Vibrational Analysis

principle of virtual work

virtual internal work

$$\delta V = \int_v \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, dv + \underbrace{\int \delta \underset{\sim}{u}^T \rho \ddot{\underset{\sim}{u}}\, dv}_{\text{work by inertia force}} + \underbrace{\int \delta \underset{\sim}{u}^T c \dot{\underset{\sim}{u}}\, dv}_{\text{work by damping force}}$$

virtual external work

$$\delta W = \delta \underset{\sim}{q}^T \underset{\sim}{P}(t) + \int_v \delta \underset{\sim}{u}^T \underset{\sim}{b}(t)\, dv$$

$$\underset{\sim}{u} = \underset{\sim}{f}\, \underset{\sim}{q}$$

$$\ddot{\underset{\sim}{u}} = \underset{\sim}{f}\, \ddot{\underset{\sim}{q}}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{B}\, \underset{\sim}{q}$$

$$\delta V = \delta W$$

$$\delta \underset{\sim}{q}^T \int \rho \underset{\sim}{f}^T \underset{\sim}{f}\, dv\, \ddot{\underset{\sim}{q}} + \delta \underset{\sim}{q}^T \int c \underset{\sim}{f}^T \underset{\sim}{f}\, dv\, \dot{\underset{\sim}{q}} + \delta \underset{\sim}{q}^T \int \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B}\, dv\, \underset{\sim}{q}$$
$$= \delta \underset{\sim}{q}^T \underset{\sim}{P} + \delta \underset{\sim}{q}^T \int \underset{\sim}{f}^T b\, dv$$

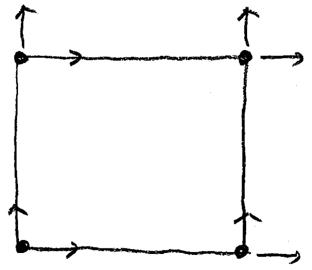
$$\underset{\sim}{M} \ddot{\underset{\sim}{q}} + \underset{\sim}{C} \dot{\underset{\sim}{q}} + \underset{\sim}{K} \underset{\sim}{q} = \underset{\sim}{P} + \underset{\sim}{P}_b$$

$$\underset{\sim}{M} = \int_v \rho\, \underset{\sim}{f}^T \underset{\sim}{f}\, dv$$ = consistent mass matrix

$\leftrightarrow$ Lumped mass matrix

$$\underset{\sim}{C} = \int_v c\, \underset{\sim}{f}^T \underset{\sim}{f}\, dv$$

$$\underset{\sim}{K} = \int_v \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B}\, dv$$



consistent mass matrix $= \begin{bmatrix} \quad & \quad \\ \quad & \quad \end{bmatrix}$

M (8×8)

Lumped mass matrix $= \begin{bmatrix} \frac{1}{4}M & & & 0 \\ & \frac{1}{4}M & & \\ & & \ddots & \\ 0 & & & \frac{1}{4}M \end{bmatrix}$

If the displacements is defined in local axes,

$$\delta' = R\,\delta$$

$$\int \delta u^T \rho \ddot{u}\, dv = \delta\delta^T R^T \int \rho f^T f\, dv\, R\, \delta$$

$$M = R^T \int \rho f^T f\, dv\, R$$

Assemble all the mass matrix to construct structural mass matrix

$$M\ddot{D} + K D = P + P_b$$

$$M\ddot{D} + C\dot{D} + K D = P + P_b$$

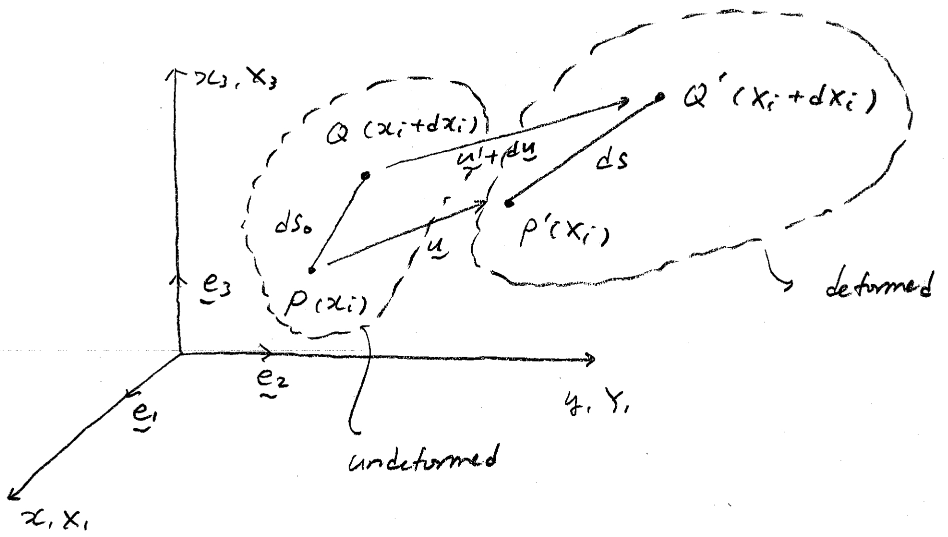
Usually, C is not defined directly, but, is defined by using Rayleigh damping method $C = a_0 K + a_1 M$

