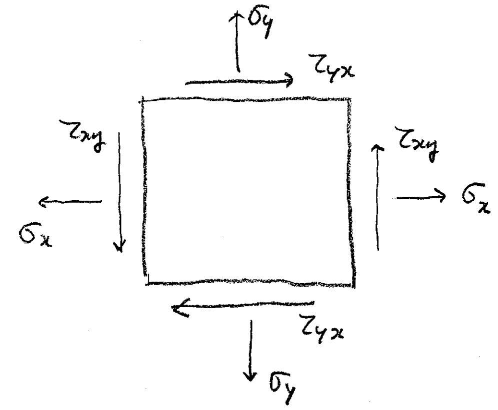
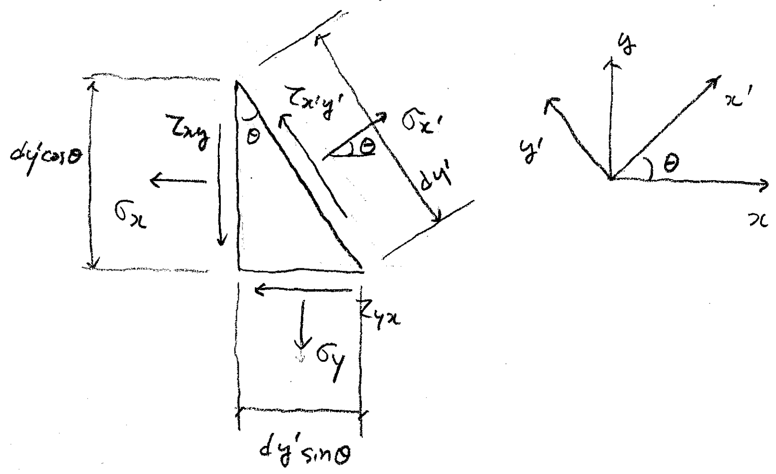
Chapter 2 Plane Stress and Plane Strain

stress - transformation





$$\sum F_{x'} = 0 \Rightarrow \sigma_{x'}\,dy' - \sigma_x \cos\theta\, dy'\cos\theta - \sigma_y \sin\theta\, dy'\sin\theta$$
$$- \tau_{xy} \sin\theta\, dy'\cos\theta - \tau_{xy}\cos\theta\, dy'\sin\theta = 0$$

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta$$

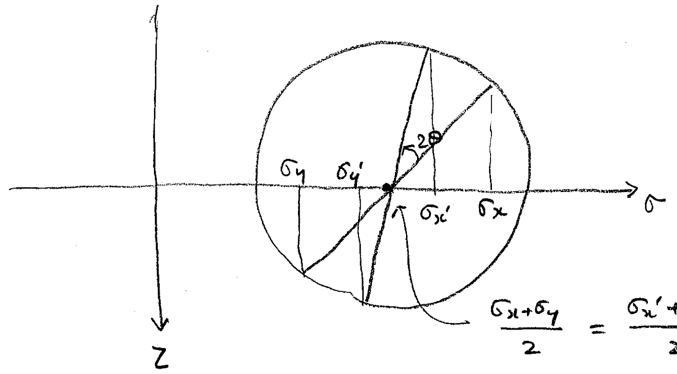
$$\sum F_{y'} = 0 \qquad \tau_{x'y'} = -(\sigma_x - \sigma_y)\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta - \sin^2\theta)$$

$$\sigma_{y'} = \sigma_{x'}(\theta + 90^\circ)$$
$$= \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy}\sin\theta\cos\theta$$

$$\underset{\sim}{\sigma}' = \underset{\sim}{T_\sigma}\, \underset{\sim}{\sigma}$$

$$\begin{bmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}}_{\underset{\sim}{T_\sigma} = T_\theta} \begin{bmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{bmatrix}$$

$$\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y \Rightarrow \text{invariants}$$



$$\frac{\sigma_x + \sigma_y}{2} = \frac{\sigma_{x'} + \sigma_{y'}}{2}$$

$$\tan 2\theta_p = \frac{2\tau_{xy}}{\sigma_x - \sigma_y} \quad : \text{principal axes}$$

Strain – displacement relationship in plane

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial u}{\partial x} \\ \frac{\partial v}{\partial y} \\ \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} \end{bmatrix}$$

$$= \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{\underset{\sim}{d}} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\underset{\sim}{u}}$$

Axis transformation of strains

Complimentary virtual strain energy densities

$$(\delta \underset{\sim}{\sigma}')^T \underset{\sim}{\varepsilon}' = (\delta \underset{\sim}{\sigma}^T)\, \underset{\sim}{\varepsilon}$$

energy is the same regardless of the definition of axis

$$\underset{\sim}{\sigma}' = \underset{\sim}{T_\theta}\, \underset{\sim}{\sigma} \Rightarrow \delta \underset{\sim}{\sigma}' = \underset{\sim}{T_\theta}\, \delta \underset{\sim}{\sigma}$$

$$\delta \underset{\sim}{\sigma}^T\, \underset{\sim}{T_\theta}^T\, \underset{\sim}{\varepsilon}' = \delta \underset{\sim}{\sigma}^T\, \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{T_\theta}^T\, \underset{\sim}{\varepsilon}' = \underset{\sim}{\varepsilon} \Rightarrow \underset{\sim}{\varepsilon}' = \underset{\sim}{T_\theta}^{-T}\, \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\varepsilon}' = \underset{\sim}{T_\varepsilon}\, \underset{\sim}{\varepsilon} \qquad \underset{\sim}{T_\varepsilon} = \underset{\sim}{T_\theta}^{-T}$$

$$\begin{bmatrix} \varepsilon_x' \\ \varepsilon_y' \\ \gamma_{xy}' \end{bmatrix} = \underbrace{\begin{bmatrix} \cos^2\theta & \sin^2\theta & \sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -\sin\theta\cos\theta \\ -2\sin\theta\cos\theta & 2\sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}}_{\underset{\sim}{T_\varepsilon}} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix}$$

or

$$\begin{bmatrix} \varepsilon_x' \\ \varepsilon_y' \\ \underbrace{\varepsilon_{xy}'}_{(\frac{1}{2}\gamma_{xy}')} \end{bmatrix} = \underbrace{\begin{bmatrix} \cos^2\theta & \sin^2\theta & 2\sin\theta\cos\theta \\ \sin^2\theta & \cos^2\theta & -2\sin\theta\cos\theta \\ -\sin\theta\cos\theta & \sin\theta\cos\theta & \cos^2\theta - \sin^2\theta \end{bmatrix}}_{\underset{\sim}{T_\varepsilon'} = \underline{T_\theta} = \underline{T_\sigma}} \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \underbrace{\varepsilon_{xy}}_{(\frac{1}{2}\gamma_{xy})} \end{bmatrix}$$

Isotropic material

plane stress ($x-y$ plane)

$$\sigma_z = \tau_{zx} = \tau_{yz} = 0 \quad . \qquad \gamma_{zx} = \gamma_{yz} = 0, \qquad \varepsilon_z \neq 0$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y) \qquad \varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_x) \qquad \varepsilon_z = \frac{-\nu}{E}(\sigma_x + \sigma_y)$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{C}\,\underset{\sim}{\sigma} \qquad \underset{\sim}{C} = \frac{1}{E}\begin{bmatrix} 1 & -\nu & 0 \\ -\nu & 1 & 0 \\ 0 & 0 & 2(1+\nu) \end{bmatrix}$$

$$\underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\varepsilon} \qquad \underset{\sim}{E} = \frac{E}{1-\nu^2}\begin{bmatrix} 1 & \nu & 0 \\ \nu & 1 & 0 \\ 0 & 0 & \lambda \end{bmatrix} \qquad \lambda = \frac{1-\nu}{2}$$

plane strain

$$\varepsilon_z = \gamma_{zx} = \gamma_{yz} = 0 \qquad \tau_{zx} = \tau_{yz} = 0 \qquad \sigma_z \neq 0$$

