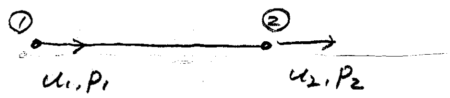
One dimensional Element

- Truss (Axial) Element
- Beam (flexural)
- Torsional Element

Truss (Axial) Element + Beam (flexural) → Beam-column element



Generic Displacement $\quad \underset{\sim}{u} = u$

body force $\quad \underset{\sim}{b} = b_x$

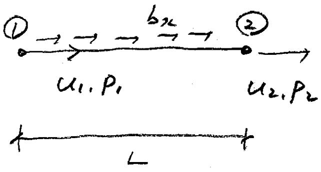
nodal displacement $\quad \underset{\sim}{q} = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$

Stress - strain $\quad \sigma = E\,\varepsilon$

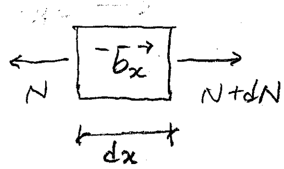
Strain - Generic displacement $\quad \underset{\sim}{\varepsilon} = \varepsilon = \frac{du}{dx} = \underbrace{\frac{d}{dx}}_{\underset{\sim}{d}} \underset{\sim}{u}$

Generic and nodal displacements $\quad u = \underset{\sim}{f}\, \underset{\sim}{q}$

Strain - nodal displacement $\quad \varepsilon = \frac{d}{dx} u = \underbrace{\frac{d}{dx} \underset{\sim}{f}}_{B}\, \underset{\sim}{q}$



$u = C_1 + C_2 x$



$N + dN - N + b_x dx = 0 \qquad \frac{dN}{dx} = -b_x$

$\frac{du}{dx} = \varepsilon \qquad N = A \cdot \sigma = A \cdot E \cdot \varepsilon = A \cdot E \frac{du}{dx}$

$\frac{dN}{dx} = A \cdot E \frac{d^2 u}{dx^2} = -b_x$

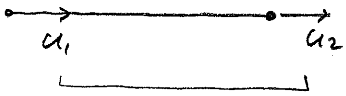
if A, E are constants and $b_x = 0$,

$\frac{d^2 u}{dx^2} = 0$

Therefore $u (= C_1 + C_2 x)$ satisfies the equilibrium equation, and $u (= C_1 + C_2 x)$ is the exact displacement function.

However, even if $u\ (= c_1 + c_2 x)$ does not the equilibrium equation, in most cases, the stiffness-based formulation can produce reliable results from the view point of engineering.

Also, practically, u should have polynomials whose numbers are the same as those of the d.o.f's so that the constants (c_1 and c_2) are expressed with the d.o.f's.



$$u = \underbrace{c_1 + c_2 x}$$

2 dof's ═══ 2 terms

$$u = c_1 + c_2 x$$

at $x = 0$, $\quad u = u_1 = c_1$

at $x = L$, $\quad u = u_2 = c_1 + c_2 L$

$$\begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

$$\begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \frac{1}{L} \begin{bmatrix} L & 0 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

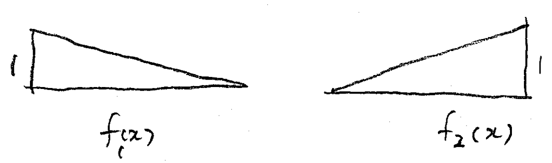
$$c_1 = \frac{u_1}{L}, \quad c_2 = \frac{1}{L}(u_2 - u_1)$$

$$u = u_1 + \left(\frac{u_2 - u_1}{L}\right) x$$

$$u = \left(1 - \frac{x}{L}\right) u_1 + \frac{x}{L} u_2 = f_1 u_1 + f_2 u_2$$

$$= \left[\left(1 - \frac{x}{L}\right) \quad \frac{x}{L}\right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix} = \underbrace{[f_1 \ f_2]}_{\underset{\sim}{f}} \underbrace{\begin{bmatrix} u_1 \\ u_2 \end{bmatrix}}_{\underset{\sim}{q}}$$

$$\underset{\sim}{f} = [f_1 \ f_2] = \left[\left(1 - \frac{x}{L}\right) \ \frac{x}{L}\right]$$



$$\underset{\sim}{\varepsilon} = \varepsilon = \frac{du}{dx} = \underbrace{\frac{d}{dx}\underset{\sim}{f}}_{\underset{\sim}{B}}\, \underset{\sim}{q}$$

$$\underset{\sim}{B} = \frac{d}{dx}\underset{\sim}{f} = \frac{d}{dx}\left[(1-\frac{x}{L}) \quad \frac{x}{L} \right]$$

$$= \left[-\frac{1}{L} \quad \frac{1}{L} \right]$$

$$\varepsilon = \underset{\sim}{B}\,\underset{\sim}{q} = \left[-\frac{1}{L} \quad \frac{1}{L} \right] \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}$$

$$\sigma = E\underset{\sim}{\varepsilon} = E\underset{\sim}{B}\,\underset{\sim}{q}$$

$$\underset{\sim}{K} = \int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dV$$

$$= \int \frac{E}{L^2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} [-1 \quad 1]\, dV$$

$$= \frac{E}{L^2} \begin{bmatrix} -1 \\ 1 \end{bmatrix} [-1 \quad 1] \int_L \int_A dA\, dx$$

$$\underline{\underset{\sim}{K} = \frac{EA}{L} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}}$$

$$\underset{\sim}{u} = c_1 + c_2 x = \underset{\sim}{g}\,\underset{\sim}{c}$$

g : Geometric matrix $= [1 \quad x]$

c : Generalized displacements $= \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$

$$\underset{\sim}{q} = \underset{\sim}{h}\,\underset{\sim}{c}$$

$$\underset{\sim}{c} = \underset{\sim}{h}^{-1}\underset{\sim}{q}$$

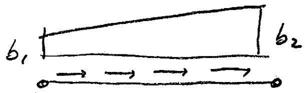
$$\underset{\sim}{u} = \underset{\sim}{g}\,\underset{\sim}{c} = \underset{\sim}{g}\,\underset{\sim}{h}^{-1}\underset{\sim}{q} = \underset{\sim}{f}\,\underset{\sim}{q}$$

$$\underset{\sim}{f} = \underset{\sim}{g}\,\underset{\sim}{h}^{-1}$$

$$= [1 \quad x] \begin{bmatrix} 1 & 0 \\ 1 & L \end{bmatrix}^{-1} = \left[(1-\frac{x}{L}) \quad (\frac{x}{L}) \right]$$

