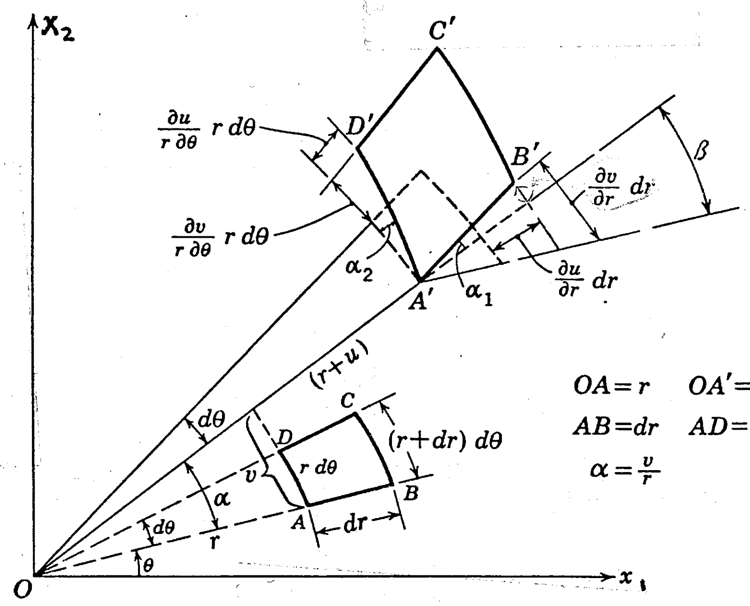


$x = r\cos\theta$

$y = r\sin\theta$

u = displacement in r direction

v = displacement in θ direction



Radial Normal strain

$$\varepsilon_r = \frac{A'B' - AB}{AB} \qquad A'B' = (1+\varepsilon_r)AB = (1+\varepsilon_r)dr$$

$$(A'B')^2 = (1+\varepsilon_r)^2 dr^2$$

$$= \left(dr + \frac{\partial u}{\partial r}dr\right)^2 + (\alpha_1\, dr)^2$$

Eliminating relatively small terms,

$$\varepsilon_r \simeq \frac{\partial u}{\partial r}$$

Hoop strain

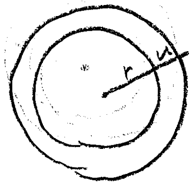
$$\varepsilon_\theta = \frac{A'D' - AD}{AD}$$

$$(A'D')^2 = (1+\varepsilon_\theta)^2 (r\,d\theta)^2$$

$$= \left[(r+u)d\theta + \frac{1}{r}\frac{\partial v}{\partial \theta} r\,d\theta \right]^2 + \left(\frac{1}{r}\frac{\partial u}{\partial \theta} r\,d\theta \right)^2$$

$$2\varepsilon_\theta + \varepsilon_\theta^2 = 2\left[\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} + \left(\frac{u}{r}\right)\left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right) \right] + \left(\frac{u}{r}\right)^2 + \left(\frac{1}{r}\frac{\partial v}{\partial \theta}\right)^2 + \left(\frac{1}{r}\frac{\partial u}{\partial \theta}\right)^2$$

$$\varepsilon_\theta = \left[\frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta} \right]$$



$$\varepsilon_\theta = \frac{2\pi u}{2\pi r} = \frac{u}{r}$$

Shear strain

$$\gamma_{r\theta} = (\alpha_1 + \alpha_2)$$

since α_1, α_2 are small, $\alpha_1 + \beta \approx \tan(\alpha_1+\beta) = \dfrac{\frac{\partial v}{\partial r}dr}{(1+\frac{\partial u}{\partial r})dr} \approx \dfrac{\partial v}{\partial r}$

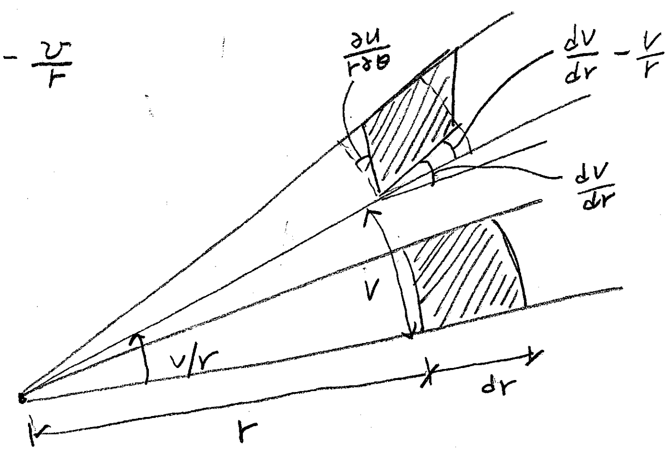
Thus $\alpha_1 = \dfrac{\partial v}{\partial r} - \beta = \dfrac{\partial v}{\partial r} - \dfrac{v}{r}$

$$\alpha_2 = \tan\alpha_2 = \frac{(\frac{1}{r}\frac{\partial u}{\partial \theta})\,r\,d\theta}{\left[(r+u)d\theta + \frac{1}{r}\frac{\partial v}{\partial \theta}\,r\,d\theta\right]}$$

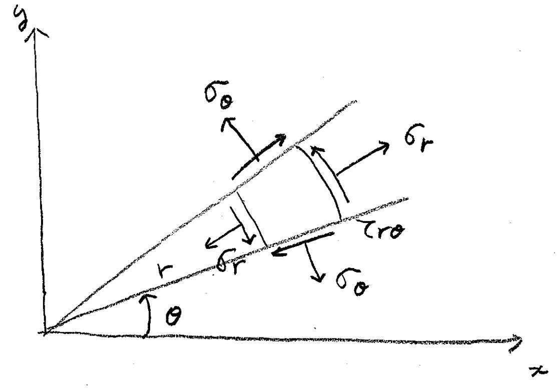
$$= \frac{(\frac{1}{r}\frac{\partial u}{\partial \theta})\,r\,d\theta}{(1 + \frac{u}{r} + \frac{1}{r}\frac{\partial v}{\partial \theta})\,r\,d\theta} \approx \frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\gamma_{r\theta} = \frac{\partial v}{\partial r} - \frac{v}{r} + \frac{1}{r}\frac{\partial u}{\partial \theta}$$

$$\begin{cases} \varepsilon_r = \dfrac{\partial u}{\partial r} \\[2ex] \varepsilon_\theta = \dfrac{\partial v}{r\,\partial \theta} + \dfrac{u}{r} \\[2ex] \gamma_{r\theta} = \dfrac{\partial u}{r\,\partial \theta} + \dfrac{\partial v}{\partial r} - \dfrac{v}{r} \end{cases}$$