Forced vibration - Modal superposition method

Step by step integration method

Fourier Transform

Analysis of Strain



x_i - initial coordinate (Lagrangian description)

X_i - current (final) coordinate (Eulerian)

Consider Reference points P and P' on the undeformed and deformed parts and neighbors Q and Q'

P and Q separated by a distance ds_0

P' and Q' separated by a distance ds

Then,
$$(ds_0)^2 = dx_1^2 + dx_2^2 + dx_3^2 = dx_i dx_i \tag{1}$$
$$(ds)^2 = dX_1^2 + dX_2^2 + dX_3^2 = dX_i dX_i$$

The displacement vector $\underset{\sim}{u}$: $P \to P'$

$\underset{\sim}{u} + d\underset{\sim}{u}$: $Q \to Q'$

Then, $$d\underset{\sim}{s}_0 + \underset{\sim}{u} + d\underset{\sim}{u} = d\underset{\sim}{s} + \underset{\sim}{u}$$

or $$d\underset{\sim}{u} = d\underset{\sim}{s} - d\underset{\sim}{s}_0 \qquad (2)$$

from Eq. (1)

$$ds^2 - ds_0^2 = dX_i\, dX_i - dx_i\, dx_i \qquad (3)$$

Consider referencing the initial coordinates

Then the final coordinates are functions of the initial ones, i.e

$$X_i = X_i(x_1, x_2, x_3) \quad \text{and}$$

$$dX_i = \frac{\partial X_i}{\partial x_1} dx_1 + \frac{\partial X_i}{\partial x_2} dx_2 + \frac{\partial X_i}{\partial x_3} dx_3$$

$$= X_{i,j}\, dx_j \qquad (4)$$

Furthermore $\underset{\sim}{u}$ is given by $\underset{\sim}{u} = u_i \underset{\sim}{e}_i$

$$\text{and} \quad u_i = X_i - x_i \qquad (5)$$

from (3) and (4)

$$(ds)^2 - (ds_0)^2 = X_{i,j}\, dx_j\, X_{i,k}\, dx_k - dx_i dx_i$$

$$= (X_{i,j} X_{i,k} - \delta_{ij}\delta_{ik})\, dx_j\, dx_k$$

from Eq. (5)

$$= [(x_i + u_i)_{,j}\ (x_i + u_i)_{,k} - \delta_{jk}]\, dx_j\, dx_k$$

$$= [(\delta_{ij} + u_{i,j})(\delta_{ik} + u_{i,k}) - \delta_{jk}]\, dx_j\, dx_k$$

$$= [\underbrace{\delta_{ij} u_{i,k}}_{u_{j,k}} + \underbrace{\delta_{ik} u_{i,j}}_{u_{k,j}} + u_{i,j}\, u_{i,k}]\, dx_j\, dx_k$$

Rearranging indices

$$ds^2 - ds_0^2 = [u_{i,j} + u_{j,i} + u_{k,i}\, u_{k,j}]\, dx_i\, dx_j \qquad (6)$$

If we define ε_{ij}^L as the components of the Lagrangian strain tensor (Green), and

$$\underline{\varepsilon_{ij}^L = \frac{1}{2}[u_{i,j} + u_{j,i} + u_{k,i}\, u_{k,j}]} \qquad (7)$$

$$\left(u_{i,j} = \frac{\partial u_i}{\partial x_j} \right)$$

Then

$$ds^2 - ds_0^2 = 2\,\varepsilon_{ij}^L\, dx_i\, dx_j \qquad (8)$$

Now consider the Eulerian description

$$x_i = x_i(x_1, x_2, x_3)$$

$$dx_i = x_{i,j}\, dX_j \qquad (9)$$

$$\underset{\sim}{u} = u_i \underset{\sim}{E}_i \qquad u_i = X_i - x_i \qquad (10)$$

$\underset{\sim}{E}_i$ = base vectors of X_i system

from Eq. (3) and (9)

$$(ds)^2 - (ds_0)^2 = dX_i dX_i - x_{i,j} dX_j\, x_{i,k} dX_k$$

$$= (\delta_{ij}\delta_{ik} - x_{i,j}\, x_{i,k})\, dX_j dX_k$$

from (10)

$$= [\delta_{jk} - (X_i - u_i)_{,j}\, (X_i - u_i)_{,k}]\, dX_j dX_k$$

$$= [\delta_{jk} - (\delta_{ij} - u_{i,j})(\delta_{ik} - u_{i,k})]\, dX_j dX_k$$

$$= [\delta_{jk} - \delta_{jk} + u_{j,k} + u_{k,j} - u_{i,j}\, u_{i,k}]\, dX_j dX_k$$

$$= [u_{j,k} + u_{k,j} - u_{i,j}\, u_{i,k}]\, dX_j dX_k$$

or

$$ds^2 - ds_0^2 = [u_{i,j} + u_{j,i} - u_{k,i}\, u_{k,j}]\, dX_i dX_j \qquad (11)$$

Introducing the Eulerian strain Tensor (Alamansi)

$$\varepsilon_{ij}^{E} = \frac{1}{2}\left[u_{i,j} + u_{j,i} - u_{k,i}u_{k,j}\right] \qquad u_{i,j} = \frac{\partial u_i}{\partial x_j}$$