P_b : due to linearly varying distributed load b_x

$$b_x = b_1 + \frac{(b_2 - b_1)x}{L}$$

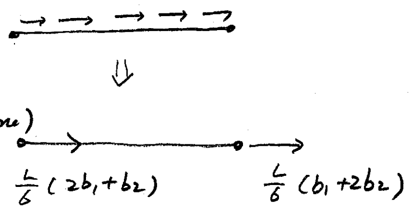


$$P_b = \int_0^L \underset{\sim}{f}^T \underset{\sim}{b}\, dx$$

$$= \int_0^L \begin{bmatrix} (1-\frac{x}{L}) \\ \frac{x}{L} \end{bmatrix} \left(b_1 + \frac{(b_2-b_1)}{L} x \right) dx$$

$$= \frac{L}{6} \begin{bmatrix} 2b_1 + b_2 \\ b_1 + 2b_2 \end{bmatrix} = -F_{EF}$$

(Fixed End force)



P_0 due to temperature change ΔT

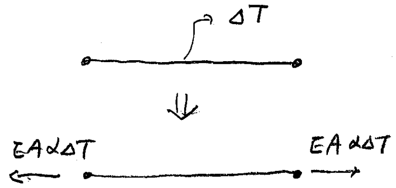
$$\varepsilon_0 = \varepsilon_T = \alpha(\Delta T) \qquad \alpha = \text{coefficient of thermal expansion}$$

$$P_0 = P_T = \int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{\varepsilon_0}\, dV$$

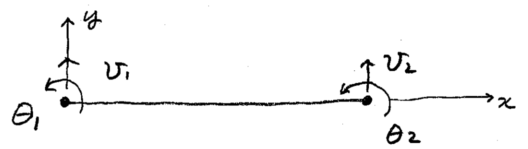
$$= \int_0^L \int_A \underset{\sim}{B}^T E\, \alpha(\Delta T)\, dA\, dx$$

$$= \frac{EA\alpha\Delta T}{L} \begin{bmatrix} -1 \\ 1 \end{bmatrix} \int_0^L dx$$

$$= EA\alpha\Delta T \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$



Flexural Element



Generic displacement $\quad \underset{\sim}{u} = v \quad \left(\theta = \frac{dv}{dx} \right)$

body force $\quad \underset{\sim}{b} = b_y$ (force per unit length)

nodal displacement $\quad \underset{\sim}{q} = \begin{bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{bmatrix}$

Generic - nodal displacement

$$v = \underset{\sim}{f}\,\underset{\sim}{q} = f_1 v_1 + f_2 \theta_1 + f_3 v_2 + f_4 \theta_2$$

strain - Generic displacement

$$\underset{\sim}{\varepsilon} = \varepsilon = -y\,\phi = -y \frac{d^2 v}{dx^2} = \underbrace{\left(-y \frac{d^2}{dx^2}\right)}_{\underset{\sim}{d} = d} v$$

strain - nodal displacement

$$\varepsilon = \underset{\sim}{d}\,\underset{\sim}{u} = \underset{\sim}{d}\,\underset{\sim}{f}\,\underset{\sim}{q}$$

$$\underset{\sim}{B} = \underset{\sim}{d}\,\underset{\sim}{f} = -y \frac{d^2}{dx^2} \left[f_1 \;\; f_2 \;\; f_3 \;\; f_4 \right]$$

stress - strain $\quad \sigma = E\,\varepsilon$

Assumed displacement function $\qquad \frac{d^2 M}{dx^2} = b_y \,,\; \frac{d^2 v}{dx^2} = \frac{M}{EI}$

$$v = C_1 + C_2 x + C_3 x^2 + C_4 x^3 \quad \Rightarrow \text{exact, satisfying } \frac{d^4 v}{dx^4} = 0$$

if I, E are constants. $b_y = 0$

$$\frac{dv}{dx} = C_2 + 2 C_3 x + 3 C_4 x^2 \qquad \left(\frac{d^2 M}{dx^2} = b_y \,,\; \frac{M}{EI} = \frac{d^2 v}{dx^2} \right)$$

$$\Rightarrow \frac{d^4 v}{dx^4} = b_y$$

Boundary conditions
$$\begin{cases} v(x=0) = v_1 \\ v'(x=0) = \theta_1 \\ v(x=L) = v_2 \\ v'(x=L) = \theta_2 \end{cases}$$

$$\underset{\sim}{q} = \begin{Bmatrix} v_1 \\ \theta_1 \\ v_2 \\ \theta_2 \end{Bmatrix} = \underbrace{\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & L & L^2 & L^3 \\ 0 & 1 & 2L & 3L^2 \end{bmatrix}}_{\underset{\sim}{h}} \underbrace{\begin{Bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{Bmatrix}}_{\underset{\sim}{c}}$$

$$\underset{\sim}{c} = \underset{\sim}{h}^{-1} \underset{\sim}{q}$$

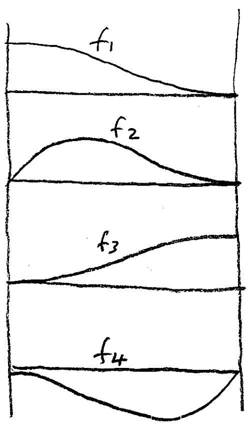
$$v = \underset{\sim}{g}\, \underset{\sim}{c} = \underbrace{\underset{\sim}{g}\, \underset{\sim}{h}^{-1}}_{\underset{\sim}{f}} \underset{\sim}{q}$$