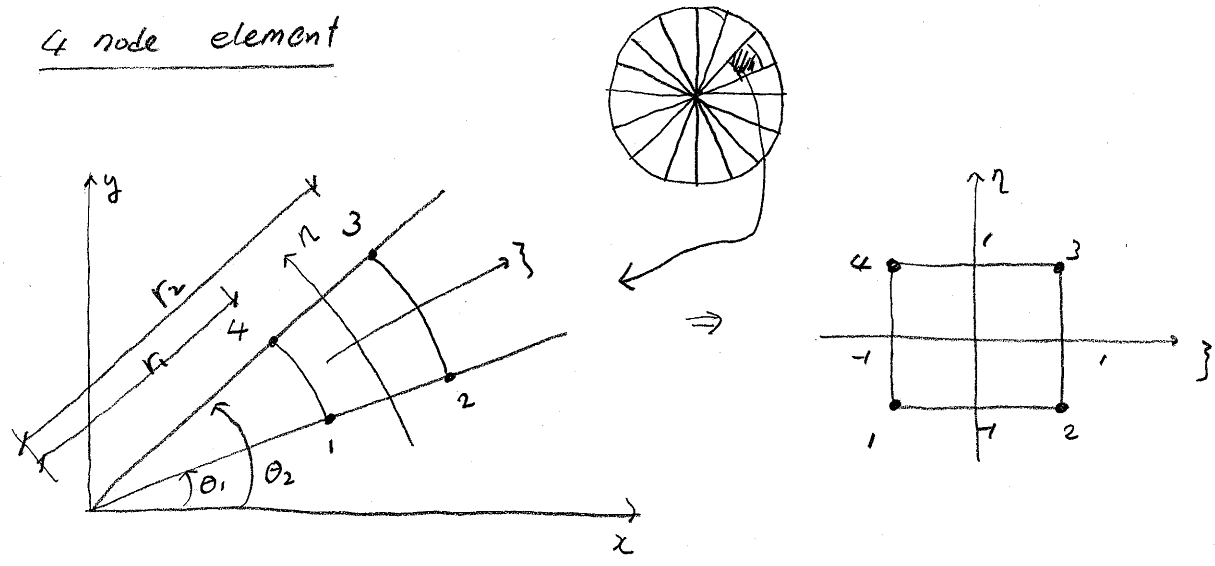


Stress - Strain relationship



$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_r \\ \sigma_\theta \\ \tau_{r\theta} \end{bmatrix} = \underset{\sim}{D}\,\underset{\sim}{\varepsilon} = \underset{\sim}{D} \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \end{bmatrix}$$

4 node element



$$u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4 = \Sigma f_i u_i$$

$$v = \Sigma v_i f_i$$

$$f_1 = \frac{1}{4}(1-\xi)(1-\eta)$$

$$f_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$f_3 = \frac{1}{4}(1+\xi)(1+\eta)$$

$$f_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$\begin{cases} r = \frac{1}{2} r_1 (1-\xi) + \frac{1}{2} r_2 (1+\xi) \\ \theta = \frac{1}{2}\theta_1 (1-\eta) + \frac{1}{2}\theta_2 (1+\eta) \end{cases}$$

$r =$ function of ξ

$\theta =$ function of η

$$\begin{cases} r = \frac{1}{2}(r_1+r_2) + \frac{1}{2}(r_2-r_1)\xi \\ \theta = \frac{1}{2}(\theta_2+\theta_1) + \frac{1}{2}(\theta_2-\theta_1)\eta \end{cases}$$

$$\frac{dr}{d\xi} = \frac{1}{2}(r_2-r_1) = \frac{1}{2}\Delta r$$

$$\frac{d\theta}{d\eta} = \frac{1}{2}(\theta_2-\theta_1) = \frac{1}{2}\Delta\theta$$

strain - generic displacement

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & \frac{\partial}{r\partial\theta} \\ \frac{\partial}{r\partial\theta} & \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{d}\,\underset{\sim}{u}$$

strain - nodal displacement

$$\underset{\sim}{\varepsilon} = \underset{\sim}{d}\,\underset{\sim}{f}\,\underset{\sim}{q}$$

$$= \underset{\sim}{B}\,\underset{\sim}{q}$$

$$\underset{\sim}{B} = \left[\begin{array}{cc|c|cc} \frac{\partial f_1}{\partial r} & 0 & & \frac{\partial f_4}{\partial r} & 0 \\ \frac{f_1}{r} & \frac{\partial f_1}{r\partial\theta} & \cdots & \frac{f_4}{r} & \frac{\partial f_4}{r\partial\theta} \\ \frac{\partial f_1}{r\partial\theta} & \left(\frac{\partial f_1}{\partial r} - \frac{f_1}{r}\right) & & \frac{\partial f_4}{r\partial\theta} & \left(\frac{\partial f_4}{\partial r} - \frac{f_4}{r}\right) \end{array}\right]$$

$$\underset{\sim}{K} = \int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dV$$

$$= t \int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dA$$

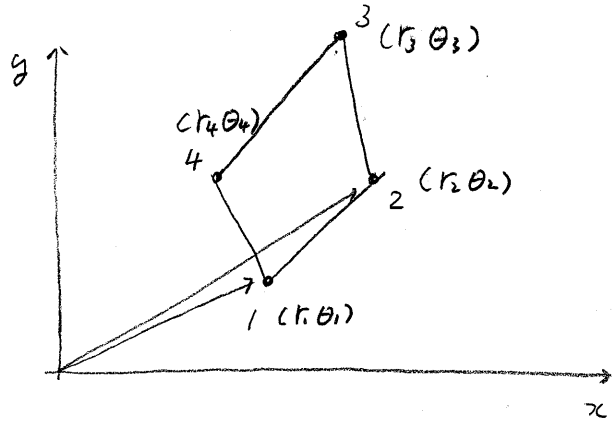
$$dA = r\,dr\,d\theta$$

$$dr = \frac{1}{2}\Delta r\, d\xi \quad , \quad d\theta = \frac{1}{2}\Delta\theta\, d\eta$$