Then

$$ds^2 - ds_0^2 = 2\,\varepsilon_{ij}^{E}\,dx_i\,dx_j$$

physical interpretation

If $dx_1 = dx_3 = 0$ and $dx_2 = ds_0$

$$ds^2 - ds_0^2 = (ds - ds_0)(ds + ds_0)$$

$$\frac{ds - ds_0}{ds_0} = \frac{(ds)^2 - (ds_0)^2}{ds_0\,(ds + ds_0)}$$

$$= \frac{2\,\varepsilon_{22}\,ds_0^2}{ds_0(ds+ds_0)}$$

$$\simeq \frac{2\,\varepsilon_{22}\,ds_0^2}{2\,ds_0^2}$$

$$= \varepsilon_{22}$$

Thus, $\varepsilon_{22} = \dfrac{ds - ds_0}{ds_0}$

Linearization

If displacement gradients are small, ie if $u_{k,i} \ll 1$

$u_{k,i} \cdot u_{k,j}$ can be neglected.

Also, the initial and final coordinate systems are identical

Thus $\frac{\partial}{\partial x_i} = \frac{\partial}{\partial X_i}$

and $\varepsilon_{ij}^L = \varepsilon_{ij}^E = \varepsilon_{ij} = \frac{1}{2}\left[u_{i,j} + u_{j,i}\right]$

ε_{ij} = small strain tensor (Cauchy)

$$\varepsilon_{11} = \frac{\partial u_1}{\partial x_1}, \quad \varepsilon_{22} = \frac{\partial u_2}{\partial x_2}, \quad \varepsilon_{33} = \frac{\partial u_3}{\partial x_3}$$

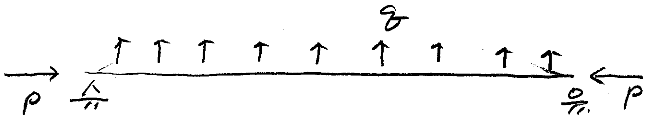
$$\varepsilon_{12} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_2} + \frac{\partial u_2}{\partial x_1}\right] = \varepsilon_{21}$$

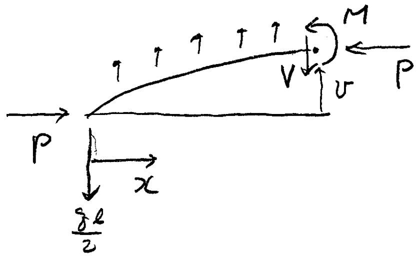
$$\varepsilon_{13} = \frac{1}{2}\left[\frac{\partial u_1}{\partial x_3} + \frac{\partial u_3}{\partial x_1}\right] = \varepsilon_{31}$$

$$\varepsilon_{23} = \frac{1}{2}\left[\frac{\partial u_2}{\partial x_3} + \frac{\partial u_3}{\partial x_2}\right] = \varepsilon_{32}$$

Geometric Nonlinear Analysis

Beam - column element





$$+\circlearrowleft M + Pv = \frac{qx^2}{2} - \frac{ql}{2}x$$

$$EI\,v'' + Pv = \frac{qx^2}{2} - \frac{ql}{2}x$$

$$EI\,v^{iv} + Pv'' = -q \Rightarrow \text{governing Eq.}$$

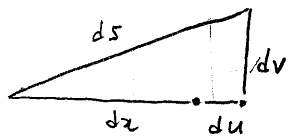
For large displacement,

$$\varepsilon_{ij} = \frac{1}{2}\left[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right]$$

$$\Rightarrow \varepsilon_x = \frac{\partial u}{\partial x} + \frac{1}{2}\left(\frac{\partial u}{\partial x}\right)^2 + \frac{1}{2}\left(\frac{\partial v}{\partial x}\right)^2$$

Interpretation

change in length due to change in geometry



$$ds^2 = (dx+du)^2 + dv^2$$

$$= dx^2\left[1 + 2\frac{du}{dx} + \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2\right]$$

$$ds = dx\sqrt{1 + 2\frac{du}{dx} + \left(\frac{du}{dx}\right)^2 + \left(\frac{dv}{dx}\right)^2}$$

$$\simeq dx\left(1 + \frac{du}{dx} + \frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2\right)$$

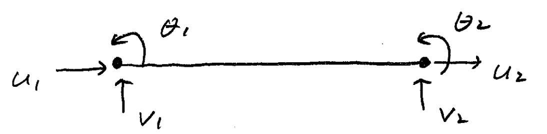
ε_x

$$= \frac{ds - dx}{dx} = \frac{du}{dx} + \underbrace{\frac{1}{2}\left(\frac{du}{dx}\right)^2 + \frac{1}{2}\left(\frac{dv}{dx}\right)^2}_{}$$

additional strain due to change in geometry total ε_x including flexural action

$$\varepsilon_x = \frac{du}{dx} + \frac{1}{2}\left(\frac{dv}{dx}\right)^2 \quad \left(\text{neglecting } \frac{1}{2}\left(\frac{du}{dx}\right)^2\right)$$