$$\underset{\sim}{g} = [\,1 \quad x \quad x^2 \quad x^3\,]$$

$$\underset{\sim}{h}^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ -\frac{3}{L^2} & -\frac{2}{L} & \frac{3}{L^2} & -\frac{1}{L} \\ \frac{2}{L^3} & \frac{1}{L^2} & -\frac{2}{L^3} & \frac{1}{L^2} \end{bmatrix}$$

$$\underset{\sim}{f} = [\,f_1 \quad f_2 \quad f_3 \quad f_4\,]$$

Hermittian polynomials
$$\begin{cases} f_1 = 1 - \frac{3}{L^2}x^2 + \frac{2}{L^3}x^3 \\ f_2 = x - \frac{2}{L}x^2 + \frac{x^3}{L^2} \\ f_3 = \frac{3}{L^2}x^2 - \frac{2}{L^3}x^3 \\ f_4 = -\frac{x^2}{L} + \frac{x^3}{L^2} \end{cases}$$



$f_1(0) = 1$ $f_1'(0) = f_1(L) = f_1'(L) = 0$

⋮

$f_4'(L) = 1$

$f_4(0) = f_4'(0) = f_4(L) = 0$

$$\varepsilon = -y\,\phi = -y\,\frac{d^2 v}{dx^2} = \underbrace{\left(-y\,\frac{d^2 \mathbf{f}}{dx^2}\right)}_{\mathbf{B}} \mathbf{q}$$

$$\mathbf{B} = -y\left[\, f_1'' \quad f_2'' \quad f_3'' \quad f_4'' \,\right] = -y\,\bar{\mathbf{B}}$$

$$f_1'' = \left(-\frac{6}{L^2} + \frac{12}{L^3}x\right)$$

$$f_2'' = \left(-\frac{4}{L} + \frac{6}{L^2}x\right)$$

$$f_3'' = \left(\frac{6}{L^2} - \frac{12}{L^3}x\right)$$

$$f_4'' = \left(-\frac{2}{L} + \frac{6}{L^2}x\right)$$

$$\delta\varepsilon = -y\,\delta\phi = \mathbf{B}\,\delta\mathbf{q} = -y\,\bar{\mathbf{B}}\,\delta\mathbf{q}$$

$$\sigma = E\varepsilon = -Ey\phi = +E\,\mathbf{B}\,\mathbf{q} = -E\,y\,\bar{\mathbf{B}}\,\mathbf{q}$$

$$\delta U = \int \delta\boldsymbol{\varepsilon}^T \boldsymbol{\sigma}\, dV$$

$$= \int \delta\mathbf{q}^T \mathbf{B}\, E\, \mathbf{B}\, \mathbf{q}\, dV$$

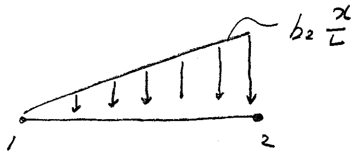
$$= \delta\mathbf{q}^T \left[\int \mathbf{B}\, E\, \mathbf{B}\, dV\right] \mathbf{q}$$

$$= \delta\mathbf{q}^T \left[\iint E y^2\, \bar{\mathbf{B}}^T \bar{\mathbf{B}}\, dA\, dx\right] \mathbf{q}$$

$$= \delta\mathbf{q}^T \underbrace{\left[EI \int \bar{\mathbf{B}}^T \bar{\mathbf{B}}\, dx\right]}_{\mathbf{K}} \mathbf{q}$$

$$\mathbf{K} = \frac{EI}{L^3}\begin{bmatrix} 12 & 6L & -12 & 6L \\ 6L & 4L^2 & -6L & 2L^2 \\ -12 & -6L & 12 & -6L \\ 6L & 2L^2 & -6L & 4L^2 \end{bmatrix}$$

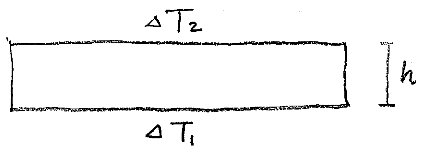
Body force $\quad b_y = b_2 \frac{x}{L}$

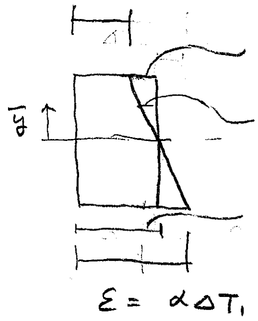


$$P_b = \int_0^L \underset{\sim}{f}^T b_y \, dx$$

$$= \begin{bmatrix} \int_0^L b_2 \frac{x}{L} f_1(x)\, dx \\ \int_0^L b_2 \frac{x}{L} f_2(x)\, dx \\ \int_0^L b_2 \frac{x}{L} f_3(x)\, dx \\ \int_0^L b_2 \frac{x}{L} f_4(x)\, dx \end{bmatrix}$$

Initial Strain





$\varepsilon = \frac{\alpha}{2}(\Delta T_2 - \Delta T_1)$

$\varepsilon_T = \frac{-\alpha \bar{y}}{h} (\Delta T_1 - \Delta T_2)$

$\varepsilon_{avg} = \frac{\alpha}{2} (\Delta T_1 + \Delta T_2)$

↳ produces equivalent nodal force ~~for~~ truss element.