$$\underset{\sim}{K} = \frac{t\,\Delta r\,\Delta\theta}{4} \int_{-1}^{1}\int_{-1}^{1} \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, r\, d\xi\, d\eta$$

$$\left\{\begin{aligned} \frac{\partial f_i}{\partial r} &= \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial f_i}{\partial \eta}\cancelto{0}{\frac{\partial \eta}{\partial r}} \\ &= \frac{\partial f_i}{\partial \xi}\frac{2}{\Delta r} \\ \frac{\partial f_i}{\partial \theta} &= \frac{\partial f_i}{\partial \xi}\cancelto{0}{\frac{\partial \xi}{\partial \theta}} + \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial \theta} \\ &= \frac{\partial f_i}{\partial \eta}\frac{2}{\Delta\theta} \end{aligned}\right.$$

Isoparametric 4 node Element



$$\begin{cases} r = \sum_i^4 f_i r_i \\ \theta = \sum f_i \theta_i \end{cases} \qquad \begin{cases} u = \sum f_i u_i \\ v = \sum f_i v_i \end{cases}$$

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & \frac{\partial}{r\partial\theta} \\ \frac{\partial}{r\partial\theta} & \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$\underset{\sim}{B} = \left[\begin{array}{cc|c} \frac{\partial f_1}{\partial r} & 0 & \\ \frac{f_1}{r} & \frac{\partial f_1}{r\partial\theta} & \cdots\cdots \\ \frac{\partial f_1}{r\partial\theta} & \left(\frac{\partial f_1}{\partial r} - \frac{f_1}{r}\right) & \end{array}\right]$$

$$\frac{\partial f_i}{\partial r} = \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial r}$$

$$\frac{\partial f_i}{\partial \theta} = \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial \theta} + \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial \theta}$$

$$\begin{bmatrix} \frac{\partial}{\partial \xi} \\ \frac{\partial}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial \theta}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial \theta}{\partial \eta} \end{bmatrix}}_{\underset{\sim}{J}} \begin{bmatrix} \frac{\partial}{\partial r} \\ \frac{\partial}{\partial \theta} \end{bmatrix}$$

$$\frac{\partial r}{\partial \xi} = \Sigma f_{i,\xi} r_i \qquad \frac{\partial \theta}{\partial \xi} = \Sigma f_{i,\xi} \theta_i$$

$$\frac{\partial r}{\partial \eta} = \Sigma f_{i,\eta} r_i \qquad \frac{\partial \theta}{\partial \eta} = \Sigma f_{i\eta} \theta_i$$

$$\frac{\partial f_i}{\partial r} = J_{11}^* \frac{\partial f_i}{\partial \xi} + J_{12}^* \frac{\partial f_i}{\partial \eta}$$

$$\frac{\partial f_i}{\partial \theta} = J_{21}^* \frac{\partial f_i}{\partial \xi} + J_{22}^* \frac{\partial f_i}{\partial \eta}$$

$$\underset{\sim}{K} = \int \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, dV$$

$$= t \int \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, dA = t \int \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, r dr d\theta$$

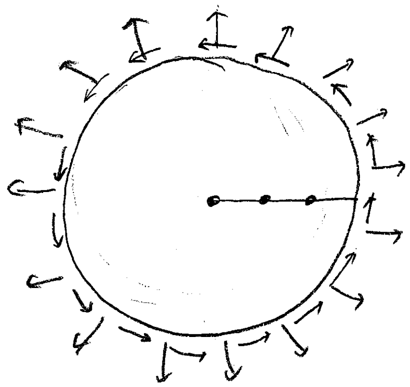
$$= t \int \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, r |J| \, d\xi d\eta$$

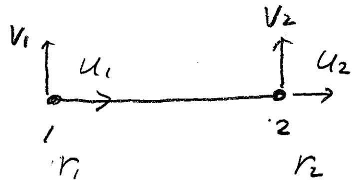
if $r_4 = r_1 \qquad \theta_1 = \theta_2$

$r_3 = r_2 \qquad \theta_4 = \theta_3$

$$\underset{\sim}{J} = \begin{bmatrix} \frac{r_2 - r_1}{2} & 0 \\ 0 & \frac{\theta_2 - \theta_1}{2} \end{bmatrix} = \frac{1}{4} \Delta r \Delta \theta$$

Axisymmetric Geometry and loadings





$$u = f_1 u_1 + f_2 u_2 \qquad V = f_1 V_1 + f_2 V_2$$

$$r = f_1 r_1 + f_2 r_2$$

$$f_1 = \frac{1}{2}(1-\xi) \qquad f_2 = \frac{1}{2}(1+\xi)$$

$$r = \frac{1}{2}(1-\xi) r_1 + \frac{1}{2}(1+\xi) r_2$$

$$= \frac{1}{2}(r_1 + r_2) + \frac{1}{2}(r_2 - r_1)\xi$$

$$\frac{dr}{d\xi} = \frac{1}{2}(r_2 - r_1) = \frac{1}{2}\Delta r$$

Strain - General displ.