$$= \underbrace{\frac{du_0}{dx}}_{\text{Axial}} \underbrace{- y\frac{d^2v}{dx^2}}_{\text{flexural}} + \underbrace{\frac{1}{2}\left(\frac{dv}{dx}\right)^2}_{\text{large deformation}}$$



$$\underset{\sim}{q}_u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} \qquad \underset{\sim}{q}_v = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$$

$$\underset{\sim}{f}_1 = [f_{u1} \ f_{u2}] \qquad \underset{\sim}{f}_2 = [f_{v1} \ f_{v2} \ f_{v3} \ f_{v4}]$$

$$u_0 = \underset{\sim}{f}_1^T \underset{\sim}{q}_u$$

$$v = \underset{\sim}{f}_2^T \underset{\sim}{q}_v$$

Approximation : the shape function is the same after large deformation.
$\Rightarrow$ This is not true. true shape function should satisfy the force equilibrium eq.

$$\varepsilon_x = \underset{\sim}{f}_1'^T \underset{\sim}{q}_u - y\, \underset{\sim}{f}_2''^T \underset{\sim}{q}_v + \frac{1}{2} \underset{\sim}{q}_v^T \underset{\sim}{f}_2' \underset{\sim}{f}_2'^T \underset{\sim}{q}_v$$

$$\varepsilon_x = u_0' - y v'' + \frac{1}{2}(v')^2$$

$$\delta\varepsilon_x = \delta u_0' - y\,\delta v'' + v'\delta v'$$

$$\delta\varepsilon_x \sigma = (\delta u_0' - y\delta v'' + v'\delta v')\, E\, (u_0' - y v'' + \tfrac{1}{2}(v')^2)$$

$$= E(u_0' - y v'' + \tfrac{1}{2}v'^2)\,\delta u_0' - E y\,\delta v''(u_0' - y v'' + \tfrac{1}{2}v'^2)$$

$$+ E v'\delta v'(u_0' - y v'' + \tfrac{1}{2}v'^2)$$

$$\int y\, dA = 0 \ , \quad \int y^2 dA = I$$

$$\int \delta\varepsilon_x \sigma\, dA = EA(u_0' + \cancel{\tfrac{1}{2}v'^2})\,\delta u_0' + EI\, v''\delta v'' + EA\, v'\delta v'(u_0' + \cancel{\tfrac{1}{2}v'^2})$$

$$\simeq EA\, u_0'\,\delta u_0' + EI\, v''\delta v'' + EA\, v'\delta v' u_0'$$

$$u_0 = \mathbf{f}_1^T \mathbf{q}_u \quad \text{and} \quad v = \mathbf{f}_2^T \mathbf{q}_v$$

$$\int \delta\varepsilon\sigma\, dA = \delta\mathbf{q}_u^T EA\, \mathbf{f}_1'^T \mathbf{f}_1' \mathbf{q}_u + \delta\mathbf{q}_v^T EI\, \mathbf{f}_2''^T \mathbf{f}_2'' \mathbf{q}_v$$

$$+ \delta\mathbf{q}_v^T \underbrace{(EA u_0')}_{\hookrightarrow \text{ current axial force } P_0} \mathbf{f}_2'^T \mathbf{f}_2' \mathbf{q}_v$$

$$\delta U = \int \delta\varepsilon\sigma\, dV$$

$$= \delta\mathbf{q}_u^T \left[EA \int \mathbf{f}_1'^T \mathbf{f}_1' dx\right] \mathbf{q}_u + \delta\mathbf{q}_v^T \left[\left[EI \int \mathbf{f}_2''^T \mathbf{f}_2'' dx\right] \mathbf{q}_u\right.$$

$$\left. + P_0 \int \mathbf{f}_2'^T \mathbf{f}_2\, dx\right] \mathbf{q}_v$$

$$\delta U = \delta W$$

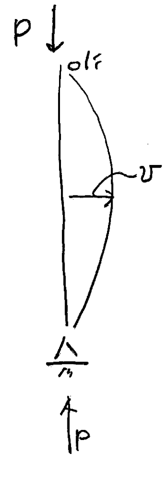
$$\underbrace{\left[EA \int \mathbf{f}_1'^T \mathbf{f}_1' dx\right]}_{\mathbf{K}_A} \mathbf{q}_u + \left[\underbrace{EI \int \mathbf{f}_2''^T \mathbf{f}_2'' dx}_{\mathbf{K}_B} + \underbrace{P_0 \int \mathbf{f}_2'^T \mathbf{f}_2' dx}_{\mathbf{G}}\right] \mathbf{q}_v = \mathbf{P}$$

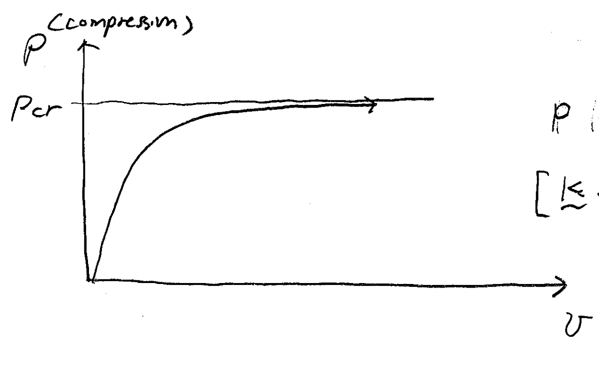
$\mathbf{K}_A$ = elastic stiffness for axial action

