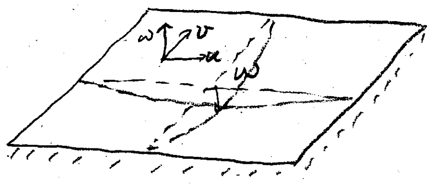
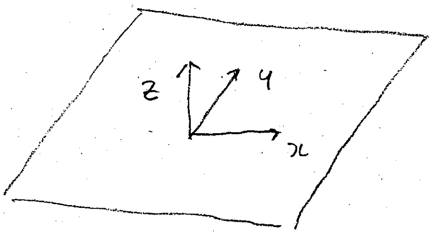
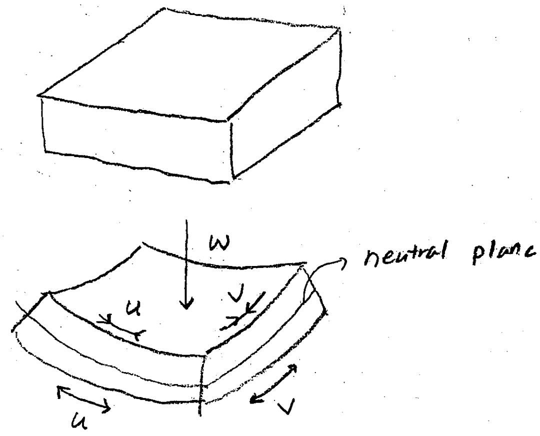
plate Bending

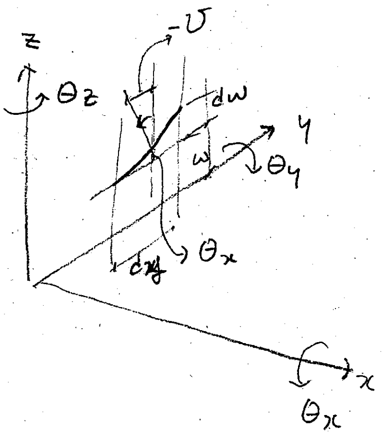
$$
\begin{cases}
\varepsilon_x = \dfrac{\partial u}{\partial x} \\[2ex]
\varepsilon_y = \dfrac{\partial v}{\partial y} \\[2ex]
\gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x}
\end{cases}
$$



vertical Displacement w



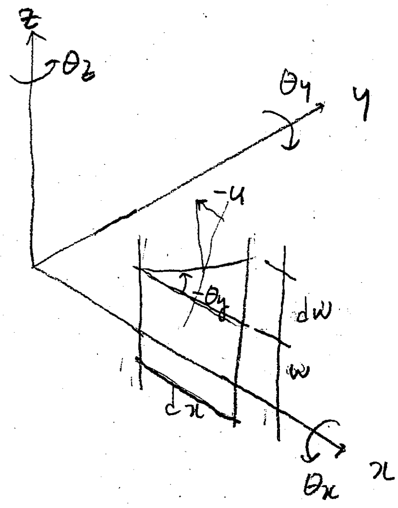




vertical displacement w develops displacements u and v in the planes distant from the neutral plane, and then plane stresses in the planes.

$$\theta_x = \frac{\partial w}{\partial y}$$

$$v = -\theta_x \cdot z = -z \frac{\partial w}{\partial y}$$



$$-\theta_y = \frac{\partial w}{\partial x}$$

$$-u = -\theta_y \cdot z$$

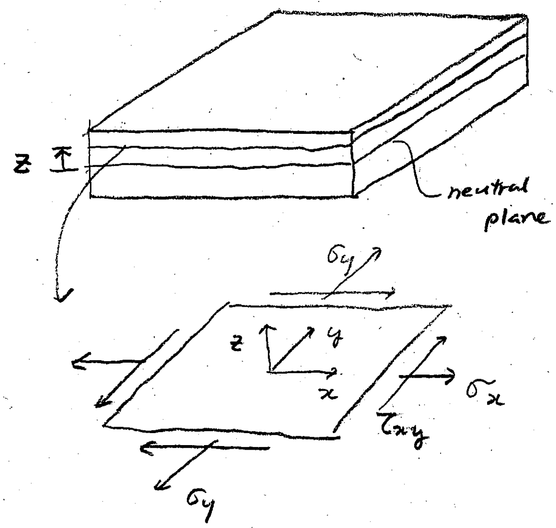
$$u = -z \cdot \frac{\partial w}{\partial x}$$