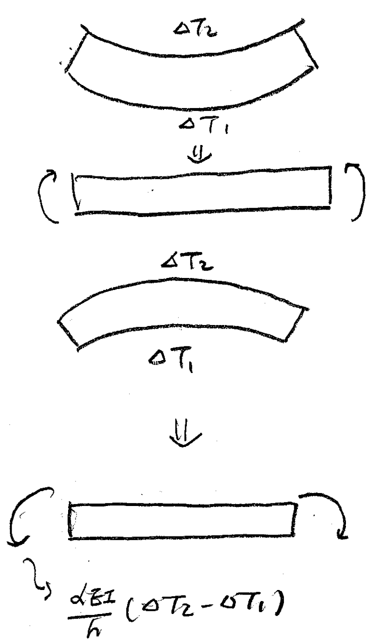
$$P_0 = \int \underset{\sim}{B}^T E \, \varepsilon_0 \, dV \qquad P_0 = P_T \qquad \varepsilon_0 = \varepsilon_T$$

$$P_T = \int \underset{\sim}{B}^T E \frac{\alpha \bar{y}}{h} (\Delta T_2 - \Delta T_1)\, dA\, dx$$

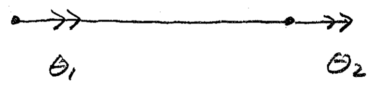
$$= \int \bar{\underset{\sim}{B}}^T E \frac{\alpha}{h} (-\bar{y}^2) (\Delta T_2 - \Delta T_1)\, dA\, dx$$

$$= \frac{\alpha EI}{h} (\Delta T_2 - \Delta T_1) \begin{bmatrix} -\int f_1'' dx \\ -\int f_2'' dx \\ -\int f_3'' dx \\ -\int f_4'' dx \end{bmatrix}$$

$$= \frac{\alpha EI}{h} (\Delta T_2 - \Delta T_1) \begin{bmatrix} 0 \\ +1 \\ 0 \\ -1 \end{bmatrix}$$



Torsional Element

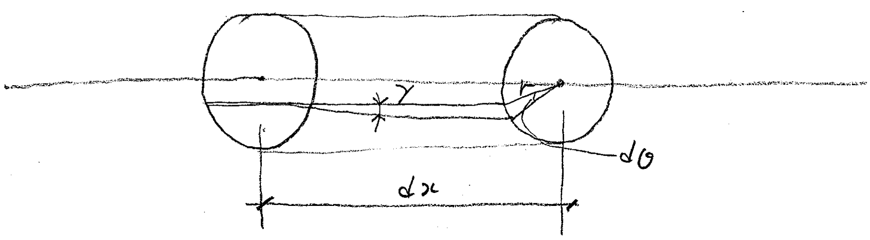


Generic Displacement $\quad \underset{\sim}{u} = \theta_x$

Body force $\quad \underset{\sim}{b} = m_x$

Stress - strain $\quad \tau = G\gamma \quad$ (St. Venant Torsion only)
not including warping torsion

Strain - Generic Displacement



$$\gamma\, dx = r\, d\theta \qquad \underline{\gamma = r\frac{d\theta}{dx}} = \underline{r\frac{d}{dx}}\,\theta$$
$$\underline{d}$$

Generic Displ. - nodal Displ.

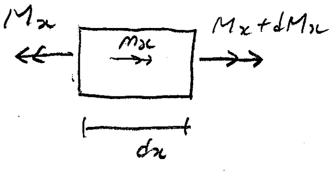
$$\theta_x = \underset{\sim}{f}\,\underset{\sim}{q} = [f_1 \;\; f_2]\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$$

$$\theta = C_1 + C_2 x$$
$$= [1 \quad x]\begin{bmatrix}C_1\\ C_2\end{bmatrix}$$

$$\left\{\begin{matrix}\theta(x=0) = \theta_1\\ \theta(x=L) = \theta_2\end{matrix}\right\} \Rightarrow$$

$$\theta = \left[(1-\tfrac{x}{L}) \quad (\tfrac{x}{L})\right]\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}$$

Equilibrium



$$-M_x + (M_x + dM_x) + m_x\, dx = 0$$

$$\frac{dM_x}{dx} = -m_x \qquad \left(M = GJ\frac{d\theta}{dx}\right)$$

$$\tau = \frac{M_x r}{J} \qquad M_x = \frac{\tau \cdot J}{r}$$

if G, J are constants and

$m_x = 0$,

$$\frac{d^2\theta}{dx^2} = 0$$

strain - nodal displacement

$$\gamma = r\frac{d\theta}{dx} = r\frac{d\underset{\sim}{f}}{dx}\cdot\underset{\sim}{q}$$

$$= \underbrace{r\begin{bmatrix}-\frac{1}{L} & \frac{1}{L}\end{bmatrix}}_{\underset{\sim}{B}}\underbrace{\begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}}_{\underset{\sim}{q}} = r\bar{\underset{\sim}{B}}\underset{\sim}{q}$$

$$K = \int \underset{\sim}{B}^T \underset{\sim}{E}\underset{\sim}{B}\, dV$$

$$= \int_L\int_A r^2 G\, \bar{\underset{\sim}{B}}^T\bar{\underset{\sim}{B}}\, dA\, dx$$

$$= GJ\int_L \bar{\underset{\sim}{B}}^T\bar{\underset{\sim}{B}}\, dx$$

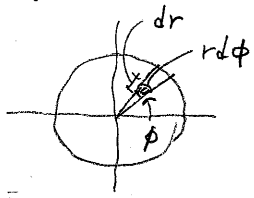
$$= \frac{GJ}{L}\begin{bmatrix}1 & -1\\ -1 & 1\end{bmatrix}$$

for circular cross-section

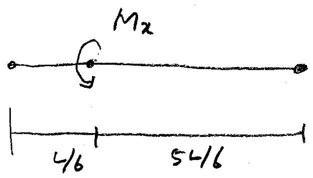
$$\int r^2 dA = \iint r^2\, dr\cdot r\,d\phi$$