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ \frac{1}{r} & \frac{\partial}{r\partial\theta} \\ \frac{\partial}{r\partial\theta} & \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

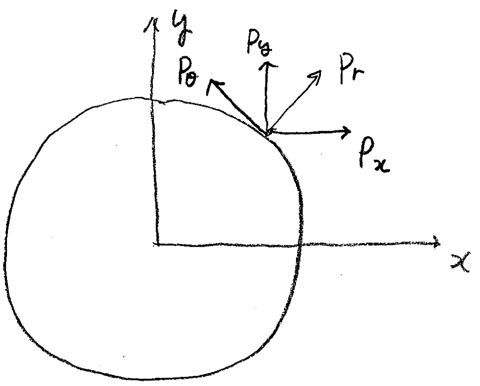
$$\underset{\sim}{B} = \left[\begin{array}{cc|cc} \frac{\partial f_1}{\partial r} & 0 & \frac{\partial f_2}{\partial r} & 0 \\ \frac{f_1}{r} & \frac{\partial f_1}{r\partial\theta} & \frac{f_2}{r} & \frac{\partial f_2}{r\partial\theta} \\ \frac{\partial f_1}{r\partial\theta} & \left(\frac{\partial f_1}{\partial r} - \frac{f_1}{r}\right) & \frac{\partial f_2}{r\partial\theta} & \left(\frac{\partial f_2}{\partial r} - \frac{f_2}{r}\right) \end{array}\right]$$

$$\frac{\partial f_1}{\partial r} = \frac{\partial f_1}{\partial \xi}\frac{\partial \xi}{\partial r} = \frac{\partial f_1}{\partial \xi}\cdot\frac{2}{\Delta r}$$

$$\underset{\sim}{K} = t\int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dr = t\int_{-1}^{1} \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, \frac{1}{2}\Delta r\, d\xi$$

(plane stress)

Axisymmetric Geometry subject to Nonaxisymmetric loadings



$$P_r = P_x \cos\theta + P_y \sin\theta$$

$$P_\theta = -P_x \sin\theta + P_y \cos\theta$$

Any load combinations can be described by Fourier Series

$$P_r = \sum_{m=0}^{\infty} \left(P_{rm} \cos m\theta + \overline{P_{rm}} \sin m\theta \right)$$

$$P_\theta = \sum_{m=0}^{\infty} \left(-P_{\theta m} \sin m\theta + \overline{P_{\theta m}} \cos m\theta \right)$$

$$\begin{cases} P_{r0} = \dfrac{1}{2\pi}\displaystyle\int_{-\pi}^{\pi} P_r\, d\theta \\ P_{rm} = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} P_r \cos m\theta\, d\theta \\ \overline{P_{rm}} = \dfrac{1}{\pi}\displaystyle\int_{-\pi}^{\pi} P_r \sin m\theta\, d\theta \\ \overline{P_{r0}} = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \cos m\theta \cos n\theta\, d\theta = 0 \qquad m \neq n$$

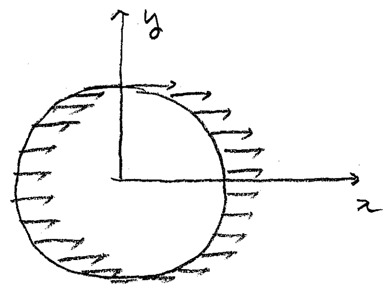
$$\int_{-\pi}^{\pi} \cos^2 m\theta\, d\theta = \begin{cases} \pi & m > 0 \\ 2\pi & m = 0 \end{cases}$$

$$\int_{-\pi}^{\pi} \sin m\theta \sin n\theta\, d\theta = 0 \qquad m \neq n$$

$$\int_{-\pi}^{\pi} \sin^2 m\theta\, d\theta = \begin{cases} \pi & m > 0 \\ 0 & m = 0 \end{cases}$$

$$\begin{cases} \overline{P_{\theta 0}} = \dfrac{1}{2\pi} \displaystyle\int_{-\pi}^{\pi} P_\theta \, d\theta \\ \overline{P_{\theta m}} = \dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} P_\theta \cos m\theta \, d\theta \\ P_{\theta m} = -\dfrac{1}{\pi} \displaystyle\int_{-\pi}^{\pi} P_\theta \sin m\theta \, d\theta \\ P_{\theta 0} = 0 \end{cases}$$

For example, $P_x = g$ $\quad P_y = 0$

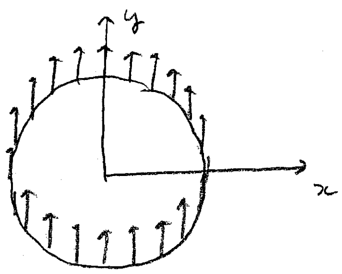


$$\begin{cases} P_r = g\cos\theta \\ P_\theta = -g\sin\theta \end{cases}$$

$P_{r1} = g$, $\quad \overline{P_{rm}} = 0$, $\quad P_{r0} = 0$. $\quad (P_{rm} = 0 \quad m \neq 1)$

$P_{\theta 1} = g$. $\quad \overline{P_{\theta m}} = 0$, $\quad (P_{\theta m} = 0 \quad m \neq 1)$

$P_x = 0$, $P_y = g$



$$\begin{cases} P_r = g\sin\theta \\ P_\theta = g\cos\theta \end{cases}$$

$$\left.\begin{array}{l} \overline{P_{r1}} = g \\ \overline{P_{\theta 1}} = g \end{array}\right\} \quad \text{others} = 0$$

For linear system,

INPUT
$$\begin{cases} P_r = \sum_{m=0}^{\infty} (P_{rm} \cos m\theta + \overline{P_{rm}} \sin m\theta) \\ P_\theta = \sum (-P_{\theta m} \sin m\theta + \overline{P_{\theta m}} \cos m\theta) \end{cases}$$

OUTPUT
$$\begin{cases} u = \sum (U_m \cos m\theta + \overline{U_m} \sin m\theta) \\ V = \sum (-V_m \sin m\theta + \overline{V_m} \cos m\theta) \end{cases}$$

Also

INPUT $\begin{cases} P_{rm} \cos m\theta \\ -P_{\theta m} \sin m\theta \end{cases}$ $\Rightarrow$ symmetric w.r.t x-axis

output $\begin{cases} U_m \cos m\theta \\ -V_m \sin m\theta \end{cases}$ $\Rightarrow$ "

INPUT $\begin{cases} \overline{P_{rm}} \sin m\theta \\ \overline{P_{\theta m}} \cos m\theta \end{cases}$ $\Rightarrow$ symmetric w.r.t y-axis

output $\begin{cases} \overline{U_m} \sin m\theta \\ \overline{V_m} \cos m\theta \end{cases}$ $\Rightarrow$ "

For response which is symmetric w.r.t x-axis

$$\underset{\sim}{U_m} = \begin{bmatrix} U_m \cos m\theta \\ -V_m \sin m\theta \end{bmatrix}$$

