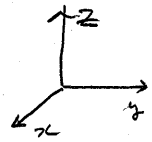
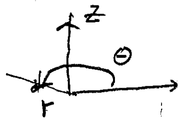
2.2 Stresses and strains in elastic continua

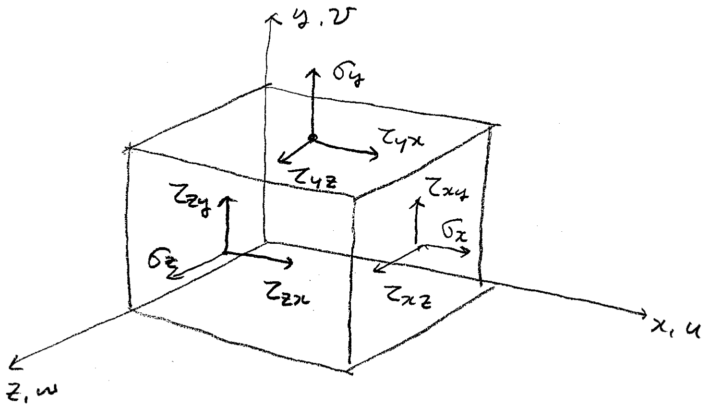
- Assumptions : linear elastic material

  small strains



- coordinate system [cartesian coordinate x, y, z
  cylindrical coordinate r, z, θ





σ = normal stress

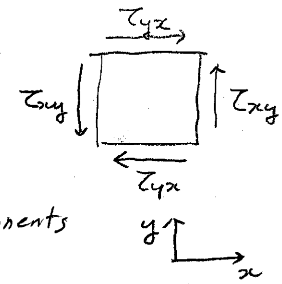
τ = shearing stress

τ_{xy} — x: direction of the plane; y: direction of the tensor (stress)

By using moment equilibrium,

$$\tau_{xy} = \tau_{yx} , \quad \tau_{yz} = \tau_{zy} , \quad \tau_{zx} = \tau_{xz}$$

9 components   6 independent components



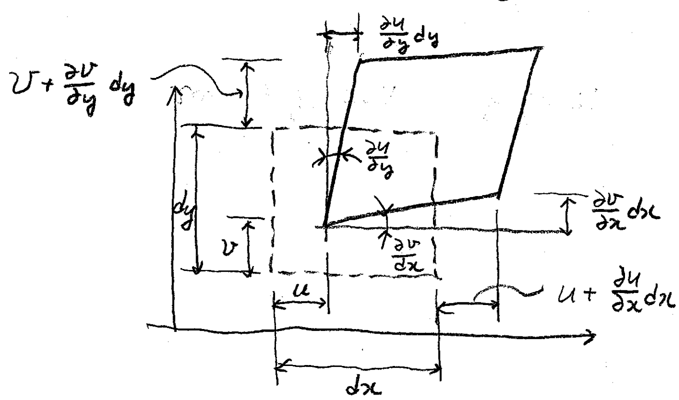
Strains

$$\varepsilon_x = \frac{\partial u}{\partial x} , \quad \varepsilon_y = \frac{\partial v}{\partial y} , \quad \varepsilon_z = \frac{\partial w}{\partial z}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \gamma_{yx} = 2\varepsilon_{xy} = 2\varepsilon_{yx}$$

$$\gamma_{yz} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y} = \gamma_{zy}$$

$$\gamma_{zx} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z} = \gamma_{xz}$$



see Fig. 2.4 p 55

$$\underset{\sim}{\sigma} = \begin{Bmatrix} \sigma_x \\ \sigma_y \\ \sigma_z \\ \tau_{xy} \\ \tau_{yz} \\ \tau_{zx} \end{Bmatrix} = \begin{Bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{Bmatrix} = \begin{Bmatrix} \sigma_1 \\ \sigma_2 \\ \sigma_3 \\ \sigma_4 \\ \sigma_5 \\ \sigma_6 \end{Bmatrix} \quad \text{or} \quad \underset{\sim}{\sigma} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}$$

$\sigma_{12} = \sigma_{21}$, $\sigma_{23} = \sigma_{32}$, $\sigma_{13} = \sigma_{31}$

6 independent components

$$\underset{\sim}{\varepsilon} = \begin{Bmatrix} \varepsilon_x \\ \varepsilon_y \\ \varepsilon_z \\ \gamma_{xy} \\ \gamma_{yz} \\ \gamma_{zx} \end{Bmatrix} \quad \text{or} \quad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$