Strain - Generic Displacement

$$\varepsilon_x = \frac{\partial u}{\partial x} = -z \frac{\partial^2 w}{\partial x^2}$$

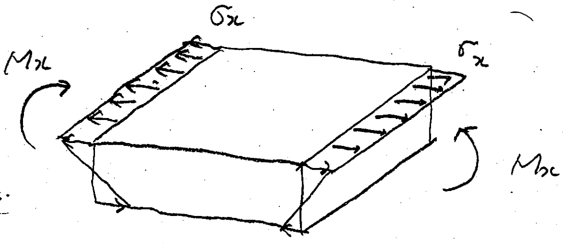
$$\varepsilon_y = \frac{\partial v}{\partial y} = -z \frac{\partial^2 w}{\partial y^2}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = -2z \frac{\partial^2 w}{\partial x \partial y}$$



$$\begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} = \begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & E_{33} \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \frac{E}{(1-\nu^2)} \begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \left(\frac{1-\nu}{2}\right) \end{bmatrix} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

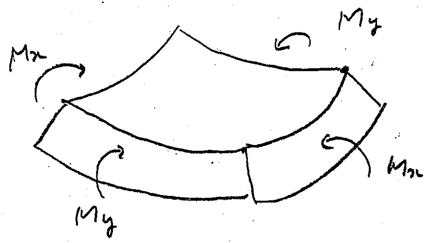
$$\begin{cases} \sigma_x = E_{11}\varepsilon_x + E_{12}\varepsilon_y = -z E_{11} \frac{\partial^2 w}{\partial x^2} - z E_{12} \frac{\partial^2 w}{\partial y^2} \\ \sigma_y = E_{21}\varepsilon_x + E_{22}\varepsilon_y = -z E_{21} \frac{\partial^2 w}{\partial x^2} - z E_{22} \frac{\partial^2 w}{\partial y^2} \\ \tau_{xy} = E_{33}\gamma_{xy} = -2E_{33} z \frac{\partial^2 w}{\partial x \partial y} \end{cases}$$



$$M_x = -\int_{-t/2}^{t/2} \sigma_x z\, dz$$

$$= \left(E_{11}\frac{\partial^2 w}{\partial x^2} + E_{12}\frac{\partial^2 w}{\partial y^2}\right)\int_{-t/2}^{t/2} z^2 dz$$

$$= \frac{t^3}{12}\left(E_{11}\frac{\partial^2 w}{\partial x^2} + E_{12}\frac{\partial^2 w}{\partial y^2}\right)$$



$$M_y = -\int_{-t/2}^{t/2} \sigma_y z\, dz$$

$$= \left(E_{21}\frac{\partial^2 w}{\partial x^2} + E_{22}\frac{\partial^2 w}{\partial y^2}\right)\int_{-t/2}^{t/2} z^2 dz$$

$$= \left(E_{21}\frac{\partial^2 w}{\partial x^2} + E_{22}\frac{\partial^2 w}{\partial y^2}\right)\frac{t^3}{12}$$

$$M_{xy} = -\int_{-t/2}^{t/2} \tau_{xy} z\, dz$$

$$= 2E_{33}\frac{\partial^2 w}{\partial x \partial y}\int_{-t/2}^{t/2} z^2 dz$$

$$= 2\frac{t^3}{12} E_{33}\frac{\partial^2 w}{\partial x \partial y}$$

$M = \{M_x, M_y \ M_{xy}\}$: Generalized Stress

$\phi = \{\phi_x, \phi_y, 2\phi_{xy}\} = \{w_{,xx} \ w_{,yy} \ 2w_{,xy}\}$: Generalized Strain

$$\bar{\underset{\sim}{E}} = \underset{\sim}{E}\frac{t^3}{12}$$

$$\begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \frac{t^3}{12}\begin{bmatrix} E_{11} & E_{12} & \\ E_{21} & E_{22} & \\ & & E_{33} \end{bmatrix}\begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}$$