Strain - Generic Displacement

$$\varepsilon_r = \frac{\partial u}{\partial r} = \frac{\partial U_m}{\partial r}\cos m\theta$$

$$\varepsilon_\theta = \frac{\partial v}{r\,\partial\theta} + \frac{u}{r} = \frac{U_m - mV_m}{r}\cos m\theta$$

$$\gamma_{r\theta} = \frac{\partial u}{r\,\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r} = \left[-\frac{\partial V_m}{\partial r} + \frac{V_m - mU_m}{r}\right]\sin m\theta$$

using Principle of Virtual Displacement

$$\int \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, dV = \int \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{E}\, \underset{\sim}{\varepsilon}\, dV$$

$$\underset{\sim}{E}\,\underset{\sim}{\varepsilon} = \begin{bmatrix} E_1 & E_2 & 0 \\ E_2 & E_1 & 0 \\ 0 & 0 & E_3 \end{bmatrix}\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \gamma_{r\theta} \end{bmatrix} = \begin{bmatrix} E_1\varepsilon_r + E_2\varepsilon_\theta \\ E_2\varepsilon_r + E_1\varepsilon_\theta \\ E_3\gamma_{r\theta} \end{bmatrix}$$

$$\delta\underset{\sim}{\varepsilon}^T \underset{\sim}{E}\,\underset{\sim}{\varepsilon} = \begin{bmatrix} \delta\varepsilon_r & \delta\varepsilon_\theta & \delta\gamma_{r\theta} \end{bmatrix}\begin{bmatrix} E_1\varepsilon_r + E_2\varepsilon_\theta \\ E_2\varepsilon_r + E_1\varepsilon_\theta \\ E_3\gamma_{r\theta} \end{bmatrix}$$

$$= E_1\underline{\delta\varepsilon_r\varepsilon_r} + E_2\underline{\varepsilon_\theta\delta\varepsilon_r} + E_2\underline{\delta\varepsilon_\theta\varepsilon_r} + E_1\underline{\delta\varepsilon_\theta\varepsilon_\theta} + E_3\underline{\delta\gamma_{r\theta}\gamma_{r\theta}}$$

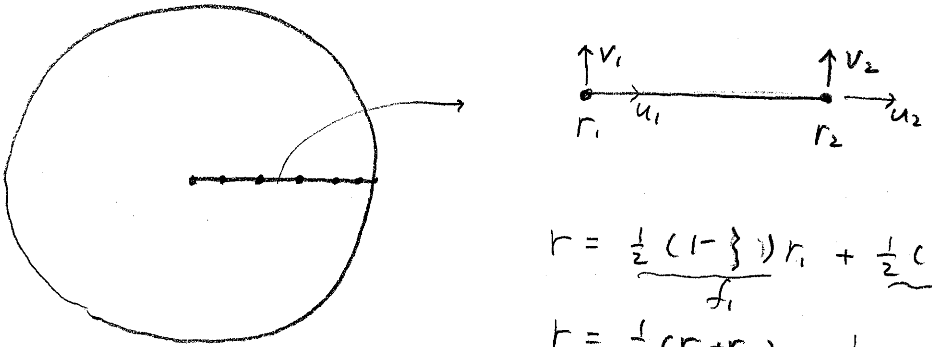
$\cos^2 m\theta$ $\qquad\qquad$ $\sin^2 m\theta$

$$\int \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, dV = t\int_0^{2\pi}\!\!\int \delta\underset{\sim}{\varepsilon}^T\underset{\sim}{\sigma}\, r\,dr\,d\theta$$

$$\int_0^{2\pi}\cos^2 m\theta\, d\theta = \begin{cases} \pi & m \neq 0 \\ 2\pi & m = 0 \end{cases} \qquad \int_0^{2\pi}\sin^2 m\theta\, d\theta = \begin{cases} \pi & m > 0 \\ 0 & m = 0 \end{cases}$$

$$\delta\underset{\sim}{q}^T \underset{\sim}{P} \Rightarrow \left\langle \begin{array}{l} \int \cos^2 m\theta \\ \int \sin^2 m\theta \end{array} \right.$$

$$\Rightarrow \int \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{E}\, \underset{\sim}{\varepsilon}\, dV = \text{function of only "}r\text{" which is independent of } \theta \text{ and } m$$



$$r = \underbrace{\tfrac{1}{2}(1-\xi)}_{f_1} r_1 + \underbrace{\tfrac{1}{2}(1+\xi)}_{f_2} r_2$$

$$r = \tfrac{1}{2}(r_1+r_2) + \tfrac{1}{2}(r_2-r_1)\xi$$

$$u = f_1 u_1 + f_2 u_2 \qquad f_1 = \tfrac{1}{2}(1-\xi)$$

$$v = f_1 v_1 + f_2 v_2 \qquad f_2 = \tfrac{1}{2}(1+\xi)$$

for m-th displacement coefficient

$$u_m = f_1 u_{1m} + f_2 u_{2m}$$

$$v_m = f_1 v_{1m} + f_2 v_{2m}$$

strain - nodal displacement

$$\underset{\sim}{\varepsilon}_m = \underset{\sim}{B}_m \underset{\sim}{q}_m$$

$$\underset{\sim}{B}_m = \left[\begin{array}{cc|cc} \frac{\partial f_1}{\partial r} & 0 & \frac{\partial f_2}{\partial r} & 0 \\ \frac{f_1}{r} & -\frac{m}{r} f_1 & \frac{1}{r} f_2 & -\frac{m}{r} f_2 \\ -\frac{m}{r} f_1 & \left(\frac{1}{r} f_1 - \frac{df_1}{dr}\right) & -\frac{m}{r} f_2 & \left(\frac{1}{r} f_2 - \frac{df_2}{dr}\right) \end{array}\right]$$

$$\frac{\partial f_i}{\partial r} = \frac{\partial f_i}{\partial \xi} \frac{d\xi}{dr}$$

$$K_m = \int_{r_1}^{r_2} \underset{\sim}{B}_m^T \underset{\sim}{D}\, \underset{\sim}{B}_m\, r\, dr \qquad m > 0$$