$\mathbf{K}_B$ = elastic stiffness for bending action

$\mathbf{G}$ = geometric stiffness (< 0 if $P_0 < 0$)

$$\left[\mathbf{K} + \mathbf{G}\right] \mathbf{q} = \mathbf{P}$$





P (comp.) $\uparrow$

$[\mathbf{K} + \mathbf{G}]$ $\downarrow$

$$[\mathbf{K} + \mathbf{G}]\, \mathbf{q} = \mathbf{P}$$

$$\Rightarrow [\mathbf{K} + \mathbf{G}]\, d\mathbf{q} = d\mathbf{P}$$

linearization (assuming the geometry is the same before buckling)

$$[\mathbf{K} + \mathbf{G}]\, d\mathbf{q} = \mathbf{0}$$

Linearized Stability Analysis

$$[\mathbf{K} + P\hat{\mathbf{G}}]\, d\mathbf{q} = \mathbf{0}$$

$\Rightarrow$ Eigenvalue problem $\Rightarrow$ solve P_{cr} (buckling force)

$d\mathbf{q}$ for P = buckling mode

$$\mathbf{G} = P_0 \int \mathbf{f}_2'^T \mathbf{f}_2' \, dx$$

$$= P_0 \begin{bmatrix} \int f_{v1}' f_{v1}' \, dx & \int f_{v1}' f_{v2}' \, dx & \int f_{v1}' f_{v3}' \, dx & \int f_{v1}' f_{v4}' \, dx \\ & \int f_{v2}' f_{v2}' \, dx & \int f_{v2}' f_{v3}' \, dx & \int f_{v2}' f_{v4}' \, dx \\ & \text{sym.} & \int f_{v3}' f_{v3}' \, dx & \int f_{v3}' f_{v4}' \, dx \\ & & & \int f_{v4}' f_{v4}' \, dx \end{bmatrix}$$

Geometric Nonlinear Analysis for plate bending

$$\varepsilon_{ij} = \frac{1}{2}\left[u_{i,j} + u_{j,i} + u_{k,i}u_{k,j}\right]$$

$$\begin{cases} \varepsilon_x = \dfrac{\partial u}{\partial x} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \quad \left(= \dfrac{1}{2}\left[u_{,x} + u_{,x} + \cancel{u_{,x}u_{,x}} + \cancel{v_{,x}v_{,x}} + w_{,x}w_{,x}\right]\right) \\ \varepsilon_y = \dfrac{\partial v}{\partial y} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \gamma_{xy} = \dfrac{\partial u}{\partial y} + \dfrac{\partial v}{\partial x} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{cases}$$

for plate bending,
$$\begin{cases} u = u_0 - z\dfrac{\partial w}{\partial x} \\ v = v_0 - z\dfrac{\partial w}{\partial y} \end{cases}$$

(membrane) plane stress

plate bending

$$\begin{cases} \varepsilon_x = \dfrac{\partial u_0}{\partial x} - z\dfrac{\partial^2 w}{\partial x^2} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial x}\right)^2 \\ \varepsilon_y = \dfrac{\partial v_0}{\partial y} - z\dfrac{\partial^2 w}{\partial y^2} + \dfrac{1}{2}\left(\dfrac{\partial w}{\partial y}\right)^2 \\ \gamma_{xy} = \dfrac{\partial u_0}{\partial y} + \dfrac{\partial v_0}{\partial x} - 2z\dfrac{\partial^2 w}{\partial x \partial y} + \dfrac{\partial w}{\partial x}\dfrac{\partial w}{\partial y} \end{cases}$$

→ additional strain due to large deformation

$$\underset{\sim}{\sigma} = \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix} \qquad \underset{\sim}{E} = \begin{bmatrix} \dfrac{E}{1-\nu^2} & \dfrac{\nu E}{1-\nu^2} & 0 \\ \dfrac{\nu E}{1-\nu^2} & \dfrac{E}{1-\nu^2} & 0 \\ 0 & 0 & G \end{bmatrix}$$

$$\sigma_x = \frac{E}{(1-\nu^2)}\left[u_{0,x} - zw_{,xx} + \nu v_{0,y} - \nu z w_{,yy} + \frac{1}{2}w_{,x}^2 + \frac{1}{2}\nu w_{,y}^2\right]$$

$$\sigma_y = \frac{E}{(1-\nu^2)}\left[\nu u_{0,x} - \nu z w_{,xx} + v_{0,y} - zw_{,yy} + \frac{1}{2}\nu w_{,x}^2 + \frac{1}{2}w_{,y}^2\right]$$

$$\tau_{xy} = G\left[u_{0,y} + v_{0,x} - 2z w_{,xy} + w_{,x}w_{,y}\right]$$

$$\int \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, dV = \int (\delta\varepsilon_x \sigma_x + \delta\varepsilon_y \sigma_y + \delta\gamma_{xy}\tau_{xy})\, dV$$