$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$$

$$\varepsilon_y = \frac{1}{E}(-\nu\sigma_z + \sigma_y - \nu\sigma_x)$$

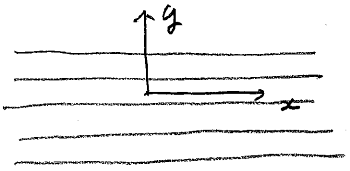
$$\varepsilon_z = \frac{1}{E}(-\nu\sigma_x - \nu\sigma_y + \sigma_z) = 0 \quad \Rightarrow \quad \sigma_z = \nu(\sigma_x + \sigma_y)$$

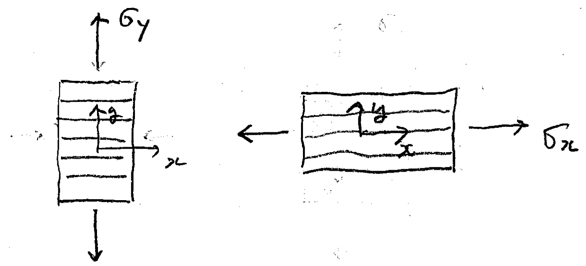
$$\gamma_{xy} = \frac{2(1+\nu)}{E}\tau_{xy}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{C}\,\underset{\sim}{\sigma} \qquad \underset{\sim}{C} = \frac{1+\nu}{E}\begin{bmatrix} (1-\nu) & -\nu & 0 \\ -\nu & (1-\nu) & 0 \\ 0 & 0 & 2 \end{bmatrix}$$

$$\underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\varepsilon} \qquad \underset{\sim}{E} = \frac{E}{(1+\nu)(1-2\nu)}\begin{bmatrix} (1-\nu) & \nu & 0 \\ \nu & (1-\nu) & 0 \\ 0 & 0 & \frac{(1-2\nu)}{2} \end{bmatrix}$$

Orthotropic material





$$\nu_{xy} = -\frac{\varepsilon_x}{\varepsilon_y} \qquad \nu_{yx} = \frac{-\varepsilon_y}{\varepsilon_x}$$

plane stress

$$\varepsilon_x = \frac{1}{E_x}\sigma_x - \frac{\nu_{xy}}{E_y}\sigma_y$$

$$\varepsilon_y = -\frac{\nu_{yx}}{E_x}\sigma_x + \frac{1}{E_y}\sigma_y$$

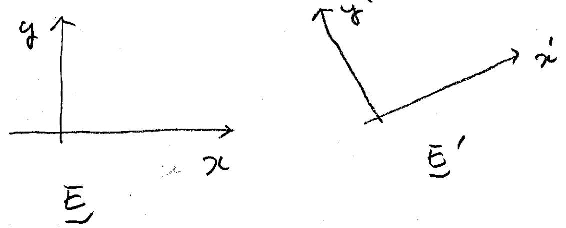
$$\gamma_{xy} = \frac{1}{G_{xy}}\tau_{xy}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{C}\,\underset{\sim}{\sigma} \qquad \underset{\sim}{C} = \begin{bmatrix} \frac{1}{E_x} & -\frac{\nu_{xy}}{E_y} & 0 \\ -\frac{\nu_{yx}}{E_x} & \frac{1}{E_y} & 0 \\ 0 & 0 & \frac{1}{G_{xy}} \end{bmatrix}$$

$$\underset{\sim}{\sigma} = \underset{\sim}{E}\,\underset{\sim}{\varepsilon} \qquad \underset{\sim}{E} = \frac{1}{1-\nu_{xy}\nu_{yx}} \begin{bmatrix} E_x & \nu_{xy}E_x & 0 \\ \nu_{yx}E_y & E_y & 0 \\ 0 & 0 & (1-\nu_{xy}\nu_{yx})G_{xy} \end{bmatrix}$$

Axis - Transformation of Constitutive Matrix

Isotropic material



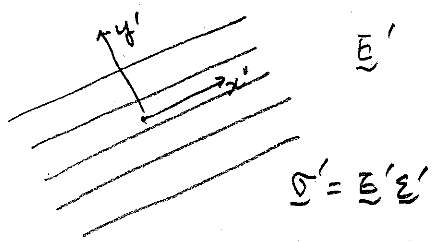
material properties are the same, regardless of axis-transformation

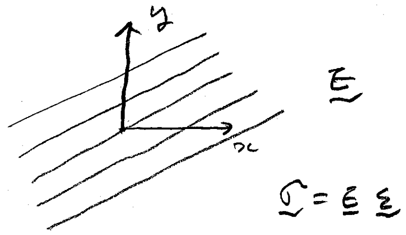
$\underset{\sim}{E} = \underset{\sim}{E}'$ for isotropic material

$$\underset{\sim}{\sigma}' = \underset{\sim}{E}'\,\underset{\sim}{\varepsilon}'$$

$$\underset{\sim}{\sigma}' = \underset{\sim}{E}\,\underset{\sim}{\varepsilon}'$$

Orthotropic material





$$\underset{\sim}{\sigma}' = \underset{\sim}{T_\sigma}\,\underset{\sim}{\sigma}$$

$$\underset{\sim}{\varepsilon}' = \underset{\sim}{T_\sigma}^{-T}\underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\sigma}' = \underset{\sim}{E}'\underset{\sim}{\varepsilon}'$$

$$\underset{\sim}{T_\sigma}\,\underset{\sim}{\sigma} = \underset{\sim}{E}'\,\underset{\sim}{T_\sigma}^{-T}\underset{\sim}{\varepsilon}$$

$$\underset{\sim}{\sigma} = \underset{\sim}{T_\sigma}^{-1}\underset{\sim}{E}'\underset{\sim}{T_\sigma}^{-T}\underset{\sim}{\varepsilon} \qquad \underset{\sim}{T_\sigma}^{-T} = \underset{\sim}{T_\varepsilon}$$

$$= \underbrace{\underset{\sim}{T_\varepsilon}^{T}\underset{\sim}{E}'\underset{\sim}{T_\varepsilon}}_{\underset{\sim}{E}}\,\underset{\sim}{\varepsilon}$$

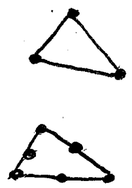
$$\underset{\sim}{E} = \underset{\sim}{T_\varepsilon}^{T}\underset{\sim}{E}'\underset{\sim}{T_\varepsilon}$$

for isotropic materials

$$\underset{\sim}{E} = \underset{\sim}{T_\varepsilon}^{T}\underset{\sim}{E}'\underset{\sim}{T_\varepsilon} = \underset{\sim}{E}'$$