Actually

$$\left\{ \begin{array}{l} \underset{\sim}{K}_m \begin{bmatrix} u_m \\ -v_m \end{bmatrix} = \begin{bmatrix} P_{rm} \\ -P_{\theta m} \end{bmatrix} \\ \overline{\underset{\sim}{K}}_m \begin{bmatrix} \bar{u}_m \\ \bar{v}_m \end{bmatrix} = \begin{bmatrix} \bar{P}_{rm} \\ \bar{P}_{\theta m} \end{bmatrix} \end{array} \right. \qquad \text{Solve } u_m,\ -v_m,\ \bar{u}_m,\ \bar{v}_m$$

$$\left\{ \begin{array}{l} u = \sum_{m=0}^{\infty} (u_m \cos m\theta + \bar{u}_m \sin m\theta) \\ v = \sum (-v_m \sin m\theta + \bar{v}_m \cos m\theta) \end{array} \right\} \Rightarrow \text{solution}$$

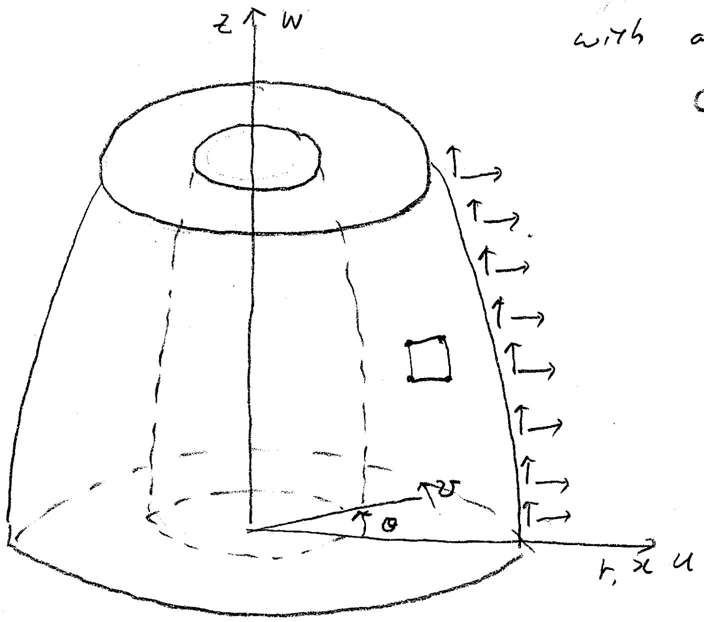
$$\underset{\sim}{B}_m = \underbrace{\begin{bmatrix} \frac{\partial f_1}{\partial r} & 0 & \frac{\partial f_2}{\partial r} & 0 \\ 0 & 0 & 0 & 0 \\ 0 & -\frac{\partial f_1}{\partial r} & 0 & -\frac{\partial f_2}{\partial r} \end{bmatrix}}_{\underset{\sim}{B}_1} + \frac{1}{r} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ f_1 & 0 & f_2 & 0 \\ 0 & f_1 & 0 & f_2 \end{bmatrix}}_{\underset{\sim}{B}_2} - \frac{m}{r} \underbrace{\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & f_1 & 0 & f_2 \\ f_1 & 0 & f_2 & 0 \end{bmatrix}}_{\underset{\sim}{B}_3}$$

$$\underset{\sim}{B}_m = \underset{\sim}{B}_1 + \frac{1}{r}\underset{\sim}{B}_2 - \frac{m}{r}\underset{\sim}{B}_3$$

$$\begin{aligned} \underset{\sim}{B}_m^T \underset{\sim}{E}\, \underset{\sim}{B}_m &= (\underset{\sim}{B}_1^T + \tfrac{1}{r}\underset{\sim}{B}_2^T - \tfrac{m}{r}\underset{\sim}{B}_3^T)\, \underset{\sim}{E}\, (\underset{\sim}{B}_1 + \tfrac{1}{r}\underset{\sim}{B}_2 - \tfrac{m}{r}\underset{\sim}{B}_3) \\ &= \underset{\sim}{B}_1^T \underset{\sim}{E}\, \underset{\sim}{B}_1 + \tfrac{1}{r}(\underset{\sim}{B}_1^T \underset{\sim}{E}\, \underset{\sim}{B}_2 + \underset{\sim}{B}_2^T \underset{\sim}{E}\, \underset{\sim}{B}_1) + \tfrac{1}{r^2}(\underset{\sim}{B}_2^T \underset{\sim}{E}\, \underset{\sim}{B}_2) \\ &\quad - \tfrac{m}{r}(\underset{\sim}{B}_3^T \underset{\sim}{E}\, \underset{\sim}{B}_1 + \underset{\sim}{B}_1^T \underset{\sim}{E}\, \underset{\sim}{B}_3) - \tfrac{m}{r^2}(\underset{\sim}{B}_3^T \underset{\sim}{E}\, \underset{\sim}{B}_3 + \underset{\sim}{B}_2^T \underset{\sim}{E}\, \underset{\sim}{B}_2) + \tfrac{m^2}{r^2}\underset{\sim}{B}_3^T \underset{\sim}{E}\, \underset{\sim}{B}_3 \\ &= \underset{\sim}{A} + m\underset{\sim}{B} + m^2 \underset{\sim}{C} \end{aligned}$$

Chapter 5. Axisymmetric Solids (Solid of revolution)

with axis-symmetric loading
(radial 방향 하중만 받을 경우)



$\underset{\sim}{u} = \begin{bmatrix} u \\ w \end{bmatrix}$ $\qquad$ $\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{bmatrix}$

if radial forces are applied to this solid,

$$\underset{\sim}{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \\ \gamma_{z\theta} \\ \gamma_{r\theta} \end{bmatrix}$$

$$\varepsilon_r = \frac{\partial u}{\partial r}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

$$\varepsilon_\theta = \frac{1}{2\pi r}\left[2\pi(r+u) - 2\pi r\right] = \frac{u}{r} + \frac{\partial v}{r\partial\theta}$$

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_z \\ \varepsilon_\theta \\ \gamma_{rz} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 \\ 0 & \frac{\partial}{\partial z} \\ \frac{1}{r} & 0 \\ \frac{\partial}{\partial z} & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u \\ w \end{bmatrix}$$