$\underset{\sim}{\varepsilon}$ 표기시에는 전체 Energy를 표현할수 있는지에 대하여 결론필요

$$\varepsilon_{12} = \varepsilon_{21} = \tfrac{1}{2}\gamma_{12} = \tfrac{1}{2}\gamma_{21} \quad \text{etc}$$

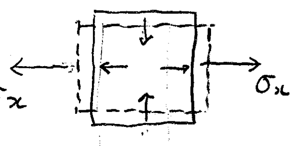
$$\text{Energy density} = \sum \sigma_{ij}\varepsilon_{ij} < \begin{matrix} 6 \text{ compont } (\gamma_{ij}) \\ 9 \text{ compont } (\varepsilon_{ij}) \end{matrix}$$

Stress-strain relationship

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_z - \nu\sigma_x)$$

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y)$$



$$\varepsilon_x = \frac{1}{E}\sigma_x, \quad \varepsilon_y = -\frac{\nu}{E}\sigma_x, \quad \varepsilon_z = -\frac{\nu}{E}\sigma_x$$

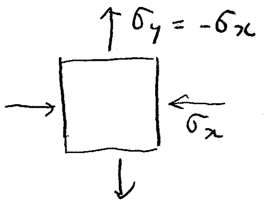
$$\gamma_{xy} = \frac{\tau_{xy}}{G}, \quad \gamma_{yz} = \frac{\tau_{yz}}{G}, \quad \gamma_{zx} = \frac{\tau_{zx}}{G}$$

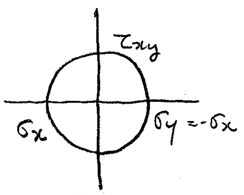
$$G = \frac{E}{2(1+\nu)}$$

E = young's modulus

G = shearing modulus

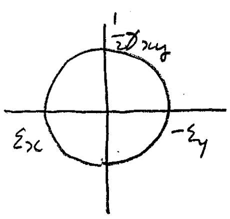
ν = Poisson's ratio

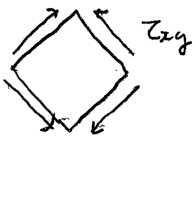




$$\varepsilon_x = \frac{\sigma_x}{E}(1+\nu)$$

$$\varepsilon_y = \frac{-\sigma_x}{E}(1+\nu)$$





$$\tau_{xy} = |\sigma_x| = |\sigma_y|$$

$$\gamma_{xy} = 2\frac{\sigma_x}{E}(1+\nu) = \frac{\tau_{xy}}{\frac{E}{2(1+\nu)}} \rightarrow G$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{C}\,\underset{\sim}{\sigma}$$

$$\underset{\sim}{C} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & -\nu & & & \\ -\nu & 1 & -\nu & & \underline{0} & \\ -\nu & -\nu & 1 & & & \\ & & & 2(1+\nu) & & \\ & \underline{0} & & & 2(1+\nu) & \\ & & & & & 2(1+\nu) \end{bmatrix}$$

$$\underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\varepsilon}$$

$$\underset{\sim}{E} = \underset{\sim}{C}^{-1} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & \nu & & & \\ \nu & (1-\nu) & \nu & & \underset{\sim}{0} & \\ \nu & \nu & (1-\nu) & & & \\ & & & \frac{(1-2\nu)}{2} & & \\ & \underset{\sim}{0} & & & \frac{(1-2\nu)}{2} & \\ & & & & & \frac{(1-2\nu)}{2} \end{bmatrix}$$

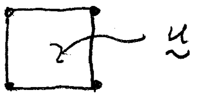
$\underset{\sim}{C}$ and $\underset{\sim}{E}$ are composed of 2 independent constants

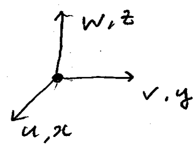
for isotropic material (등방성 재료).