$$\delta\varepsilon_x \sigma_x = \frac{E}{(1-\nu^2)} \left[\delta u_{o,x} - z\,\delta w_{,xx} + \delta w_{,x} w_{,x} \right] \times$$

$$\left[u_{o,x} - z w_{,xx} + \nu v_{o,y} - \nu z w_{,yy} + \left(\frac{1}{2} w_{,x}^2 \overset{0 \text{ small}}{\cancel{+ \frac{1}{2}\nu w_{,y}^2}} \right) \right]$$

considering $\int z\, dA = 0$ and $\int z^2 dA = I$

$$\delta\varepsilon_x \sigma_x = \frac{E}{(1-\nu^2)} \delta u_{o,x} \left[u_{o,x} + \nu v_{o,y} \right] \quad — ①$$

$$+ \frac{E z^2}{(1-\nu^2)} \left[\delta w_{,xx} w_{,xx} + \nu\, \delta w_{,xx} w_{,yy} \right] \quad — ②$$

$$+ \frac{E}{(1-\nu^2)} \delta w_{,x} w_{,x} \left[u_{o,x} + \nu v_{o,y} \right] \quad — ③$$

① = membrane action

② = plate bending action

③ = effect of axial force on bending action

$$③ \approx \sigma_{xo}\, \delta w_{,x} w_{,x} \qquad \left(\sigma_{xo} = \frac{E}{(1-\nu^2)} [u_{o,x} + \nu v_{o,y}] \right)$$

Additional terms due to P-Δ effect

$$\begin{cases} \delta\varepsilon_x \sigma_x \Rightarrow \sigma_{xo}\, \delta w_{,x} w_{,x} \\ \delta\varepsilon_y \sigma_y \Rightarrow \sigma_{yo}\, \delta w_{,y} w_{,y} \\ \delta\gamma_{xy} \tau_{xy} \Rightarrow \tau_{xyo} (\delta w_{,x} w_{,y} + \delta w_{,y} w_{,x}) \end{cases}$$

$$\delta U = \delta W$$

$$\delta \underset{\sim}{q}_1^T \underset{\sim}{K}_1 \underset{\sim}{q}_1 + \delta \underset{\sim}{q}_2^T \underset{\sim}{K}_2 \underset{\sim}{q}_2 + \delta \underset{\sim}{q}_2^T \underset{\sim}{G} \underset{\sim}{q}_2 = \delta \underset{\sim}{q}_1^T \underset{\sim}{P}_1 + \delta \underset{\sim}{q}_2 \underset{\sim}{P}_2$$

$$\underset{\sim}{q}_1 = \begin{bmatrix} u_{01} \\ v_{01} \\ u_{02} \\ v_{02} \\ \vdots \end{bmatrix} \qquad \underset{\sim}{q}_2 = \begin{bmatrix} w_1 \\ \theta_{x1} \\ \theta_{y1} \\ w_2 \\ \vdots \end{bmatrix}$$

$$[\underset{\sim}{K}_1 + \underset{\sim}{K}_2 + \underset{\sim}{G}]\, \underset{\sim}{q} = \underset{\sim}{P}$$

$\underset{\sim}{K}_1$ = membrane stiffness

$\underset{\sim}{K}_2$ = plate bending stiffness

$\underset{\sim}{G}$ = Geometric stiffness

$$\delta U_G = \int \sigma_{x0}\, \delta w_{,x}\, w_{,x}\, dV + \int \sigma_{y0}\, \delta w_{,y}\, w_{,y}\, dV$$
$$+ \int \tau_{xy0}\, \delta w_{,x}\, w_{,y}\, dV + \int \tau_{xy0}\, \delta w_{,y}\, w_{,x}\, dV$$

$$\underset{\sim}{G} = \int \sigma_{x0} \underset{\sim}{f}_{2,x}^T \underset{\sim}{f}_{2,x}\, dV + \int \sigma_{y0} \underset{\sim}{f}_{2,y}^T \underset{\sim}{f}_{2,y}\, dV$$
$$+ \int \tau_{xy0} \underset{\sim}{f}_{2,x}^T \underset{\sim}{f}_{2,y}\, dV + \int \tau_{xy0} \underset{\sim}{f}_{2,y}^T \underset{\sim}{f}_{2,x}\, dV$$

$w = \underset{\sim}{f}_2 \cdot \underset{\sim}{q}_2$ $\qquad$ $\underset{\sim}{f}_2$ = shape function for plate bending

$$\left[\underset{\sim}{K}_1 + \underset{\sim}{K}_2 + \underset{\sim}{G} \right] \underset{\sim}{q} = \underset{\sim}{P}$$

$$\Rightarrow \left[\underset{\sim}{K}_1 + \underset{\sim}{K}_2 + \underset{\sim}{G} \right] d\underset{\sim}{q} = d\underset{\sim}{P}$$ linearization

$$\left[\underset{\sim}{K}_1 + \underset{\sim}{K}_2 + \underset{\sim}{G} \right] d\underset{\sim}{q} = \underset{\sim}{0}$$ linearized stability analysis