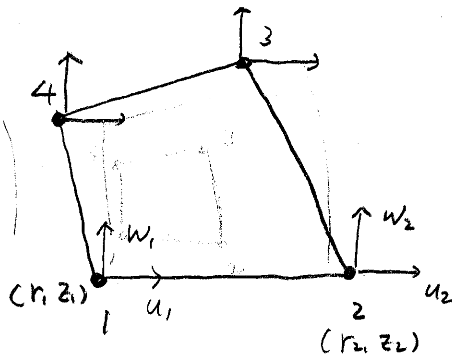
$$\underset{\sim}{\varepsilon} = \underset{\sim}{d}\,\underset{\sim}{u}$$

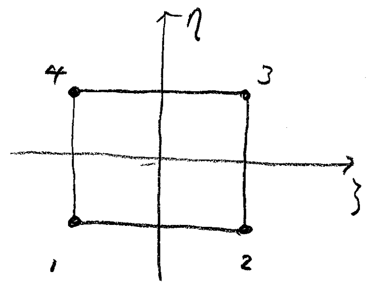
For isotropic material

$$\underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\varepsilon}$$

$$\underset{\sim}{E} = \frac{E}{(1+\nu)(1-2\nu)} \begin{bmatrix} (1-\nu) & \nu & \nu & 0 \\ \nu & (1-\nu) & \nu & 0 \\ \nu & \nu & (1-\nu) & 0 \\ 0 & 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}$$

Isoparametric Quadrilateral Element





$$\underset{\sim}{q} = \begin{bmatrix} u_1 \\ w_1 \\ u_2 \\ w_2 \\ u_3 \\ w_3 \\ u_4 \\ w_4 \end{bmatrix}$$

$$\begin{cases} u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4 \\ w = f_1 w_1 + f_2 w_2 + f_3 w_3 + f_4 w_4 \end{cases}$$

$$f_i = \frac{1}{4}(1+\xi_i \xi)(1+\eta_i \eta)$$

$$\begin{cases} r = f_1 r_1 + f_2 r_2 + f_3 r_3 + f_4 r_4 \\ z = f_1 z_1 + f_2 z_2 + f_3 z_3 + f_4 z_4 \end{cases}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{d}\, \underset{\sim}{f}\, \underset{\sim}{q}$$

$$= \underset{\sim}{B}\, \underset{\sim}{q}$$

$$\underset{\sim}{B} = \left[\begin{array}{cc|c|cc} f_{1,r} & 0 & & f_{4,r} & 0 \\ 0 & f_{1,z} & & 0 & f_{4,z} \\ \frac{f_1}{r} & 0 & \cdots\cdots & \frac{f_4}{r} & 0 \\ f_{1,z} & f_{1,r} & & f_{4,z} & f_{4,r} \end{array}\right]$$

$$\frac{\partial f_i}{\partial r} = \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial r} + \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial r} = J_{11}^{*}\frac{\partial f_i}{\partial \xi} + J_{12}^{*}\frac{\partial f_i}{\partial \eta}$$

$$\frac{\partial f_i}{\partial z} = \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial z} = J_{21}^{*}\frac{\partial f_i}{\partial \xi} + J_{22}^{*}\frac{\partial f_i}{\partial \eta}$$

$$\underset{\sim}{J} = \begin{bmatrix} \frac{\partial r}{\partial \xi} & \frac{\partial z}{\partial \xi} \\ \frac{\partial r}{\partial \eta} & \frac{\partial z}{\partial \eta} \end{bmatrix} \qquad \underset{\sim}{J}^{-1} = \underset{\sim}{J}^{*} = \begin{bmatrix} \frac{\partial \xi}{\partial r} & \frac{\partial \eta}{\partial r} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} \end{bmatrix}$$

$$\frac{\partial r}{\partial \xi} = \sum_{i=1}^{4} f_{i,\xi}\, r_i$$

$$\frac{\partial z}{\partial \xi} = \sum f_{i,\xi}\, z_i$$

$$\frac{\partial r}{\partial \eta} = \sum f_{i,\eta}\, r_i$$

$$\frac{\partial z}{\partial \eta} = \sum f_{i,\eta}\, z_i$$

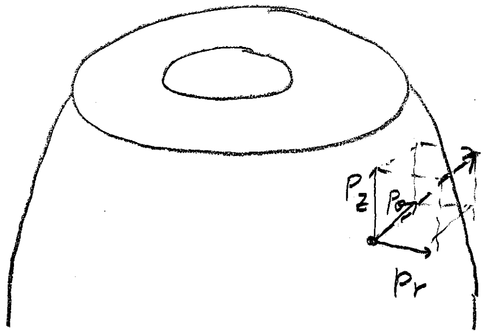
$$\underset{\sim}{K} = \iint \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \; r \, dr \, dz$$

$$= \iint \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \; r \, |J| \, d\xi \, d\eta$$

$$\underset{\sim}{P}_b = \iint \underset{\sim}{f}^T \underset{\sim}{b} \; |J| \, r \, d\xi \, d\eta$$

$$\underset{\sim}{P}_o = \iint \underset{\sim}{B}^T \underset{\sim}{E} \, \underset{\sim}{\varepsilon}_o \; |J| \, r \, d\xi \, d\eta$$

Axisymmetric Solid with non-axisymmetric loading



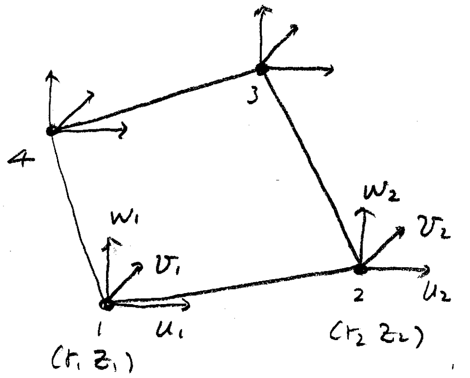
$$P_r = \sum_{m=0}^{\infty} \left[P_{rm} \cos m\theta + \overline{P_{rm}} \sin m\theta \right]$$

$$P_\theta = \sum \left[P_{\theta m} \sin m\theta + \overline{P_{\theta m}} \cos m\theta \right]$$

$$P_z = \sum \left[P_{zm} \cos m\theta + \overline{P_{zm}} \sin m\theta \right]$$