1.3 Virtual Work Basis of Finite-Element method

Definition

- Generic Displacement $\underset{\sim}{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix}$





$$\underset{\sim}{u} = \begin{bmatrix} u \\ v \\ w \\ \theta_x \\ \theta_y \\ \theta_z \end{bmatrix}$$

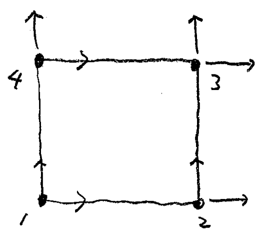
- Body forces $\underset{\sim}{b} = \begin{bmatrix} b_x \\ b_y \\ b_z \end{bmatrix}$ ex) body weights

  (loading applied to a element body)

- Nodal displacements $\underset{\sim}{\delta} = [\underset{\sim}{\delta_i}]$

$$\delta_i = \begin{bmatrix} \delta_{xi} \\ \delta_{yi} \\ \delta_{zi} \end{bmatrix} = \begin{bmatrix} u_i \\ v_i \\ w_i \end{bmatrix}$$

for 2-D element



$$\underset{\sim}{\delta} = \begin{bmatrix} \underset{\sim}{\delta_1} \\ \underset{\sim}{\delta_2} \\ \underset{\sim}{\delta_3} \\ \underset{\sim}{\delta_4} \end{bmatrix} = \begin{bmatrix} \delta_{x1} \\ \delta_{y1} \\ \vdots \\ \delta_{x4} \\ \delta_{y4} \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_4 \\ v_4 \end{bmatrix}$$

Sometimes, rotations $\left(\frac{\partial v}{\partial x}\right)$ and curvatures $\left(\frac{\partial^2 v}{\partial x^2}\right)$ can be included in nodal displacements.

- Nodal forces $\underset{\sim}{P} = [\underset{\sim}{P_i}]$ $P_i = \begin{bmatrix} P_{xi} \\ P_{yi} \\ P_{zi} \end{bmatrix}$

- Relationship between Generic Displacement and nodal displacement

$\underset{\sim}{u} = \underset{\sim}{f}\,\underset{\sim}{\delta}$ $\underset{\sim}{f}$ = displacement shape function (function of x, y, z)

$$u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4$$
$$v = f_1 v_1 + f_2 v_2 + f_3 v_3 + f_4 v_4$$

$f_i(x, y)$ for 2-D element

$$\begin{bmatrix} u \\ v \end{bmatrix} = \begin{bmatrix} f_1 & 0 & f_2 & 0 & f_3 & 0 & f_4 & 0 \\ 0 & f_1 & 0 & f_2 & 0 & f_3 & 0 & f_4 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}$$

- Strain - displacement relationship

$$\underset{\sim}{\varepsilon} = \underset{\sim}{d}\,\underset{\sim}{u}$$

$$\underbrace{\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}}_{\underset{\sim}{\varepsilon}} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{\underset{\sim}{d}} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\underset{\sim}{u}}$$

- Strain - nodal displacement

$$\underset{\sim}{\varepsilon} = \underset{\sim}{d}\,\underset{\sim}{u} = \underset{\sim}{d}\,\underset{\sim}{f}\,\underset{\sim}{q} = \underset{\sim}{B}\,\underset{\sim}{q}$$

$$\underbrace{\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}}_{\underset{\sim}{\varepsilon}} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{\underset{\sim}{d}} \underbrace{\begin{bmatrix} f_1 & 0 & f_2 & 0 & f_3 & 0 & f_4 & 0 \\ 0 & f_1 & 0 & f_2 & 0 & f_3 & 0 & f_4 \end{bmatrix}}_{\underset{\sim}{f}} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_4 \\ v_4 \end{bmatrix}}_{\underset{\sim}{q}}$$

$$= \underbrace{\left[\begin{array}{cc|cc|cc|cc} \frac{\partial f_1}{\partial x} & 0 & \frac{\partial f_2}{\partial x} & 0 & \frac{\partial f_3}{\partial x} & 0 & \frac{\partial f_4}{\partial x} & 0 \\ 0 & \frac{\partial f_1}{\partial y} & 0 & \frac{\partial f_2}{\partial y} & 0 & \frac{\partial f_3}{\partial y} & 0 & \frac{\partial f_4}{\partial y} \\ \frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial x} & \frac{\partial f_4}{\partial y} & \frac{\partial f_4}{\partial x} \end{array}\right]}_{\underset{\sim}{B}} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_4 \\ v_4 \end{bmatrix}}_{\underset{\sim}{q}}$$

- Stress - strain relationship

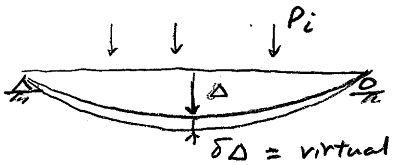
$$\underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\varepsilon}$$

- Stress - nodal displacement

$$\underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\varepsilon} = \underset{\sim}{E}\,\underset{\sim}{B}\,\underset{\sim}{q}$$

Principle of Virtual Displacement

If a general structure in equilibrium is subjected to a system of small virtual displacements within a compatible state of deformation, the virtual work of external forces is equal to the virtual strain energy of internal stresses.