$$\mathbf{M} = \bar{\mathbf{E}}\,\boldsymbol{\Phi}$$

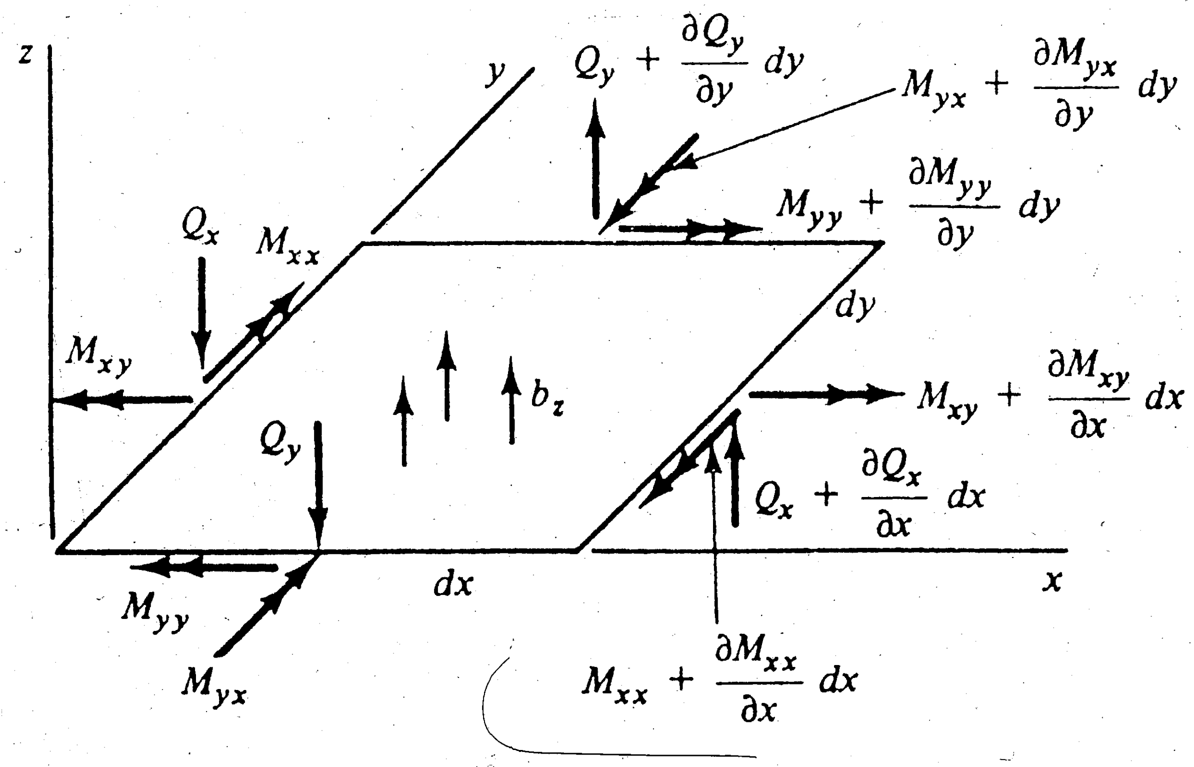
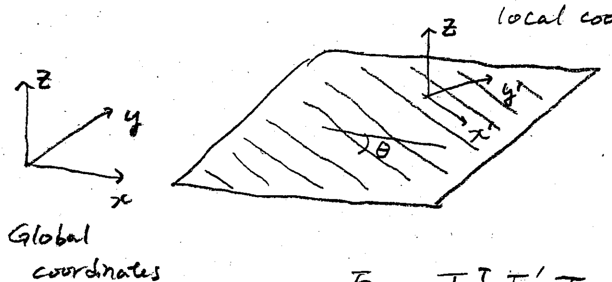


Figure 6.2 Equilibrium in a Plate

for orthotropic material

local coordinates for orthotropic material



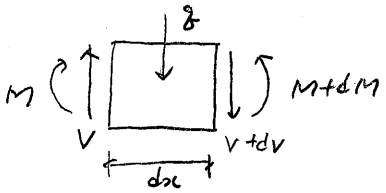
$$\mathbf{E}' = \begin{bmatrix} E_{11}' & E_{12}' & 0 \\ E_{21}' & E_{22}' & 0 \\ 0 & 0 & E_{33}' \end{bmatrix}$$

$$\mathbf{E} = \mathbf{T}_\varepsilon^T\,\mathbf{E}'\,\mathbf{T}_\varepsilon \quad \Rightarrow \quad \bar{\mathbf{E}} = \mathbf{T}_\varepsilon^T\,\bar{\mathbf{E}}'\,\mathbf{T}_\varepsilon$$