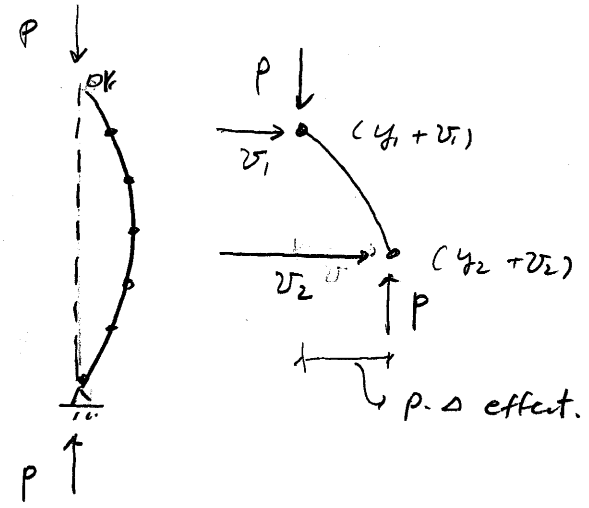
$$\left| \underset{\sim}{K}_1 + \underset{\sim}{K}_2 + \lambda \hat{\underset{\sim}{G}} \right| = 0$$ solve λ λP = buckling force

$d\underset{\sim}{q}$ = buckling mode

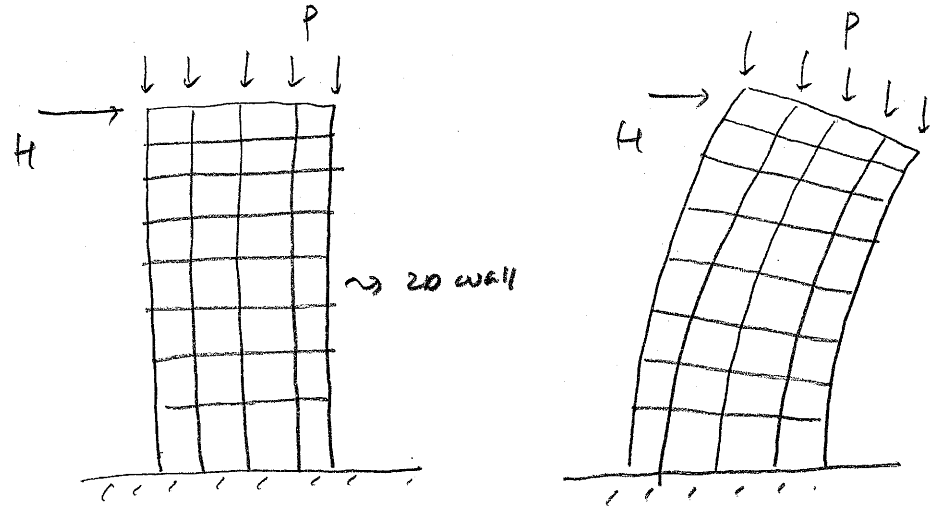
Geometric Nonlinear Analysis using updated coordinates

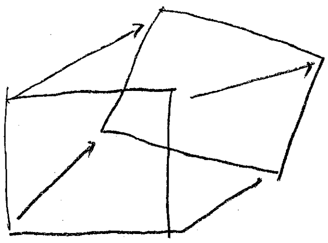
Though nonlinear analysis can be performed using FE analysis, still several assumptions are used. For example, the shape function used for small deformation is still used for large deformation. Therefore, the geometric stiffness approach can be acceptible if the deflection is limited to a moderate value.

Further, the P-Δ effect can be directly addressed when the coordinates of the structure is updated with the deflections.



Plane stress Element





$\underset{\sim}{X}^t = \underset{\sim}{X}^0 + \Delta \underset{\sim}{u}$:

$\underset{\sim}{X}^0 = \underset{\sim}{X}^t$

new $\underset{\sim}{X}^t = \underset{\sim}{X}^0 +$ new $\Delta \underset{\sim}{u}$

until $\Delta \underset{\sim}{u} \to 0$, the coordinates $\underset{\sim}{X}^t$ is updated.

Nonlinear Analysis

① General Input

② Input
- Material properties
- Geometry and node numbering
- element numbering and properties
- Load and combinations
- Nonlinear Analysis
  - control method
  - load step size $\Delta \mathbf{P}$
  - Target force or displacement $\mathbf{P}_{target}$

③ Construct element stiffness $\mathbf{k}$

Assemble $\mathbf{k}$'s to make structural stiffness $\mathbf{K}$

(in $\mathbf{K}$, geometric stiffness $\mathbf{G}$ may be included)

④ $\mathbf{K} \Delta \mathbf{U} = \Delta \mathbf{P}$

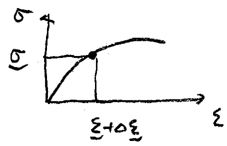
solve incremental displacement $\Delta \mathbf{U}$

⑤ Calculate incremental and/or total strains and stresses at Gauss Integration points

$\Delta \boldsymbol{\varepsilon} = \mathbf{B} \Delta \boldsymbol{\delta}$

Elastic material $\quad \boldsymbol{\sigma} = \boldsymbol{\sigma} + \Delta \boldsymbol{\sigma} \; (= \mathbf{E} \mathbf{B} \boldsymbol{\delta})$

material nonlinearity $\quad \boldsymbol{\sigma} = f(\boldsymbol{\varepsilon} + \Delta \boldsymbol{\varepsilon})$

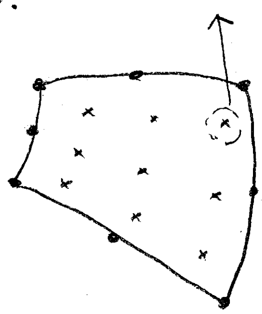


if advanced material rule is used,

σ-ε relationship is more complicated.