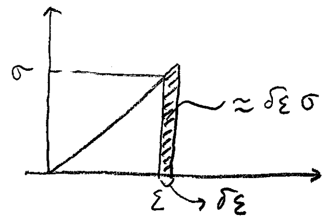


$$\delta U = \delta W$$

δU = virtual strain energy due to the virtual displacement

δW = virtual external work

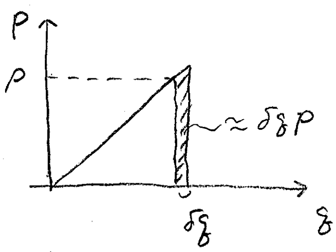
$$\delta U = \int \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, dV$$



$$\left\{\begin{aligned} \underset{\sim}{\sigma} &= \underset{\sim}{E}\,\underset{\sim}{B}\,\underset{\sim}{\delta} \\ \delta\underset{\sim}{\varepsilon} &= \underset{\sim}{B}\,\delta\underset{\sim}{\delta} \\ \delta\underset{\sim}{\varepsilon}^T &= \delta\underset{\sim}{\delta}^T\,\underset{\sim}{B}^T \end{aligned}\right.$$

$$\delta U = \int \delta\underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}\, dV = \int \delta\underset{\sim}{\delta}^T \underset{\sim}{B}^T \underset{\sim}{E}\,\underset{\sim}{B}\,\underset{\sim}{\delta}\, dV$$

$$= \delta\underset{\sim}{\delta}^T \left[\int \underset{\sim}{B}^T \underset{\sim}{E}\,\underset{\sim}{B}\, dV\right] \underset{\sim}{\delta}$$



$$\delta W = \delta\underset{\sim}{\delta}^T \underset{\sim}{P} + \int \delta\underset{\sim}{u}^T \underset{\sim}{b}\, dV$$

$$= \delta\underset{\sim}{\delta}^T \underset{\sim}{P} + \delta\underset{\sim}{\delta}^T \int \underset{\sim}{f}^T \underset{\sim}{b}\, dV$$

$\underset{\sim}{P}$: nodal forces

$\underset{\sim}{b}$: body forces

$$\delta U = \delta W$$

$$\cancel{\delta q^T}\left[\int B^T E\, B\, dV\right] q = \cancel{\delta q^T} P + \cancel{\delta q^T} \int f^T b\, dV$$

$$K\, q = P + P_b$$

$$K = \int B^T E\, B\, dV = \text{element stiffness matrix}$$

$$P_b = \int f^T b\, dV = \text{equivalent nodal loads due to body forces}$$

Dynamic Equilibrium Equation $\quad M\ddot{q} + C\dot{q} + K q = P(t)$

virtual energy due to inertia force

$$\int \delta u^T \underbrace{(\rho\, \ddot{u})}_{\text{inertia force}} dV = \delta q^T \underbrace{\left[\int f^T \rho\, f\, dV\right]}_{} \ddot{q} \qquad \rho = \text{density}$$

Generalized mass, Consistent mass $= M$

virtual energy due to damping force

$$\int \delta u^T \underbrace{(c\, \dot{u})}_{\text{damping force}} dV = \delta q^T \underbrace{\left[\int f^T c\, f\, dV\right]}_{} \dot{q}$$

Generalized damping force, consistent damping $= C$

Effect of initial strain

$$\varepsilon = \varepsilon_0 + c\,\sigma \qquad \varepsilon_0 = \text{initial strain (= strains that have no relation with stress)}$$

$$\sigma = c^{-1}(\varepsilon - \varepsilon_0)$$

$$= E\,(\varepsilon - \varepsilon_0) \qquad \varepsilon - \varepsilon_0 = \text{net strain}$$

$$\delta U = \int \delta\varepsilon^T \sigma\, dV$$

$$= \delta q^T \int B^T E\,(\varepsilon - \varepsilon_0)\, dV$$

$$= \delta q^T \int B^T E\, \varepsilon\, dV - \delta q^T \int B\, E\, \varepsilon_0\, dV$$

$$= \delta q^T \left[\int B^T E\, B\, dV\right] q - \delta q^T \underbrace{\left[\int B\, E\, \varepsilon_0\, dV\right]}_{}$$