Equilibrium with respect to z forces

$$q\,dx\,dy - Q_x\,dy + \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy - Q_y\,dx$$

$$+ \left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)dx = 0$$



$$q\,dx\,dy + \frac{\partial Q_x}{\partial x}dx\,dy + \frac{\partial Q_y}{\partial y}dx\,dy = 0$$

$$\Rightarrow \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} + q = 0 \quad \text{——} \; (1)$$

for beams $\frac{dV}{dx} = -q$

Equilibrium of moments w.r.t y-axis

$$-q\,dx\,dy \cdot \frac{dx}{2} - \left(Q_x + \frac{\partial Q_x}{\partial x}dx\right)dy \cdot dx + Q_y\,dx\frac{dx}{2}$$

$$-\left(Q_y + \frac{\partial Q_y}{\partial y}dy\right)dx\frac{dx}{2} + M_x\,dy - \left(M_x + \frac{\partial M_x}{\partial x}dx\right)dy$$

$$+ M_{yx}\,dx - \left(M_{yx} + \frac{\partial M_{yx}}{\partial y}dy\right)dx = 0$$

neglecting higher order terms

$$\Rightarrow \frac{\partial M_x}{\partial x} + \frac{\partial M_{yx}}{\partial y} + Q_x = 0 \quad \text{——} \; (2)$$

for beams $\frac{dM}{dx} = V$

Equilibrium of moments w.r.t x-axis

$$\Rightarrow \frac{\partial M_y}{\partial y} + \frac{\partial M_{xy}}{\partial x} + Q_y = 0 \quad \text{——} \; (3)$$

from ② and ③, and ①

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{yx}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} + \frac{\partial Q_x}{\partial x} + \frac{\partial Q_y}{\partial y} = 0$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_y}{\partial y^2} = q$$

for beams $\frac{d^2 M}{dx^2} = q$

$$M_x = \frac{t^3}{12}\left(E_{11}\frac{\partial^2 w}{\partial x^2} + E_{12}\frac{\partial^2 w}{\partial y^2}\right)$$

for beams, $M = EI\phi$
$= EIv''$

$$= \frac{t^3}{12}\frac{E}{(1-\nu^2)}\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)$$

$$M_y = \frac{t^3}{12}\left(E_{21}\frac{\partial^2 w}{\partial x^2} + E_{22}\frac{\partial^2 w}{\partial y^2}\right)$$

$$= \frac{t^3}{12}\frac{E}{(1-\nu^2)}\left(\nu\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2}\right)$$

$$M_{xy} = 2\frac{t^3}{12}E_{33}\frac{\partial^2 w}{\partial x \partial y} = \frac{(1-\nu)E\,t^3}{(1-\nu^2)12}\frac{\partial^2 w}{\partial x \partial y}$$

$$\frac{\partial^2 M_x}{\partial x^2} + 2\frac{\partial^2 M_{xy}}{\partial x \partial y} + \frac{\partial^2 M_x}{\partial y^2}$$

$$= \frac{t^3}{12}\frac{E}{(1-\nu^2)}\left[\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4}\right]$$