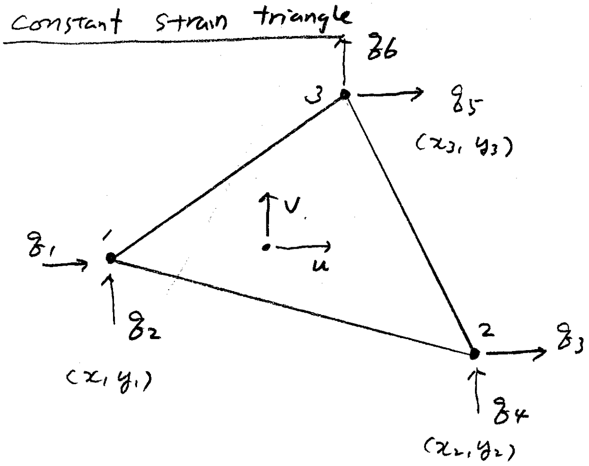
2.2 Triangular Elements

$\begin{cases} \text{constant strain triangle} \\ \text{linear strain triangle} \end{cases}$


3-nodes

6-nodes

constant strain triangle



Generic Displacement $\quad \underset{\sim}{u} = \begin{bmatrix} u \\ v \end{bmatrix}$

nodal Displacement $\quad \underset{\sim}{q} = \begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \end{bmatrix} = \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$

Generic - nodal displacement

$$\left.\begin{aligned} u &= c_1 + c_2 x + c_3 y \\ v &= c_4 + c_5 x + c_6 y \end{aligned}\right] \rightarrow \text{same form}$$

$$u = \underbrace{[1 \quad x \quad y]}_{\underset{\sim}{g}_u} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\underset{\sim}{c}_u}$$

Generally, the assumed displacement function is not the exact displacement that satisfies the force equilibrium.

1) However, the principle of virtual displacement results in an acceptable solution, approximately satisfying the force-equilibrium through the energy method.
2) Since the displacement function is of a fixed form, the resulting stiffness matrix can be used for any structures with complex conditions as long as a large number of elements are used.
3) The fixed form of stiffness automatizes the procedure of FE analysis. Therefore, the FEA procedure can be fitted to analysis using computer.

at node 1 (x_1, y_1) $\quad u = u_1$

$$u_1 = C_1 + C_2 x_1 + C_3 y_1$$

$$u_2 = C_1 + C_2 x_2 + C_3 y_2$$

$$u_3 = C_1 + C_2 x_3 + C_3 y_3$$

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}}_{\underset{\sim}{g}_u} = \underbrace{\begin{bmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{bmatrix}}_{\underset{\sim}{h}_u} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}}_{\underset{\sim}{c}_u}$$

$$u = \underset{\sim}{g}_u \underset{\sim}{c}_u = \underset{\sim}{g}_u \underset{\sim}{h}_u^{-1} \underset{\sim}{g}_u$$

$$= \underset{\sim}{f}_u \underset{\sim}{g}_u$$

$$\underset{\sim}{h}_u^{-1} = \frac{1}{\det(\underset{\sim}{h}_u)} \underbrace{\begin{bmatrix} (x_2 y_3 - x_3 y_2) & (x_3 y_1 - x_1 y_3) & (x_1 y_2 - x_2 y_1) \\ (y_2 - y_3) & (y_3 - y_1) & (y_1 - y_2) \\ (x_3 - x_2) & (x_1 - x_3) & (x_2 - x_1) \end{bmatrix}}_{\text{constants}}$$

$$\det(\underset{\sim}{h}_u) = \begin{vmatrix} 1 & x_1 & y_1 \\ 1 & x_2 & y_2 \\ 1 & x_3 & y_3 \end{vmatrix}$$

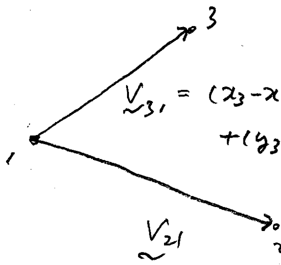
$$= x_2 y_3 - y_2 x_3 - x_1 y_3 + y_1 x_3 + x_1 y_2 - y_1 x_2$$

$$= (x_2 - x_1) y_3 + (x_1 - x_3) y_2 + (x_3 - x_2) y_1$$

$$= (x_2 - x_1) y_3 + (x_1 - x_3) y_2 + (x_3 - x_1) y_1 + (x_1 - x_2) y_1$$

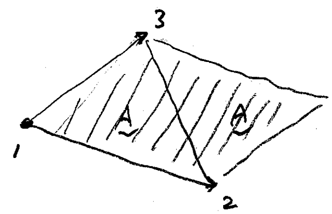
$$= -(x_3 - x_1)(y_2 - y_1) + (x_2 - x_1)(y_3 - y_1)$$

$$= V_{21} \times V_{31} = 2A$$



$\underset{\sim}{V}_{31} = (x_3 - x_1) i + (y_3 - y_1) j$

$\underset{\sim}{V}_{21} = (x_2 - x_1) i + (y_2 - y_1) j$



$$\underset{\sim}{f}_u = \underset{\sim}{g}_u \underset{\sim}{h}_u^{-1} = [\, f_1 \quad f_2 \quad f_3 \,]$$

$$f_1 = \frac{1}{2A}\left[(x_2 y_3 - x_3 y_2) + (y_2 - y_3)x + (x_3 - x_2) y \right]$$

$$f_2 = \frac{1}{2A}\left[(x_3 y_1 - x_1 y_3) + (y_3 - y_1)x + (x_1 - x_3) y \right]$$

$$f_3 = \frac{1}{2A}\left[(x_1 y_2 - x_2 y_1) + (y_1 - y_2)x + (x_2 - x_1) y \right]$$

$$u = f_1 u_1 + f_2 u_2 + f_3 u_3 = \underset{\sim}{f}_u \underset{\sim}{g}_u$$

$$v = f_1 v_1 + f_2 v_2 + f_3 v_3 = \underset{\sim}{f}_v \underset{\sim}{g}_v \qquad \underset{\sim}{f}_v = \underset{\sim}{f}_u$$

$$\underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\underset{\sim}{u}} = \underbrace{\begin{bmatrix} f_1 & 0 & f_2 & 0 & f_3 & 0 \\ 0 & f_1 & 0 & f_2 & 0 & f_3 \end{bmatrix}}_{\underset{\sim}{f}} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}}_{\underset{\sim}{q}}$$

Strain - Generic displacement

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}}_{\underset{\sim}{d}} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\underset{\sim}{u}}$$

Strain - nodal displacement

$$\underset{\sim}{\varepsilon} = \underset{\sim}{d}\,\underset{\sim}{u} = \underset{\sim}{d}\,\underset{\sim}{f}\,\underset{\sim}{q} = \underset{\sim}{B}\,\underset{\sim}{q} \qquad \underset{\sim}{B} = \underset{\sim}{d}\,\underset{\sim}{f}$$

$$\underset{\sim}{B} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} f_1 & 0 & f_2 & 0 & f_3 & 0 \\ 0 & f_1 & 0 & f_2 & 0 & f_3 \end{bmatrix}$$

$$
\underset{\sim}{B} = \left[\begin{array}{cc|cc|cc}
\frac{\partial f_1}{\partial x} & 0 & \frac{\partial f_2}{\partial x} & 0 & \frac{\partial f_3}{\partial x} & 0 \\
0 & \frac{\partial f_1}{\partial y} & 0 & \frac{\partial f_2}{\partial y} & 0 & \frac{\partial f_3}{\partial y} \\
\frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial x} & \frac{\partial f_2}{\partial y} & \frac{\partial f_2}{\partial x} & \frac{\partial f_3}{\partial y} & \frac{\partial f_3}{\partial x}
\end{array}\right]
$$

$$
= \underbrace{\frac{1}{2A} \left[\begin{array}{cc|cc|cc}
(y_2-y_3) & 0 & (y_3-y_1) & 0 & (y_1-y_2) & 0 \\
0 & (x_3-x_2) & 0 & (x_1-x_3) & 0 & (x_2-x_1) \\
(x_3-x_2) & (y_2-y_3) & (x_1-x_3) & (y_3-y_1) & (x_2-x_1) & (y_1-y_2)
\end{array}\right]}_{\text{constant}}
$$