$$= \int_0^{2\pi}\int_0^R r^3 dr\, d\phi =$$

$$= \frac{\pi R^4}{2} = J \text{ (torsional constant)}$$



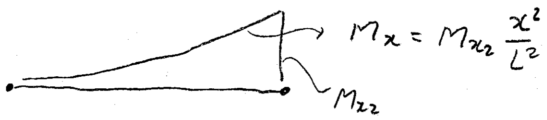
Body force



$$\delta\underset{\sim}{u}^T\underset{\sim}{b} = (\delta\theta_{x=L/6})\cdot M_x$$

$$= \delta\underset{\sim}{q}^T\underset{\sim}{f}^T_{x=L/6}\cdot M_x$$

$$P_b = (\underset{\sim}{f}^T_{x=L/6})M_x = M_x\begin{bmatrix}1-\frac{1}{6}\\ \frac{1}{6}\end{bmatrix} = \frac{M_x}{6}\begin{bmatrix}5\\ 1\end{bmatrix}$$



$$P_b = \int \underset{\sim}{f}^T\underset{\sim}{b}\, dx = \int \underset{\sim}{f}^T M_x\, dx$$

$$= \int_0^L \underset{\sim}{f}^T M_{x2}\frac{x^2}{L^2}\, dx$$

Generalized Stress and Strains

flexural Member

Generalized stress $= M = EI\phi$

Generalized strain $= \phi$

$$\delta U = \int \delta \boldsymbol{\varepsilon}^T \boldsymbol{\sigma} \, dV$$

$$= \int \delta \phi \, M \, dx$$

$$\phi = v'' = \frac{d^2 v}{dx^2} = \frac{d^2 \boldsymbol{f}}{dx^2} \boldsymbol{q} = \bar{\boldsymbol{B}} \boldsymbol{q}$$

$$M = EI\phi = EI \bar{\boldsymbol{B}} \boldsymbol{q}$$

$$\delta U = \int \delta \phi \, M \, dx \qquad P_0 = \int_0^L \bar{\boldsymbol{B}}^T EI \phi_0 \, dx$$

$$= \delta \boldsymbol{q}^T \Big[\underbrace{\int \bar{\boldsymbol{B}}^T EI \bar{\boldsymbol{B}} \, dx}_{\boldsymbol{K}} \Big] \boldsymbol{q}$$

Torsional member

Generalized stress $= M_x = GJ\psi = GJ\left(\frac{d\theta}{dx}\right)$

Generalized stress $= \psi \; \left(= \frac{d\theta}{dx}\right)$

$$\psi = \frac{d\theta}{dx} = \frac{d\boldsymbol{f}}{dx} \boldsymbol{q} = \bar{\boldsymbol{B}} \boldsymbol{q}$$

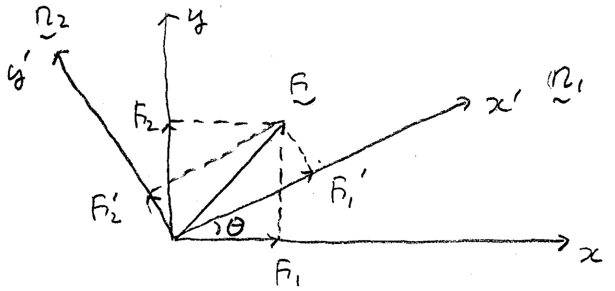
$$M_x = GJ\psi = GJ \bar{\boldsymbol{B}} \boldsymbol{q}$$

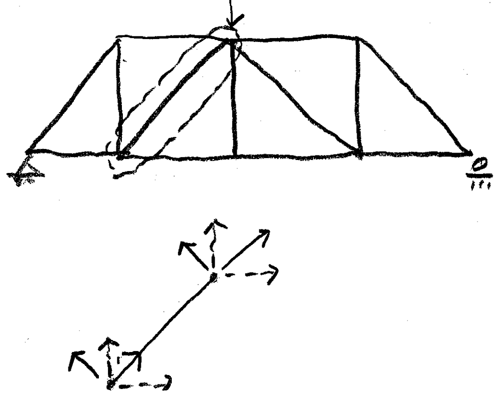
$$\delta U = \int \delta \psi^T M_x \, dx$$

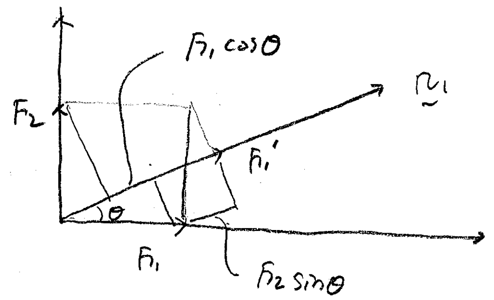
$$= \delta \boldsymbol{q}^T \Big[\underbrace{\int \bar{\boldsymbol{B}}^T GJ \bar{\boldsymbol{B}} \, dx}_{\boldsymbol{K}} \Big] \boldsymbol{q}$$

$$P_0 = \int_0^L \bar{\boldsymbol{B}}^T GJ \psi_0 \, dx$$

Axis - Transformations







$$F_1' = F_1\cos\theta + F_2\sin\theta \qquad F_1' = \underset{\sim}{n_1}\cdot\underset{\sim}{F} = \underset{\sim}{n_1}^T\underset{\sim}{F}$$

$$F_2' = -F_1\sin\theta + F_2\cos\theta \qquad F_2' = \underset{\sim}{n_2}\cdot\underset{\sim}{F} = \underset{\sim}{n_2}^T\underset{\sim}{F}$$

$$\begin{bmatrix} F_1' \\ F_2' \end{bmatrix} = \begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$= \begin{bmatrix} n_{11} & n_{12} \\ n_{21} & n_{22} \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