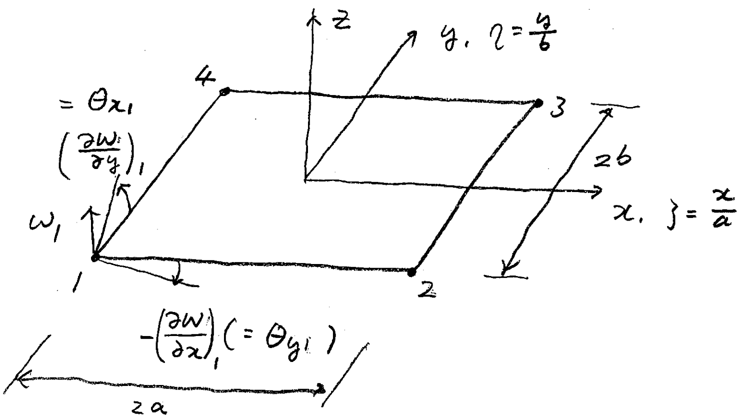
$$= D\,\nabla^4 w = q$$

for beams $EIv'''' = q$

∇^4 : bi Laplace operator

$\Rightarrow$ Governing Differential Equation of Plate

M. Z. C Element



nodal displacements

$$\underset{\sim}{\delta_i} = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \end{bmatrix} = \begin{bmatrix} w_i \\ \left(\frac{\partial w}{\partial y}\right)_i \\ \left(-\frac{\partial w}{\partial x}\right)_i \end{bmatrix} = \begin{bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \end{bmatrix}$$

$$\underset{12\times 1}{\underset{\sim}{\delta}} = \begin{bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ \vdots \\ w_4 \\ \theta_{x4} \\ \theta_{y4} \end{bmatrix}$$

nodal forces

$$\underset{\sim}{P_i} = \begin{bmatrix} P_{i1} \\ P_{i2} \\ P_{i3} \end{bmatrix} = \begin{bmatrix} P_i \\ M_{xi} \\ M_{yi} \end{bmatrix}$$

Displacement function

1

$\xi \quad \eta$

$\xi^2 \quad \xi\eta \quad \eta^2$ → 12

$\xi^3 \quad \eta\xi^2 \quad \eta^2\xi \quad \eta^3$

$\xi^4 \quad \xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3 \quad \eta^4$

$$w = C_1 + C_2\xi + C_3\eta + C_4\xi^2 + C_5\xi\eta + C_6\eta^2 + C_7\xi^3 + C_8\xi^2\eta + C_9\xi\eta^2 + C_{10}\eta^3 + C_{11}\xi^3\eta + C_{12}\xi\eta^3$$

$$w = \underset{\sim}{g}\,\underset{\sim}{c} \qquad \underset{\sim}{c} = \begin{bmatrix} C_1 \\ C_2 \\ \vdots \\ C_{11} \\ C_{12} \end{bmatrix}$$

$$\frac{\partial w}{\partial \eta} = C_3 + C_5 \xi + 2C_6 \eta + C_8 \xi^2 + 2C_9 \xi \eta + 3C_{10}\eta^2 + C_{11}\xi^3 + 3C_{12}\xi\eta^2$$

$$-\frac{\partial w}{\partial \xi} = -C_2 - 2C_4 \xi - C_5 \eta - 3C_7 \xi^2 - 2C_8 \xi \eta - C_9 \eta^2 - 3C_{11}\xi^2\eta - C_{12}\eta^3$$

using boundary conditions

at node 1 $(\xi_1, \eta_1) = (-1, -1)$

$$w = w_1$$

$$\frac{\partial w}{\partial y} = \frac{\partial w}{\partial \eta}\frac{d\eta}{dy} = \frac{1}{b}\frac{\partial w}{\partial \eta} = \theta_{x1}$$

$$-\frac{\partial w}{\partial x} = -\frac{\partial w}{\partial \xi}\frac{\partial \xi}{\partial x} = -\frac{1}{a}\frac{\partial w}{\partial \xi} = \theta_{y1}$$

$$\underset{\sim}{q} = \underset{\sim}{h}\,\underset{\sim}{c}$$

$$\underset{\sim}{c} = \underset{\sim}{h}^{-1}\underset{\sim}{q}$$

$$\underset{\sim}{w} = \underset{\sim}{g}\,\underset{\sim}{c} = \underbrace{\underset{\sim}{g}\,\underset{\sim}{h}^{-1}}_{\underset{\sim}{f}}\underset{\sim}{q} = \underset{\sim}{f}\,\underset{\sim}{q}$$

$$\underset{\sim}{w} = \sum_{i=1}^{4} (w_i f_{i1} + \theta_{xi} f_{i2} + \theta_{yi} f_{i3})$$