$\underset{\sim}{\varepsilon} = \underset{\sim}{B}\, \underset{\sim}{d} \Rightarrow$ constant strain $\Longrightarrow$ constant strain triangle

$$
\underset{\sim}{K} = \int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dV
$$

$$
= \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B} \int dV
$$

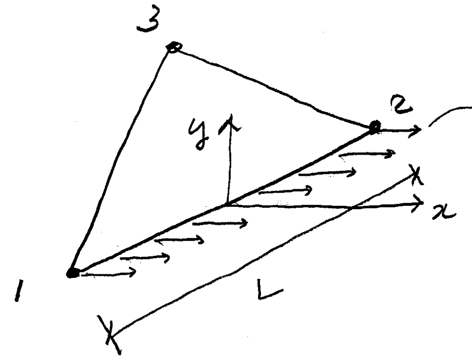
$$
= \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, A t \quad \Rightarrow \text{constant}
$$

$\underset{\sim}{E}$ — plane stress

└ plane strain

Equivalent nodal force

uniformly distributed loading



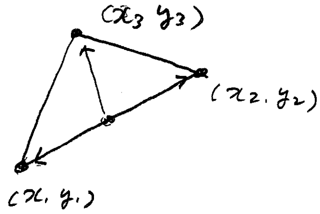
$$\underset{\sim}{P}_{bx} = \int \underset{\sim}{f_u}^T b_x \, dL$$

$$= \int \underset{\sim}{h_u}^{-T} \underset{\sim}{g_u}^T b_x \, dL$$

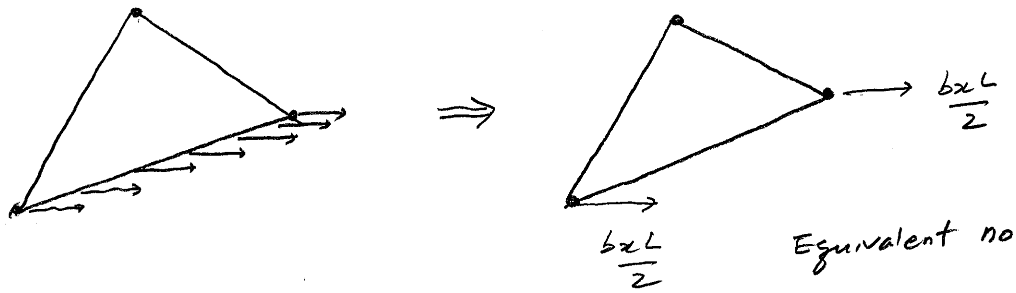
$$= \underset{\sim}{h_u}^{-T} \int \underset{\sim}{g_u}^T b_x \, dL \qquad \underset{\sim}{g_u} = [\, 1 \quad x \quad y \,]$$

$$= \underset{\sim}{h_u}^{-T} \begin{bmatrix} \int b_x \, dL \\ \int x b_x \, dL \\ \int y b_x \, dL \end{bmatrix} = h_u^{-T} b_x \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{b_x L}{2A} \begin{bmatrix} \overset{\frac{A}{2} \times 2}{(x_2 y_3 - x_3 y_2)} \\ (x_3 y_1 - y_3 x_1) \\ (x_1 y_2 - x_2 y_1) \end{bmatrix}$$

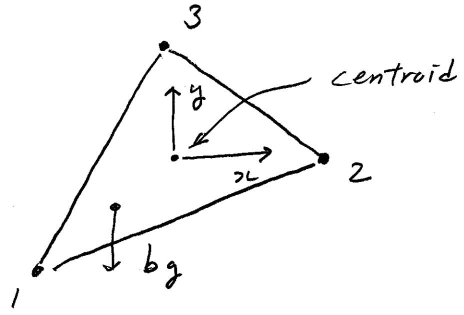


$$= \frac{b_x L}{2A} \begin{bmatrix} A \\ A \\ 0 \end{bmatrix} = \frac{b_x L}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$



Equivalent nodal load

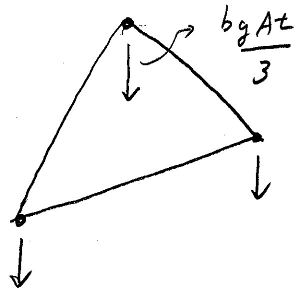
Gravity loading



$$\underset{\sim}{P}_{by} = \int \underset{\sim}{f}_v^T(-bg)\,dV = \int \underset{\sim}{h}_v^{-T}\, \underset{\sim}{g}_v^T(-bg)\,dV$$

$$= \underset{\sim}{h}_v^{-T} \begin{bmatrix} \int -bg\,t\,dA \\ \int -x\,bg\,t\,dA \\ \int -y\,bg\,t\,dA \end{bmatrix} = -h_v^{-T}\, bg\,t \begin{bmatrix} A \\ 0 \\ 0 \end{bmatrix}$$

$$= \frac{-bgt}{2} \begin{bmatrix} (x_2y_3 - x_3y_2) \\ (x_3y_1 - y_3x_1) \\ (x_1y_2 - x_2y_1) \end{bmatrix} = \frac{-bgt}{2} \begin{bmatrix} 2A/3 \\ 2A/3 \\ 2A/3 \end{bmatrix}$$



$$= \frac{-bgAt}{3} \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$$

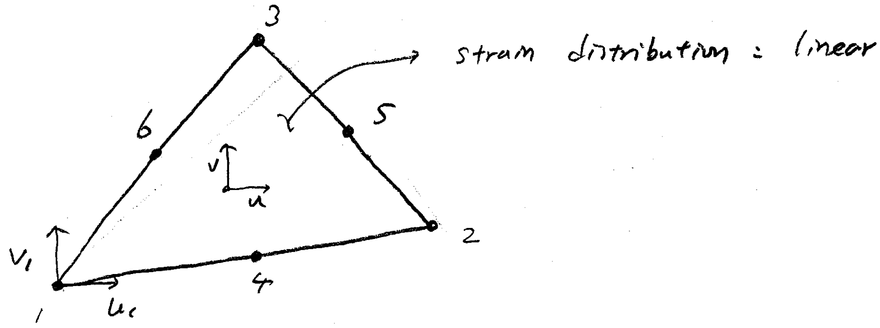
equivalent nodal forces

uniform Temperature change ΔT

$$\underset{\sim}{\varepsilon}_0 = \alpha \Delta T \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{P}_T = \underset{\sim}{P}_0 = \int_v \underline{B}^T \underset{\sim}{E}\, \underset{\sim}{\varepsilon}_0\, dV = \underline{B}^T \underset{\sim}{E}\, \underset{\sim}{\varepsilon}_0\, At$$

Linear Strain Triangle



nodal displacement $\underset{\sim}{q} = \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_6 \\ v_6 \end{bmatrix}$