From $\delta U = \delta W$

$$\underset{\sim}{K}\,\underset{\sim}{u} = \underset{\sim}{P} + \underset{\sim}{P}_b + \underset{\sim}{P}_o$$

$$P_o = \int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{\varepsilon}_o\, dv$$ = Equivalent nodal force due to initial strains

Principle of stationary total potential Energy

of all the possible systems of compatible displacements, the one that produces forces which are in equilibrium is the one that makes the total potential energy stationary

$$\pi = U + V$$

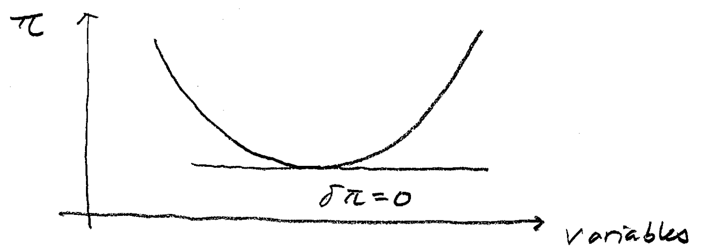
π = total potential energy

U = strain energy

V = change in potential

(− work done by the specified forces)

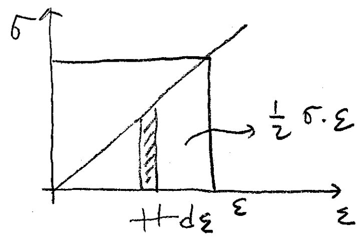
$\delta\pi = 0$ and $\delta^2\pi > 0$ generates equilibrium equation.



$$U = \int\int_0^{\varepsilon} \underset{\sim}{\sigma}^T d\underset{\sim}{\varepsilon}\, dV \qquad \underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\varepsilon}$$

$$= \frac{1}{2}\int \underbrace{(\underset{\sim}{E}\,\underset{\sim}{\varepsilon})^T}_{\sigma} \underset{\sim}{\varepsilon}\, dV$$

$$= \frac{1}{2}\int \underset{\sim}{\varepsilon}^T \underset{\sim}{E}\,\underset{\sim}{\varepsilon}\, dV$$



$$V = -\underset{\sim}{g}^T \underset{\sim}{P}$$

$$U = \frac{1}{2}\int \boldsymbol{\varepsilon}^T \mathbf{E}\, \boldsymbol{\varepsilon}\, dV$$

$$= \frac{1}{2} \boldsymbol{\delta}^T \left[\int \mathbf{B}^T \mathbf{E} \mathbf{B}\, dV\right] \boldsymbol{\delta}$$

$$\delta U = \frac{1}{2}\delta\boldsymbol{\delta}^T \left[\int \mathbf{B}^T \mathbf{E} \mathbf{B}\, dV\right] \boldsymbol{\delta} + \frac{1}{2}\boldsymbol{\delta}^T \left[\int \mathbf{B}^T \mathbf{E} \mathbf{B}\, dV\right] \delta\boldsymbol{\delta}$$

$$= \delta\boldsymbol{\delta}^T \left[\int \mathbf{B}^T \mathbf{E} \mathbf{B}\, dV\right] \boldsymbol{\delta}$$

$$\delta V = -\delta\boldsymbol{\delta}^T \mathbf{P}$$

$$\delta\pi = 0 \Rightarrow \delta U = -\delta V$$

$$\cancel{\delta\boldsymbol{\delta}^T} \left[\underbrace{\int \mathbf{B}^T \mathbf{E} \mathbf{B}\, dV}_{\mathbf{K}}\right] \boldsymbol{\delta} = \cancel{\delta\boldsymbol{\delta}^T} \cdot \mathbf{P}$$

$$\mathbf{K}\,\boldsymbol{\delta} = \mathbf{P}$$

$$\delta^2\pi \simeq \delta\boldsymbol{\delta}^T \left[\int \mathbf{B}^T \mathbf{E} \mathbf{B}\, dV\right] \delta\boldsymbol{\delta}$$

$$= \delta\boldsymbol{\delta}^T \mathbf{K}\, \delta\boldsymbol{\delta} > 0$$

Since $\mathbf{K}$ is positive definite.

if $\mathbf{K}$ is symmetric and diagonals > 0, $\mathbf{a}^T \mathbf{K} \mathbf{a} > 0$

Therefore from a definition of matrix calculation

$$\delta^2\pi = \delta\boldsymbol{\delta}^T \mathbf{K}\, \delta\boldsymbol{\delta} > 0$$