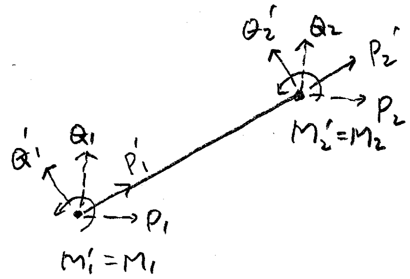
$$= \begin{bmatrix} \underset{\sim}{n_1}^T \\ \underset{\sim}{n_2}^T \end{bmatrix}\begin{bmatrix} F_1 \\ F_2 \end{bmatrix}$$

$$\underset{\sim}{F}' = \underset{\sim}{R}\,\underset{\sim}{F}$$

$$\underset{\sim}{F} = \underset{\sim}{R}^{-1}\underset{\sim}{F}' = \underset{\sim}{R}^T\underset{\sim}{F}'$$

R : Rotational matrix (orthogonal matrix)

$$\underset{\sim}{q}' = \underset{\sim}{R}\,\underset{\sim}{q}, \quad \underset{\sim}{q} = \underset{\sim}{R}^T\underset{\sim}{q}'$$



$$\underset{\sim}{P}' = \underset{\sim}{K}'\underset{\sim}{q}'$$

$$\underset{\sim}{R}^T\underset{\sim}{P}' = \underset{\sim}{R}^T\underset{\sim}{K}'\underset{\sim}{R}\,\underset{\sim}{q}$$

$$\underset{\sim}{P} = \underset{\sim}{R}^T\underset{\sim}{K}'\underset{\sim}{R}\,\underset{\sim}{q}$$

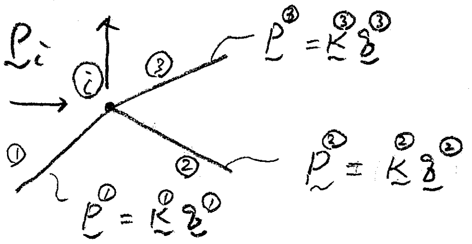
$$= \underset{\sim}{K}\,\underset{\sim}{q}$$

$$\underset{\sim}{K} = \underset{\sim}{R}^T\underset{\sim}{K}'\underset{\sim}{R}$$

for Beam - column element

$$\begin{bmatrix} P_1' \\ Q_1' \\ M_1' \\ P_2' \\ Q_2' \\ M_2' \end{bmatrix} = \underbrace{\begin{bmatrix} c & s & & & & \\ -s & c & & & & \\ & & 1 & & & \\ & & & c & s & \\ & & & -s & c & \\ & & & & & 1 \end{bmatrix}}_{\underset{\sim}{R}} \begin{bmatrix} P_1 \\ Q_1 \\ M_1 \\ P_2 \\ Q_2 \\ M_2 \end{bmatrix}$$

Assemblage of Elements



External force $\downarrow$ internal force $\downarrow$

$$\underset{\sim}{P}_1 = \underset{\sim}{P}_1^{①} + \underset{\sim}{P}_1^{②} + \underset{\sim}{P}_1^{③} = (\underset{\sim}{k}_1^{①} + \underset{\sim}{k}_1^{②} + \underset{\sim}{k}_1^{③})\,\underset{\sim}{\delta}$$

$$\underset{\sim}{P}_2 = \underset{\sim}{P}_2^{①} + \underset{\sim}{P}_2^{②} + \underset{\sim}{P}_2^{③} = (\underset{\sim}{k}_2^{①} + \underset{\sim}{k}_2^{②} + \underset{\sim}{k}_2^{③})\,\underset{\sim}{\delta}$$

$$\vdots$$

$$\underset{\sim}{P} = (\Sigma \underset{\sim}{k})\,\underset{\sim}{\delta} = \underset{\sim}{K}\,\underset{\sim}{\delta}$$

$\Sigma \underset{\sim}{P} = \Sigma \underset{\sim}{K}\,\underset{\sim}{\delta}$ at node ⓘ

$\underset{\sim}{A} = \underset{\sim}{S}\,\underset{\sim}{D}$ for the structure

$$\underset{\sim}{A} = \underset{\sim}{A}_n + \underset{\sim}{A}_b + \underset{\sim}{A}_o$$

$$\underset{\sim}{S} = \underbrace{\Sigma \underset{\sim}{K}}_{\text{Assemblage}}$$

$\underset{\sim}{S}$: Global stiffness matrix

$\underset{\sim}{K}$: Element stiffness matrix

Solution

$\underset{\sim}{S}\,\underset{\sim}{D} = \underset{\sim}{A}$ (n equations)

$$\begin{bmatrix} \underset{\sim}{S}_{FF} & \underset{\sim}{S}_{FR} \\ \underset{\sim}{S}_{RF} & \underset{\sim}{S}_{RR} \end{bmatrix} \begin{bmatrix} \underset{\sim}{D}_F \\ \underset{\sim}{D}_R \end{bmatrix} = \begin{bmatrix} \underset{\sim}{A}_F \\ \underset{\sim}{A}_R \end{bmatrix}$$

$\left\{\begin{array}{l} \text{knowns : } \underset{\sim}{D}_R,\ \underset{\sim}{A}_F \\ \text{unknowns : } \underset{\sim}{D}_F,\ \underset{\sim}{A}_R \\ \quad (n \text{ unknowns}) \end{array}\right.$

F: free

R: restraint

In the first equation,

$$\underset{\sim}{S}_{FF}\underset{\sim}{D}_F + \underset{\sim}{S}_{FR}\underset{\sim}{D}_R = \underset{\sim}{A}_F$$

$$\underset{\sim}{S}_{FF}\underset{\sim}{D}_F = \underset{\sim}{A}_F - \underbrace{\underset{\sim}{S}_{FR}\underset{\sim}{D}_R}_{}$$ solve $\underset{\sim}{D}_F$

↳ equivalent nodal force due to boundary constraint.