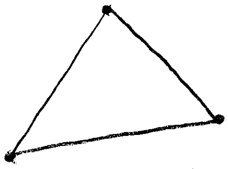
$$\left\{ \begin{aligned} u &= c_1 + c_2 x + c_3 y + c_4 x^2 + c_5 xy + c_6 y^2 \\ v &= c_7 + c_8 x + c_9 y + c_{10} x^2 + c_{11} xy + c_{12} y^2 \end{aligned} \right\} \text{ 12 constants}$$

$$\begin{array}{ccccc} & & 1 & & \\ & x & & y & \\ x^2 & & xy & & y^2 \end{array}$$

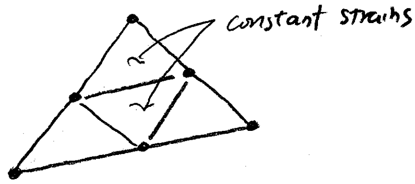
pascal triangle

It is convenient to use the area coordinate system for triangular elements

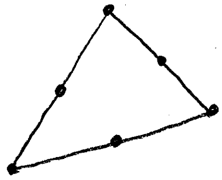
Triangular Element



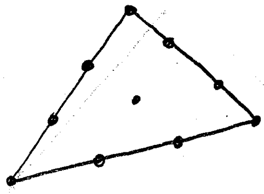
constant strain triangle
(CST) 3 nodes



CST in CST



Linear strain triangle
(LST) 6 nodes



Quadratic strain triangle
10 nodes

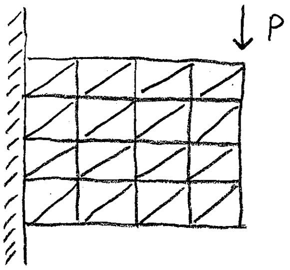
Advantages of triangular element

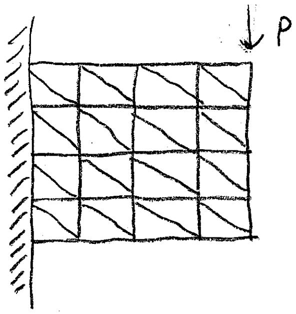
1) Simplicity in element stiffness

2) usefulness in the description of the geometry of complex structures

Disadvantages

1) Lack of geometric isotropy

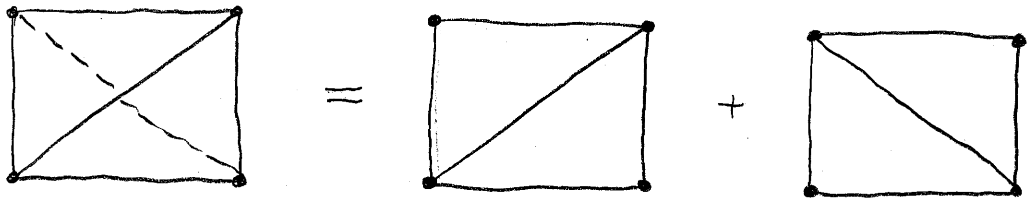




same number of elements
but different solutions

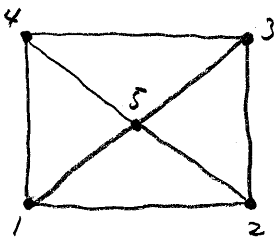
Modification of modeling

1)

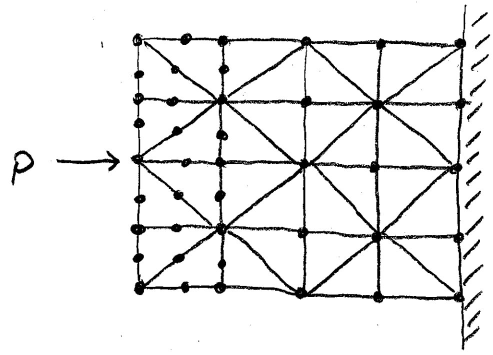


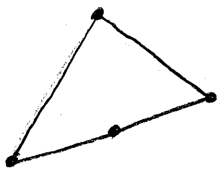
1/2 thickness   1/2 thickness

2)

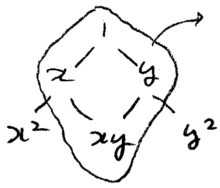


Transition Element





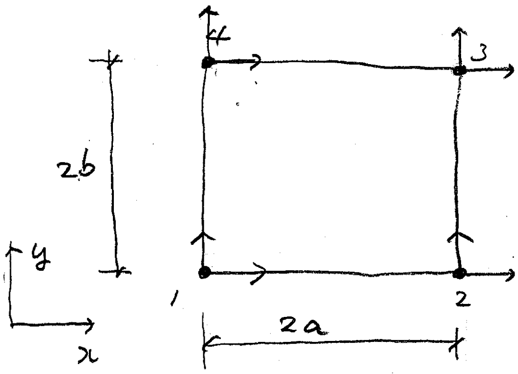
Transition Element



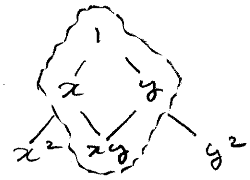
elements of shape function.

Rectangular Elements

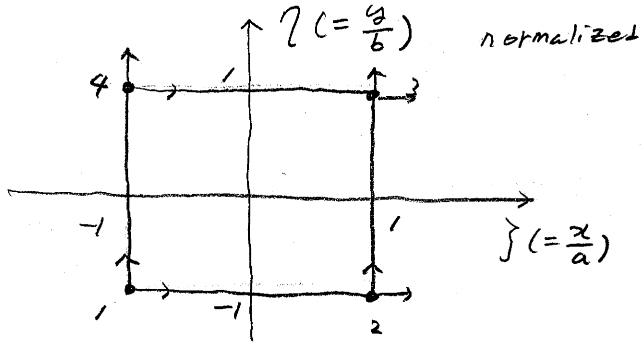
Bilinear Displacement Rectangle by Melosh



$$\begin{cases} u = c_1 + c_2 x + c_3 y + c_4 xy \\ v = c_5 + c_6 x + c_7 y + c_8 xy \end{cases}$$



$\Downarrow$



normalized coordinates (ξ, η)

$$\begin{cases} u = c_1 + c_2 \xi + c_3 \eta + c_4 \xi\eta \\ v = c_5 + c_6 \xi + c_7 \eta + c_8 \xi\eta \end{cases}$$

$$u = c_1 + c_2 \xi + c_3 \eta + c_4 \xi \eta$$

$$= \underbrace{[1 \quad \xi \quad \eta \quad \xi\eta]}_{\underset{\sim}{g}_u} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}}_{\underset{\sim}{c}_u}$$

using boundary condition at the nodes

$$\underbrace{\begin{bmatrix} u_1 \\ u_2 \\ u_3 \\ u_4 \end{bmatrix}}_{\underset{\sim}{q}_u} = \underbrace{\begin{bmatrix} 1 & -1 & -1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}}_{\underset{\sim}{h}_u} \underbrace{\begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}}_{\underset{\sim}{c}_u}$$