$\downarrow$ symmetric w.r.t x-axis

$\downarrow$ symmetric w.r.t y-axis



$$r = \sum f_i r_i$$

$$z = \sum f_i z_i$$

$$\underset{\sim}{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{r\theta} \\ \gamma_{\theta z} \\ \gamma_{zr} \end{bmatrix}$$

$$u = \sum f_i u_i$$

$$v = \sum f_i v_i$$

$$w = \sum f_i w_i$$

$$\varepsilon_r = \frac{\partial u}{\partial r}$$

$$\varepsilon_\theta = \frac{\partial v}{r\partial\theta} + \frac{u}{r}$$

$$\varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{r\theta} = \frac{\partial u}{r\partial\theta} + \frac{\partial v}{\partial r} - \frac{v}{r}$$

$$\gamma_{\theta z} = \frac{\partial w}{r\partial\theta} + \frac{\partial v}{\partial z}$$

$$\gamma_{zr} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r}$$

$$\begin{bmatrix} \varepsilon_r \\ \varepsilon_\theta \\ \varepsilon_z \\ \gamma_{r\theta} \\ \gamma_{\theta z} \\ \gamma_{zr} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial r} & 0 & 0 \\ \frac{1}{r} & \frac{\partial}{r\partial\theta} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{r\partial\theta} & \left(\frac{\partial}{\partial r} - \frac{1}{r}\right) & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{r\partial\theta} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial r} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

INPUT output

$$\left.\begin{matrix} P_{rm} \cos m\theta \\ P_{\theta m} \sin m\theta \\ P_{zm} \cos m\theta \end{matrix}\right\} \Longrightarrow \left\{\begin{matrix} u_m \cos m\theta \\ v_m \sin m\theta \\ w_m \cos m\theta \end{matrix}\right.$$

$$\left.\begin{matrix} \overline{P}_{rm} \sin m\theta \\ \overline{P}_{\theta m} \cos m\theta \\ \overline{P}_{zm} \sin m\theta \end{matrix}\right\} \Longrightarrow \left\{\begin{matrix} \overline{u}_m \sin m\theta \\ \overline{v}_m \cos m\theta \\ \overline{w}_m \sin m\theta \end{matrix}\right.$$

$$\underset{\sim}{\varepsilon}_m = \begin{bmatrix} \frac{\partial u_m}{\partial r}\cos m\theta & 0 & 0 \\ \frac{u_m \cos m\theta}{r} & \frac{m v_m \cos m\theta}{r} & 0 \\ 0 & 0 & \frac{\partial w_m}{\partial z}\cos m\theta \\ \frac{m u_m \sin m\theta}{r} & \left(\frac{\partial v_m}{\partial r} - \frac{v_m}{r}\right)\sin m\theta & 0 \\ 0 & \frac{\partial v_m}{\partial z}\sin m\theta & \frac{-m w_m}{r}\sin m\theta \\ \frac{\partial u_m}{\partial z}\cos m\theta & 0 & \frac{\partial w_m}{\partial r}\cos m\theta \end{bmatrix}$$

$$\underset{\sim}{\varepsilon}_m = \underset{\sim}{B}_m \, \underset{\sim}{\delta}_m$$

$$\underset{6\times 12}{\underset{\sim}{B}_m} = \left[\begin{array}{ccc|c} f_{1,r}\cos m\theta & 0 & 0 & \\ \frac{f_1}{r}\cos m\theta & \frac{m f_1}{r}\cos m\theta & 0 & \\ 0 & 0 & f_{1,z}\cos m\theta & \cdots \\ \frac{m f_1}{r}\sin m\theta & \left(f_{1,r} - \frac{f_1}{r}\right)\sin m\theta & 0 & \\ 0 & f_{1,z}\sin m\theta & -\frac{m f_1}{r}\sin m\theta & \\ f_{1,z}\cos m\theta & 0 & f_{1,r}\cos m\theta & \end{array} \right]$$

$$\underset{6\times 12}{\bar{\mathbf{B}}_m} = \left[\begin{array}{ccc|c} f_{1,r} & 0 & 0 & \\ \frac{f_1}{r} & \frac{mf_1}{r} & 0 & \\ \frac{mf_1}{r} & \left(f_{1,r} - \frac{f_1}{r}\right) & 0 & - - - - \\ 0 & f_{1,z} & -\frac{mf_1}{r} & \\ f_{1,z} & 0 & f_{1,r} & \end{array}\right]$$

$$\int \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, dV = \delta \boldsymbol{\delta}^T \mathbf{P}$$

$$\Rightarrow \cancel{\delta \boldsymbol{\delta}_m^T} \int \mathbf{B}_m^T \mathbf{E} \mathbf{B}_m \, dA \; \boldsymbol{\delta}_m = \cancel{\delta \boldsymbol{\delta}_m^T} \mathbf{P}_m \cdot k\pi$$

$$\cancel{k\pi} \underbrace{\int \bar{\mathbf{B}}_m^T \mathbf{E} \bar{\mathbf{B}}_m \, dA}_{\mathbf{K}_m} \; \boldsymbol{\delta}_m = \mathbf{P}_m \cdot \cancel{k\pi}$$

$$\begin{cases} m = 0 \quad k = 2 \\ m = 1, 2, \cdots, m \quad k = 1 \\ \int_0^{2\pi} \cos^2 m\theta \, d\theta = \int_0^{2\pi} \sin^2 m\theta \, d\theta = \pi \end{cases}$$

$$\left.\begin{cases} \mathbf{K}_m \mathbf{U}_m = \mathbf{P}_m \\ \mathbf{K}_m \bar{\mathbf{U}}_m = \bar{\mathbf{P}}_m \end{cases}\right\} \Rightarrow \text{solve} \begin{bmatrix} u_{mi} \\ v_{mi} \\ w_{mi} \end{bmatrix} \quad \begin{bmatrix} \bar{u}_{mi} \\ \bar{v}_{mi} \\ \bar{w}_{mi} \end{bmatrix}$$

$$\begin{cases} u_i = \sum \left(u_{mi} \cos m\theta + \bar{u}_{mi} \sin m\theta \right) \\ v_i = \sum \left(v_{mi} \sin m\theta + \bar{v}_{mi} \cos m\theta \right) \\ w_i = \sum \left(w_{mi} \cos m\theta + \bar{w}_{mi} \sin m\theta \right) \end{cases}$$