In the second equation

$$\underset{\sim}{S}_{RF}\, \underset{\sim}{D}_F + \underset{\sim}{S}_{RR}\, \underset{\sim}{D}_R = \underset{\sim}{A}_R$$ solve $\underset{\sim}{A}_R$ (= reactions)

with the known $\underset{\sim}{D}_F$

$$\left.\begin{aligned} \underset{\sim}{\varepsilon} &= \underset{\sim}{B}\, \underset{\sim}{g} \\ \underset{\sim}{\sigma} &= \underset{\sim}{E}\, \underset{\sim}{B}\, \underset{\sim}{g} \end{aligned}\right\}$$ calculate $\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\sigma}$

Solution without rearrangement of stiffness matrix

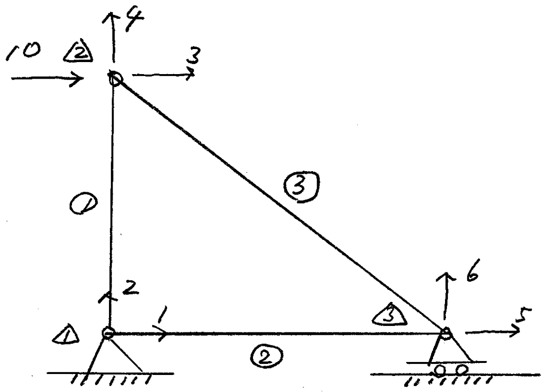
$$\begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} \\ S_{21} & S_{22} & S_{23} & S_{24} \\ S_{31} & S_{32} & S_{33} & S_{34} \\ S_{41} & S_{42} & S_{43} & S_{44} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} A_1 \\ A_2 \\ A_3 \\ A_4 \end{bmatrix}$$

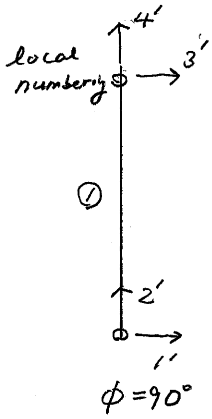
if $D_3 = D_R$, $A_3 = A_R$

$$\begin{bmatrix} S_{11} & S_{12} & 0 & S_{14} \\ S_{21} & S_{22} & 0 & S_{24} \\ 0 & 0 & 1 & 0 \\ S_{41} & S_{42} & 0 & S_{44} \end{bmatrix} \begin{bmatrix} D_1 \\ D_2 \\ D_3 \\ D_4 \end{bmatrix} = \begin{bmatrix} A_1 - S_{13} D_3 \\ A_2 - S_{23} D_3 \\ D_3 \\ A_4 - S_{43} D_3 \end{bmatrix}$$

solve D_1, D_2 and D_4

The reaction can be calculated from element nodal forces.

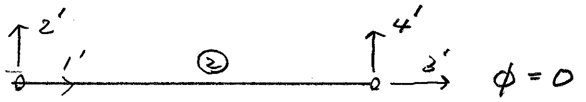




$$\begin{array}{cccccc} & 1 & 2 & 3 & 4 & \rightarrow \text{Global} \\ & 1' & 2' & 3' & 4' & \rightarrow \text{local} \end{array}$$

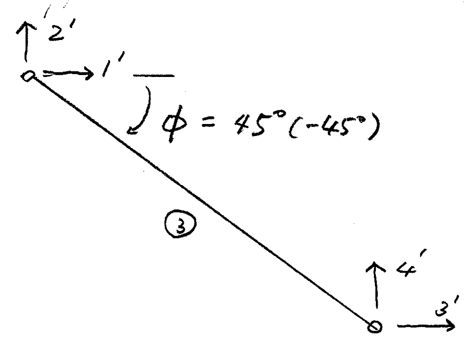
$$\begin{vmatrix} F_1^{①} \\ F_2^{①} \\ F_3^{①} \\ F_4^{①} \end{vmatrix} = \begin{vmatrix} k_{11}^{①} & k_{12}^{①} & k_{13}^{①} & k_{14}^{①} \\ & \ddots & & \\ & & & \\ & & & k_{44}^{①} \end{vmatrix} \begin{vmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{vmatrix}$$

$$\Downarrow \; \underset{\sim}{k}^{①}$$



$$\begin{array}{cccc} 1 & 2 & 5 & 6 \\ 1' & 2' & 3' & 4' \end{array}$$

$$\underset{\sim}{k}^{②} = \begin{vmatrix} k_{11}^{②} & k_{12}^{②} & k_{13}^{②} & k_{14}^{②} \\ & \ddots & & \\ & & & \\ & & & k_{44}^{②} \end{vmatrix}$$



$$\begin{array}{cccc} 3 & 4 & 5 & 6 \\ 1' & 2' & 3' & 4' \end{array}$$

$$\underset{\sim}{k}^{③} = \begin{vmatrix} k_{11}^{③} & k_{12}^{③} & k_{13}^{③} & k_{14}^{③} \\ & \ddots & & \\ & & & \\ & & & k_{44}^{③} \end{vmatrix}$$