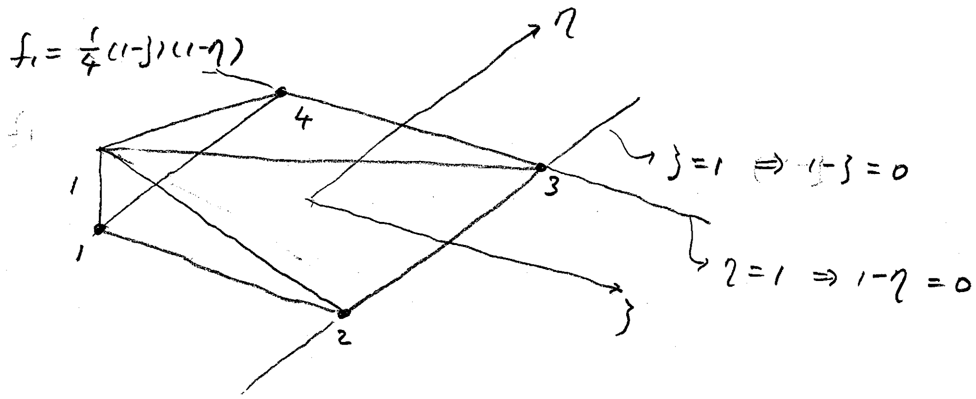
$$\underset{\sim}{h}_4^{-1} = \frac{1}{4}\begin{bmatrix} 1 & 1 & 1 & 1 \\ -1 & 1 & 1 & -1 \\ -1 & -1 & 1 & 1 \\ 1 & -1 & 1 & -1 \end{bmatrix}$$

$$\underset{\sim}{f}_4 = \underset{\sim}{g}_4 \underset{\sim}{h}_4^{-1}$$

$$\underset{\sim}{f}_4 = [\, f_1 \quad f_2 \quad f_3 \quad f_4 \,]$$

$$\left.\begin{aligned} f_1 &= \tfrac{1}{4}(1-\xi)(1-\eta) \\ f_2 &= \tfrac{1}{4}(1+\xi)(1-\eta) \\ f_3 &= \tfrac{1}{4}(1+\xi)(1+\eta) \\ f_4 &= \tfrac{1}{4}(1-\xi)(1+\eta) \end{aligned}\right\} \quad f_i = \tfrac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$$

$(\xi_i, \eta_i) \Rightarrow$ coordinate of node i



$$\begin{cases} f_1 = 1 \text{ at node } 1 \;\; (\xi_1, \eta_1) = (-1, -1) \\ f_1 = 0 \text{ at other nodes} \end{cases} \Rightarrow \quad f_1 = a(1-\xi)(1-\eta) \qquad a = \tfrac{1}{4}$$

$$u = \underset{\sim}{f_u}\, \underset{\sim}{q_u}$$

$$v = \underset{\sim}{f_v}\, \underset{\sim}{q_v} \qquad \underset{\sim}{f_v} = \underset{\sim}{f_u}$$

$$\underset{\sim}{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \underset{\sim}{f}\, \underset{\sim}{q}$$

$$= \underbrace{\begin{bmatrix} f_1 & 0 & f_2 & 0 & f_3 & 0 & f_4 & 0 \\ 0 & f_1 & 0 & f_2 & 0 & f_3 & 0 & f_4 \end{bmatrix}}_{\underset{\sim}{f}} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \\ u_4 \\ v_4 \end{bmatrix}}_{\underset{\sim}{q}}$$

Strain - Generic Displacement

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial u}{\partial \xi}\frac{\partial \xi}{\partial x} = \frac{1}{a}\frac{\partial u}{\partial \xi} \qquad \xi = \frac{x}{a}$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \frac{\partial v}{\partial \eta}\frac{\partial \eta}{\partial y} = \frac{1}{b}\frac{\partial v}{\partial \eta} \qquad \eta = \frac{y}{b}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \frac{\partial u}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial v}{\partial \xi}\frac{\partial \xi}{\partial x}$$

$$= \frac{1}{b}\frac{\partial u}{\partial \eta} + \frac{1}{a}\frac{\partial v}{\partial \xi}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{1}{a}\frac{\partial}{\partial \xi} & 0 \\ 0 & \frac{1}{b}\frac{\partial}{\partial \eta} \\ \frac{1}{b}\frac{\partial}{\partial \eta} & \frac{1}{a}\frac{\partial}{\partial \xi} \end{bmatrix}}_{\underset{\sim}{d}} \underbrace{\begin{bmatrix} u \\ v \end{bmatrix}}_{\underset{\sim}{u}}$$

Strain – nodal displacement

$$\underset{\sim}{\varepsilon} = \underset{\sim}{d}\,\underset{\sim}{u} = \underset{3\times2}{\underset{\sim}{d}}\;\underset{2\times8}{\underset{\sim}{f}}\;\underset{\sim}{q} \qquad \underset{3\times8}{\underset{\sim}{B}} = \underset{\sim}{d}\,\underset{\sim}{f} \qquad f_{i,\xi} = \frac{\partial f}{\partial \xi}\ ,\ f_{i,\eta} = \frac{\partial f}{\partial \eta}$$

$$\underset{\sim}{B} = \left[\begin{array}{c|c|c|c|c|c|c|c} \frac{1}{a} f_{1,\xi} & 0 & \frac{1}{a} f_{2,\xi} & 0 & \frac{1}{a} f_{3,\xi} & 0 & \frac{1}{a} f_{4,\xi} & 0 \\ \hline 0 & \frac{1}{b} f_{1,\eta} & 0 & \frac{1}{b} f_{2,\eta} & 0 & \frac{1}{b} f_{3,\eta} & 0 & \frac{1}{b} f_{4,\eta} \\ \hline \frac{1}{b} f_{1,\eta} & \frac{1}{a} f_{1,\xi} & \frac{1}{b} f_{2,\eta} & \frac{1}{a} f_{2,\xi} & \frac{1}{b} f_{3,\eta} & \frac{1}{a} f_{3,\xi} & \frac{1}{b} f_{4,\eta} & \frac{1}{a} f_{4,\xi} \end{array}\right]$$

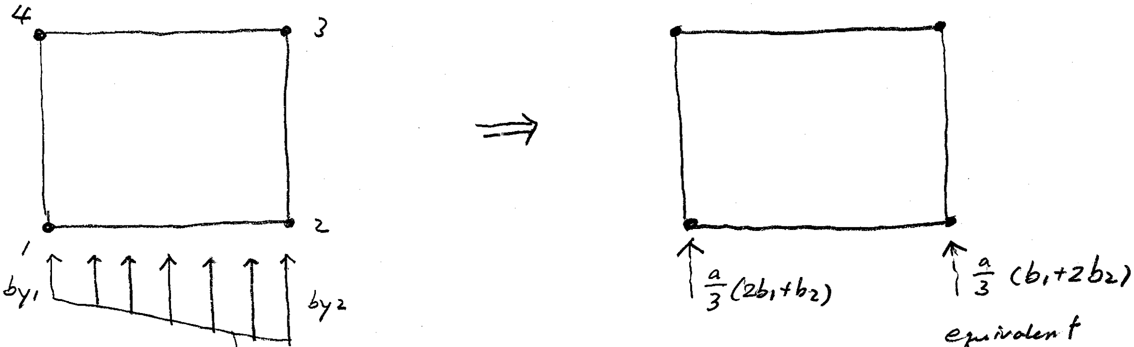
$$= \left[\begin{array}{c|c|c|c|c|c|c|c} -\frac{1}{a}(1-\eta) & 0 & \frac{1}{a}(1-\eta) & 0 & \frac{1}{a}(1+\eta) & 0 & -\frac{1}{a}(1+\eta) & 0 \\ \hline 0 & -\frac{1}{b}(1-\xi) & 0 & -\frac{1}{b}(1+\xi) & 0 & \frac{1}{b}(1+\xi) & 0 & \frac{1}{b}(1-\xi) \\ \hline -\frac{1}{b}(1-\xi) & -\frac{1}{a}(1-\eta) & -\frac{1}{b}(1+\xi) & \frac{1}{a}(1-\eta) & \frac{1}{b}(1+\xi) & \frac{1}{a}(1+\eta) & \frac{1}{b}(1-\xi) & -\frac{1}{a}(1+\eta) \end{array}\right]$$

$$\underset{\sim}{K} = \int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dV$$

$$= t \int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dx\,dy \qquad dx = a\,d\xi\ ,\ dy = b\,d\eta$$

$$= abt \int_{-1}^{1}\!\!\int_{-1}^{1} \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, d\xi\, d\eta \qquad \underset{\sim}{E} < \begin{array}{l}\text{plane stress} \\ \text{plane strain}\end{array}$$