$$\begin{cases} f_{i1} = \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(2+\xi_i\xi+\eta_i\eta-\xi^2-\eta^2) \\ f_{i2} = -\frac{1}{8} b\eta_i (1+\xi_i\xi)(1-\eta_i\eta)(1+\eta_i\eta)^2 \\ f_{i3} = \frac{1}{8} a\xi_i (1-\xi_i\xi)(1+\eta_i\eta)(1+\xi_i\xi)^2 \end{cases}$$

$$\underset{\sim}{w} = \begin{bmatrix} f_{11} & f_{12} & f_{13} & f_{21} & f_{22} & f_{23} & f_{31} & f_{32} & f_{33} & f_{41} & f_{42} & f_{43} \end{bmatrix} \begin{bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ \vdots \\ w_4 \\ \theta_{x4} \\ \theta_{y4} \end{bmatrix}$$

Generalized strain – nodal displacement

use $\underset{\sim}{\Phi} = \bar{\underset{\sim}{B}}\,\underset{\sim}{q}$ instead of $\underset{\sim}{\varepsilon} = \underset{\sim}{B}\,\underset{\sim}{q}$

$$\underset{\sim}{\Phi} = \begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}$$

$$\bar{\underset{\sim}{B}} = \left[\begin{array}{ccc|c|ccc} f_{11,xx} & f_{12,xx} & f_{13,xx} & & f_{41,xx} & f_{42,xx} & f_{43,xx} \\ f_{11,yy} & f_{12,yy} & f_{13,yy} & \cdots & f_{41,yy} & f_{42,yy} & f_{43,yy} \\ 2f_{11,xy} & 2f_{12,xy} & 2f_{13,xy} & & 2f_{41,xy} & 2f_{42,xy} & 2f_{43,xy} \end{array}\right]$$

moment – nodal displacement

$$\underset{\sim}{M} = \begin{bmatrix} M_x \\ M_y \\ M_{xy} \end{bmatrix} = \underbrace{\frac{t^3}{12}\begin{bmatrix} E_{11} & E_{12} & 0 \\ E_{21} & E_{22} & 0 \\ 0 & 0 & E_{33} \end{bmatrix}}_{\bar{\underset{\sim}{E}}} \begin{bmatrix} w_{,xx} \\ w_{,yy} \\ 2w_{,xy} \end{bmatrix}$$

$$\underset{\sim}{M} = \bar{\underset{\sim}{E}}\,\underset{\sim}{\Phi} \qquad (\underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\varepsilon})$$

$$= \bar{\underset{\sim}{E}}\,\bar{\underset{\sim}{B}}\,\underset{\sim}{q}$$

Principle of virtual displacement

$$\int \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{E}\,\underset{\sim}{\varepsilon}\, dV = \int \delta\underset{\sim}{\Phi}^T \bar{\underset{\sim}{E}}\,\underset{\sim}{\Phi}\, dA$$

$$\Rightarrow \quad \underset{\sim}{K} = \int \underset{\sim}{B}^T \underset{\sim}{E}\,\underset{\sim}{B}\, dV = \int \bar{\underset{\sim}{B}}^T \bar{\underset{\sim}{E}}\,\bar{\underset{\sim}{B}}\, dA$$

$$\underset{\sim}{K} = \int \bar{\underset{\sim}{B}}^T \underset{\sim}{E} \bar{\underset{\sim}{B}} \, dA$$

$$= ab \int_{-1}^{1}\int_{-1}^{1} \bar{\underset{\sim}{B}}^T \underset{\sim}{E} \bar{\underset{\sim}{B}} \, d\xi d\eta$$

Calculation of Bending Moments

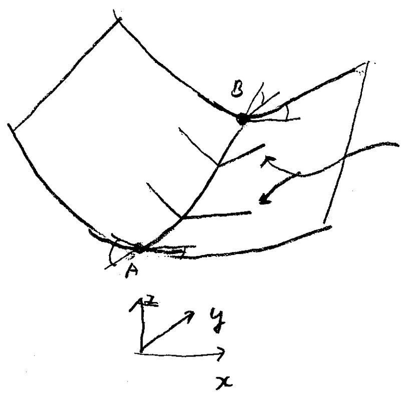
$$\underset{\sim}{M} = \underset{\sim}{E} \bar{\underset{\sim}{B}} \underset{\sim}{q}$$