⑥ Calculate internal forces

$\mathbf{Q} = \Sigma \int \mathbf{B}^T \boldsymbol{\sigma} \, dV$



⑩ update data

$\mathbf{X} = \mathbf{X} + \Delta \mathbf{U}$

$\mathbf{U} = \mathbf{U} + \Delta \mathbf{U}$

$\boldsymbol{\varepsilon} = \boldsymbol{\varepsilon} + \Delta \boldsymbol{\varepsilon}$

$\boldsymbol{\sigma} = \boldsymbol{\sigma} + \Delta \boldsymbol{\sigma}$

Go to ③

⑦ Calculate unbalance force

$$\underset{\sim}{P}\ (=\underset{\sim}{P}+\Delta\underset{\sim}{P}) - \underset{\sim}{Q} = \underset{\sim}{R}$$

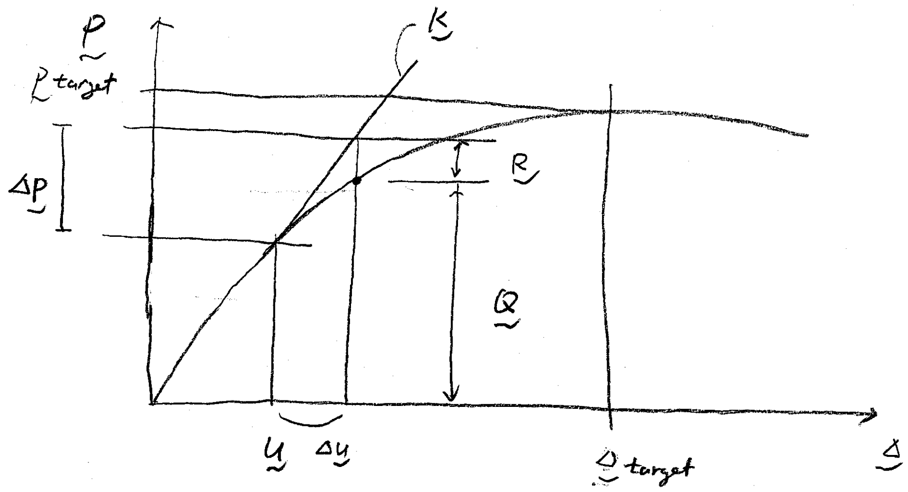
⑧ If $\underset{\sim}{R}$ is small enough, Go to ⑨

other wise $\Delta\underset{\sim}{P} = \underset{\sim}{R}$ Go to 10

⑨ CHECK $\underset{\sim}{P} \rightarrow \underset{\sim}{P}_{target}$

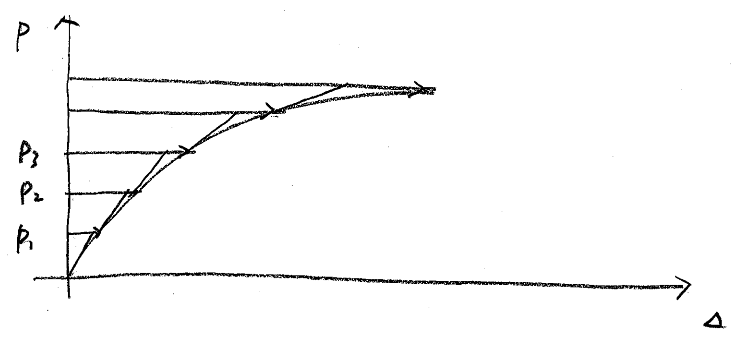
If yes ⇒ finish

If not $\Delta\underset{\sim}{P} = \Delta\underset{\sim}{P}$ Go to 10.

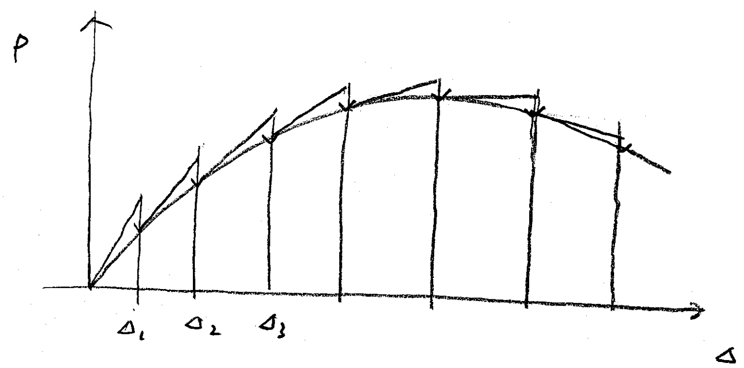


Solution scheme for nonlinear analysis

Force - control



Displacement - control



Arc-length method