Equivalent nodal force



$$b_y = \frac{(b_{y1}+b_{y2})}{2} + \frac{(b_{y2}-b_{y1})}{2}\,\xi$$

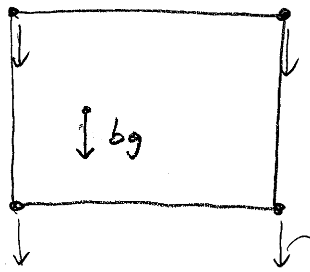
$$\underset{\sim}{P}_b = \int \underset{\sim}{f}^T \underset{\sim}{b}\, dx$$

$$\underset{\sim}{f}^T(\eta=-1)\, \underset{\sim}{b}$$

$$= \begin{bmatrix} \frac{1}{2}(1-\xi) & 0 \\ 0 & \frac{1}{2}(1-\xi) \\ \frac{1}{2}(1+\xi) & 0 \\ 0 & \frac{1}{2}(1+\xi) \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ b_y \end{bmatrix} = \begin{bmatrix} 0 \\ \frac{1}{2}(1-\xi)b_y \\ 0 \\ \frac{1}{2}(1+\xi)b_y \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\underset{\sim}{P}_b = \begin{bmatrix} 0 \\ \int_{-1}^{1} \frac{1}{2}(1-\xi)\, b_y\, a\, d\xi \\ 0 \\ \int_{-1}^{1} \frac{1}{2}(1+\xi)\, b_y\, a\, d\xi \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{a}{3} \begin{bmatrix} 0 \\ (2b_{y1} + b_{y2}) \\ 0 \\ (b_{y1} + 2b_{y2}) \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Gravity load

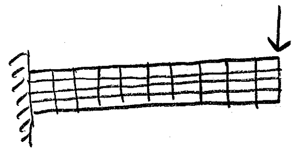


$$\underset{\sim}{b} = \begin{bmatrix} 0 \\ -b_g \end{bmatrix}$$

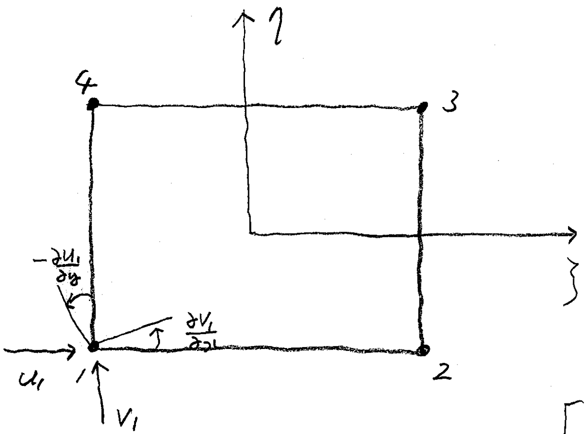
equivalent nodal load $= -b_g\, abt$

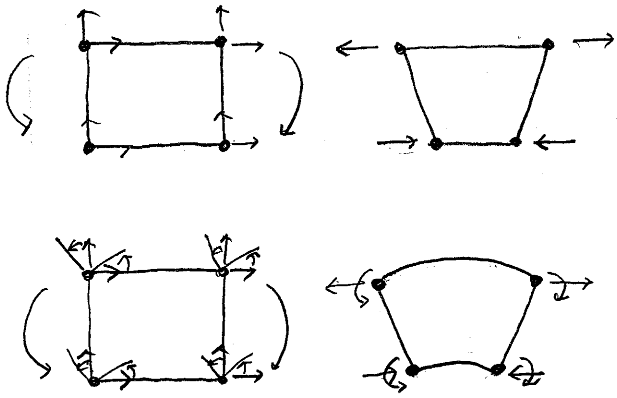
$$\underset{\sim}{P_b} = \int \underset{\sim}{f}^T \underset{\sim}{b}\, dV = abt \int_{-1}^{1}\int_{-1}^{1} \underset{\sim}{f}^T \underset{\sim}{b}\, d\xi\, d\eta$$

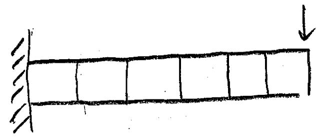
$$= -b_g\, abt \begin{bmatrix} 0 \\ \iint \frac{1}{4}(1-\xi)(1-\eta)\, d\xi\, d\eta \\ 0 \\ \iint \frac{1}{4}(1+\xi)(1-\eta)\, d\xi\, d\eta \\ 0 \\ \iint \frac{1}{4}(1+\xi)(1+\eta)\, d\xi\, d\eta \\ 0 \\ \iint \frac{1}{4}(1-\xi)(1+\eta)\, d\xi\, d\eta \end{bmatrix} = -b_g\, abt \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$$



Linear cubic Rectangle (Barber)



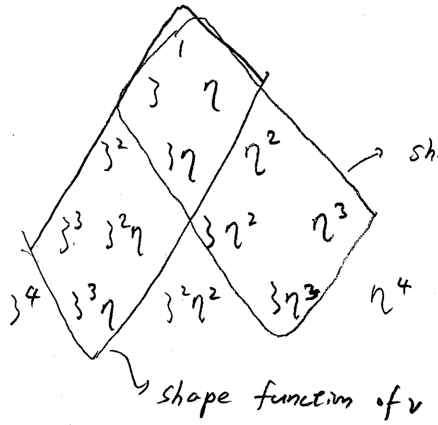




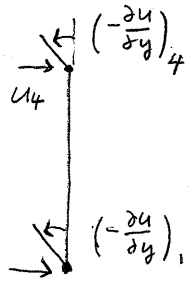
nodal displacement $q = \begin{bmatrix} u_1 \\ v_1 \\ \left(\frac{\partial v}{\partial x}\right)_1 \\ \left(-\frac{\partial u}{\partial y}\right)_1 \\ \vdots \\ u_4 \\ v_4 \\ \left(\frac{\partial v}{\partial x}\right)_4 \\ \left(-\frac{\partial u}{\partial y}\right)_4 \end{bmatrix}$

$$u = c_1 + c_2\xi + c_3\eta + c_4\xi\eta + c_5\eta^2 + c_6\xi\eta^2 + c_7\eta^3 + c_8\xi\eta^3 \quad -①$$

$$v = c_9 + c_{10}\xi + c_{11}\eta + c_{12}\xi\eta + c_{13}\xi^2 + c_{14}\xi^2\eta + c_{15}\xi^3 + c_{16}\xi^3\eta \quad -②$$



in case of u,



4 conditions

⇒ 3rd order eq. with respect to η (or y)