Calculation of strains and stresses

$$\underset{\sim}{\varepsilon} = -z \underset{\sim}{\phi} = -z \bar{\underset{\sim}{B}} \underset{\sim}{q}$$

$$\underset{\sim}{\sigma} = \underset{\sim}{E} \underset{\sim}{\varepsilon} = -z \underset{\sim}{E} \bar{\underset{\sim}{B}} \underset{\sim}{q}$$

The MZC rectangle is nonconforming because normal slopes are not compatible, and discontinuities occur at adjoining edges. The normal slope along an edge varies cubically and is not uniquely defined by two slopes at the ends of the edges.



discontinuity in slopes

At Edge $\overline{AB}$

The cubic displacement function is defined with nodal displacements

w_A, w_B, θ_{xA} θ_{xB}

$\Rightarrow$ compatible

But the cubic slope function is defined only with θ_{yA} and θ_{yB}

$\Rightarrow$ ~~is~~ not uniquely defined

$\Rightarrow$ incompatible

BFG Rectangle

nodal displacement

$$\underset{\sim}{\delta_i} = \begin{bmatrix} \delta_{i1} \\ \delta_{i2} \\ \delta_{i3} \\ \delta_{i4} \end{bmatrix} = \begin{bmatrix} w_i \\ (w_{,y})_i \\ -(w_{,x})_i \\ (w_{,xy})_i \end{bmatrix} = \begin{bmatrix} w_i \\ \theta_{xi} \\ \theta_{yi} \\ (w_{,xy})_i \end{bmatrix}$$

$$\underset{\sim}{\delta} = \begin{bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ (w_{,xy})_1 \\ \vdots \\ w_4 \\ \theta_{x4} \\ \theta_{y4} \\ (w_{,xy})_4 \end{bmatrix}$$

nodal forces

$$\underset{\sim}{P} = \begin{bmatrix} P_1 \\ M_{x1} \\ M_{y1} \\ M_{xy1} \\ \vdots \\ P_4 \\ M_{x4} \\ M_{y4} \\ M_{xy4} \end{bmatrix}$$

$$1$$
$$\xi \quad \eta$$
$$\xi^2 \quad \xi\eta \quad \eta^2$$
$$\xi^3 \quad \xi^2\eta \quad \xi\eta^2 \quad \eta^3$$
$$\xi^3\eta \quad \xi^2\eta^2 \quad \xi\eta^3$$
$$\xi^3\eta^2 \quad \xi^2\eta^3$$
$$\xi^3\eta^3$$

16

$$w = \sum_{i=1}^{4} \left(f_{i1} w_{i1} + f_{i2} \theta_{xi} + f_{i3} \theta_{yi} + f_{i4} w_{,xy\,i} \right)$$

Generalized strain

$$\underset{\sim}{\Phi} = \begin{bmatrix} w,_{xx} \\ w,_{yy} \\ 2w,_{xy} \end{bmatrix} \qquad \underset{\sim}{\Phi} = \bar{\underset{\sim}{B}}\,\underset{\sim}{\delta}$$

$$\underset{3\times16}{\bar{\underset{\sim}{B}}} = \left[\begin{array}{cccc|c} f_{11,xx} & f_{12,xx} & f_{13,xx} & f_{14,xx} & \\ f_{11,yy} & f_{12,yy} & f_{13,yy} & f_{14,yy} & \cdots \\ 2f_{11,xy} & 2f_{12,xy} & 2f_{13,xy} & 2f_{14,xy} & \end{array}\right]$$

$$\underset{\sim}{K} = \int \bar{\underset{\sim}{B}}^T \bar{\underset{\sim}{E}}\, \bar{\underset{\sim}{B}}\, dA$$

$$= ab \iint \bar{\underset{\sim}{B}}^T \bar{\underset{\sim}{E}}\, \bar{\underset{\sim}{B}}\, d\xi\, d\eta$$

The BFS rectangle is said to be conforming because it has normal-slope compatibility at all edges. That is, the normal slope along a given edge has a cubic variation that is controlled by four parameters, which are the normal slope and warp at each end.

This element gives greater accuracy than the MZC rectangle because of a higher-order displacement function and a larger number of nodal displacements. The improvement is not due to the fact that the BFS rectangle has normal-slope compatibility.

Triangular elements

annular elements

⇒ see test book