By force - equilibrium

At node △1

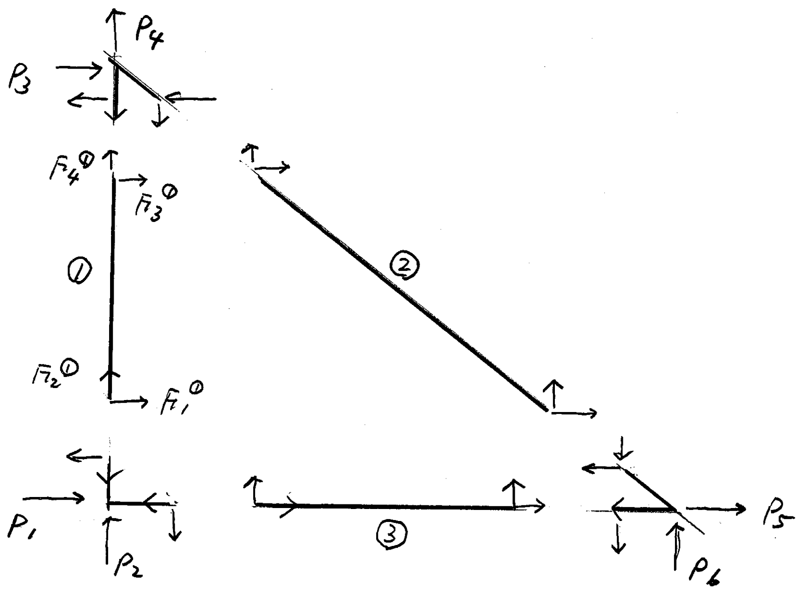
$$\begin{vmatrix} P_1 \\ P_2 \end{vmatrix} = \begin{vmatrix} F_1^{①} + F_1^{②} \\ F_2^{①} + F_2^{②} \end{vmatrix} = \begin{vmatrix} (k_{11}^{①} + k_{11}^{②}) u_1 + (k_{12}^{①} + k_{12}^{②}) u_2 + k_{13}^{①} u_3 + k_{14}^{①} u_4 \\ + k_{13}^{②} u_5 + k_{14}^{②} u_6 \\ (k_{21}^{①} + k_{21}^{②}) u_1 + (k_{22}^{①} + k_{22}^{②}) u_2 + k_{23}^{①} u_3 + k_{24}^{①} u_4 \\ + k_{23}^{②} u_5 + k_{24}^{②} u_6 \end{vmatrix}$$

At node △2

$$\begin{vmatrix} P_3 \\ P_4 \end{vmatrix} = \begin{vmatrix} F_3^{①} + F_1^{③} \\ F_4^{①} + F_4^{③} \end{vmatrix}$$

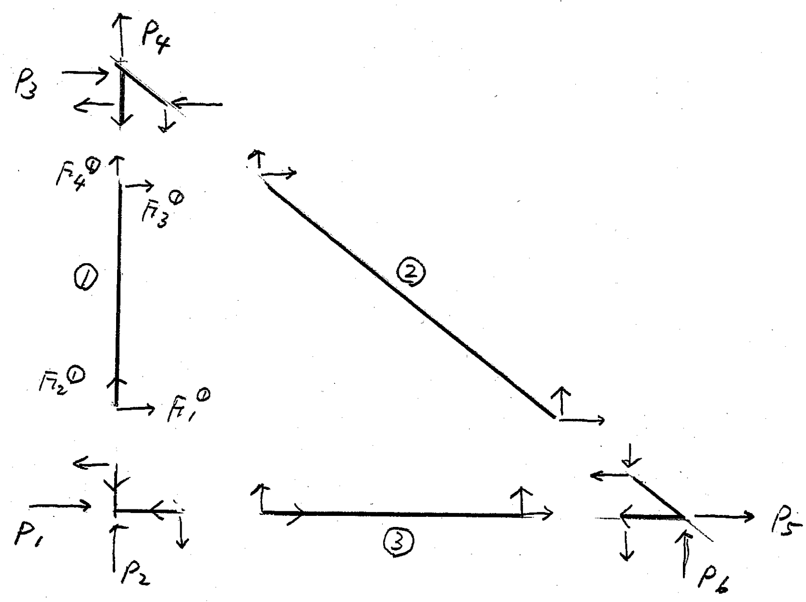
At node △3

$$\begin{vmatrix} P_5 \\ P_6 \end{vmatrix} = \begin{vmatrix} F_3^{②} + F_3^{③} \\ F_4^{②} + F_4^{③} \end{vmatrix}$$



$$
\begin{bmatrix} P_1 \\ P_2 \\ P_3 \\ P_4 \\ P_5 \\ P_6 \end{bmatrix}
=
\begin{array}{cccccc}
1 & 2 & 3 & 4 & 5 & 6 \\
\end{array}
$$

$$
\begin{bmatrix}
k_{11}^{①} + k_{11}^{②} & k_{12}^{①} + k_{12}^{②} & k_{13}^{①} & k_{14}^{①} & k_{13}^{②} & k_{14}^{②} \\
k_{21}^{①} + k_{21}^{②} & k_{22}^{①} + k_{22}^{②} & k_{23}^{①} & k_{24}^{①} & k_{23}^{②} & k_{24}^{②} \\
k_{31}^{①} & k_{32}^{①} & k_{33}^{①} + k_{11}^{③} & k_{34}^{①} + k_{12}^{③} & k_{13}^{③} & k_{14}^{③} \\
k_{41}^{①} & k_{42}^{①} & k_{43}^{①} + k_{21}^{③} & k_{44}^{①} + k_{22}^{③} & k_{23}^{③} & k_{24}^{③} \\
k_{31}^{②} & k_{32}^{②} & k_{31}^{③} & k_{32}^{③} & k_{33}^{②} + k_{33}^{③} & k_{34}^{②} + k_{34}^{③} \\
k_{41}^{②} & k_{42}^{②} & k_{41}^{③} & k_{42}^{③} & k_{43}^{②} + k_{43}^{③} & k_{44}^{②} + k_{44}^{③}
\end{bmatrix}
\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \\ u_5 \\ u_6 \end{bmatrix}
$$



	1	2	3	4	5	6		
P_1							0	EQ for reactions
P_2							0	
10	$k_{31}^{①}$	$k_{32}^{①}$	$k_{33}^{①} + k_{11}^{③}$	$k_{34}^{①} + k_{12}^{③}$	$k_{13}^{③}$	$k_{14}^{③}$	u_3	Equations for free displaceme
0	$k_{41}^{①}$	$k_{42}^{①}$	$k_{43}^{①} + k_{21}^{③}$	$k_{44}^{①} + k_{22}^{③}$	$k_{23}^{③}$	$k_{24}^{③}$	u_4	
0	$k_{31}^{②}$	$k_{32}^{②}$	$k_{31}^{③}$	$k_{32}^{③}$	$k_{33}^{②} + k_{33}^{③}$	$k_{34}^{②} + k_{34}^{③}$	u_5	
P_6							0	EQ for reaction.