* condition for selecting polynomial eqns.
  8 terms, 3rd order eqns

$$u = [\,f_1 \ f_2 \ f_3 \ f_4 \ f_5 \ f_6 \ f_7 \ f_8\,] \begin{bmatrix} u_1 \\ (-\frac{\partial u}{\partial y})_1 \\ \vdots \\ u_4 \\ (-\frac{\partial u}{\partial y})_4 \end{bmatrix} \qquad u = \underset{\sim}{f_u}\, \underset{\sim}{q_u}$$

$$v = [\,f_9 \ f_{10} \ f_{11} \ f_{12} \ f_{13} \ f_{14} \ f_{15} \ f_{16}\,] \begin{bmatrix} v_1 \\ (\frac{\partial v}{\partial x})_1 \\ \vdots \\ v_4 \\ (\frac{\partial v}{\partial x})_4 \end{bmatrix} \qquad v = \underset{\sim}{f_v}\, \underset{\sim}{q_v}$$

f_i's can be defined with c_i's by using B/c :

at node 1, $u = u_1$, $(-\frac{\partial u}{\partial y})_\cdot = (\frac{\partial u}{\partial y})_1$ etc

$$\underset{\sim}{u} = \begin{bmatrix} u \\ v \end{bmatrix} = \underbrace{\left[\begin{array}{cccc|cccc|cccc|cccc} f_1 & 0 & 0 & f_2 & f_3 & 0 & 0 & f_4 & f_5 & 0 & 0 & f_6 & f_7 & 0 & 0 & f_8 \\ 0 & f_9 & f_{10} & 0 & 0 & f_{11} & f_{12} & 0 & 0 & f_{13} & f_{14} & 0 & 0 & f_{15} & f_{16} & 0 \end{array}\right]}_{\underset{\sim}{f}} \underbrace{\begin{bmatrix} u_1 \\ v_1 \\ (\frac{\partial v}{\partial x})_1 \\ (\frac{\partial u}{\partial y})_1 \\ \vdots \end{bmatrix}}_{\underset{\sim}{q}}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{d}\,\underset{\sim}{u} = \underset{\sim}{d}\,\underset{\sim}{f}\,\underset{\sim}{q} = \underset{\sim}{B}\,\underset{\sim}{q}$$

$$\underset{\sim}{B} = \underset{\sim}{d}\,\underset{\sim}{f} \qquad \underset{\sim}{d} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix}$$

$$\underset{\sim}{B} = \left[\begin{array}{cccc|c|cccc} f_{1,x} & 0 & 0 & f_{2,x} & & f_{7,x} & 0 & 0 & f_{8,x} \\ 0 & f_{9,y} & f_{10,y} & 0 & \cdots & 0 & f_{15,y} & f_{16,y} & 0 \\ f_{1,y} & f_{9,x} & f_{10,x} & f_{2,y} & & f_{7,y} & f_{15,x} & f_{16,x} & f_{8,y} \end{array}\right]$$

$$K = \int_V \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dV \qquad f_{i,x} = \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial x} \qquad f_{i,y} = \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

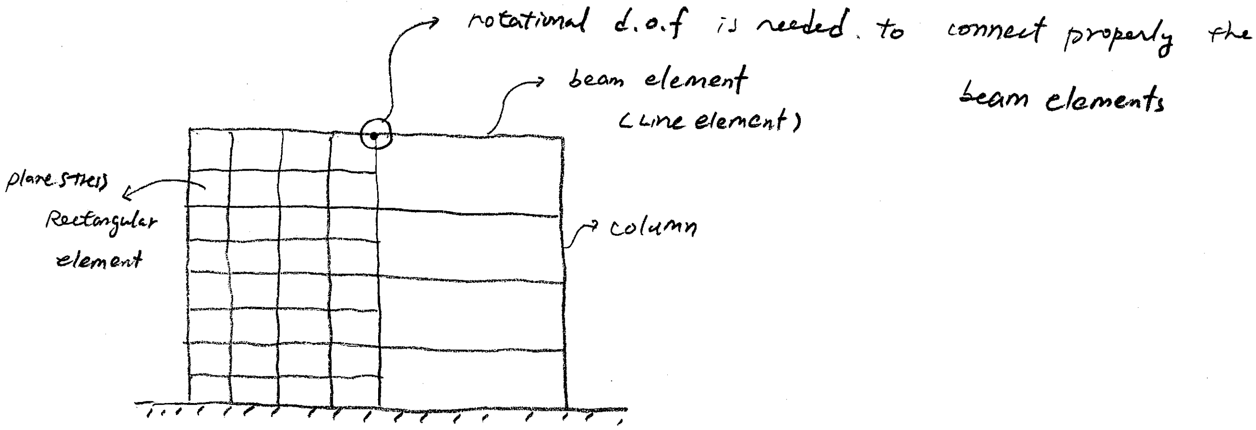
The shape function of u is linear in ξ and cubic in η

The shape function of v is linear in η and cubic in ξ.

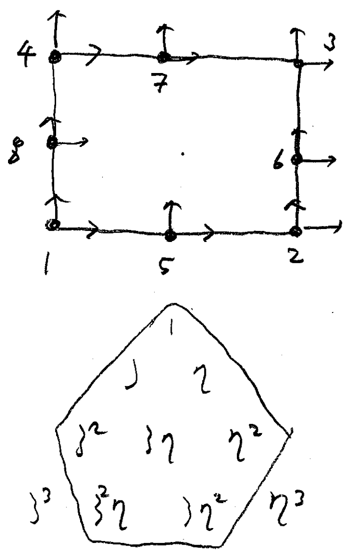
$\Rightarrow$ Linear - cubic Rectangle

The cubic functions allow compatibility between any edge of the rectangle and a bordering flexural element

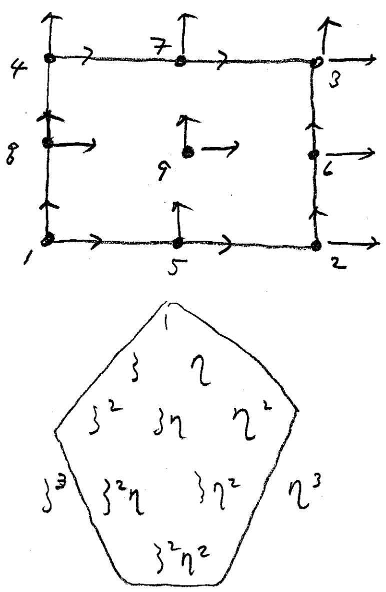
For example

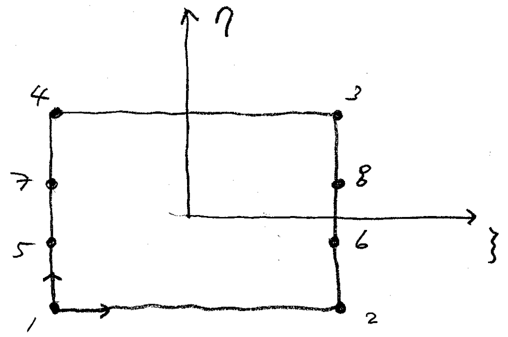


Eight-node Element



nine-node Element





$$\begin{cases} \text{four conditions in } \eta \text{ direction} \\ \text{two conditions in } \xi \text{ direction} \end{cases}$$

$$\begin{cases} u = c_1 + c_2\xi + c_3\eta + c_4\xi\eta + c_5\eta^2 + c_6\xi\eta^2 + c_7\eta^3 + c_8\xi\eta^3 \\ v = c_9 + c_{10}\xi + c_{11}\eta + c_{12}\xi\eta + c_{13}\eta^2 + c_{14}\xi\eta^2 + c_{15}\eta^3 + c_{16}\xi\eta^3 \end{cases}$$

Nodal displacements should be defined differently in the η direction.