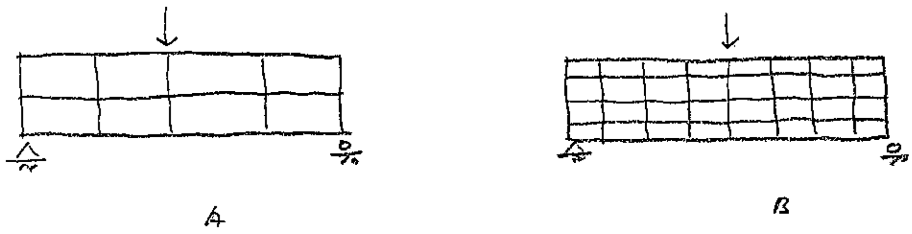
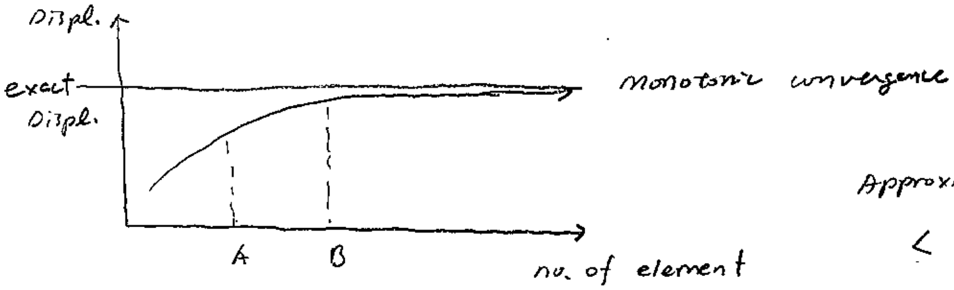
Convergence





Approximate displ. (Lower order)
< exact displ.
(higher order)

- Low order displacement function ⇒ large stiffness (stiff structure)
  small displacement

$$\underline{K}\,\underline{U} = \underline{P} \ (= \text{constant})$$

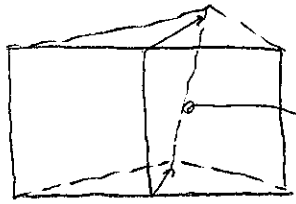
- (Higher order displacement function
  or
  Combination of a large number of elements

  ⇒ small stiffness (flexible structure),
  large displacement

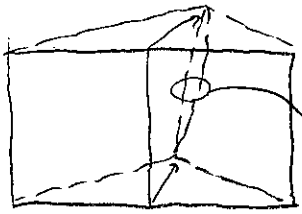
The approximate solution resulting from a limited number of elements always underestimates the exact displacement Δ_{exact}.

A approximate solution ($\Delta_{approx.}$) is referred to as the lower bound solution.

conforming element

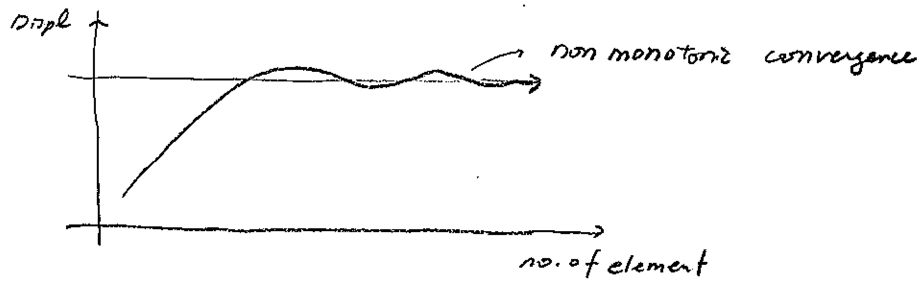


non-conforming element



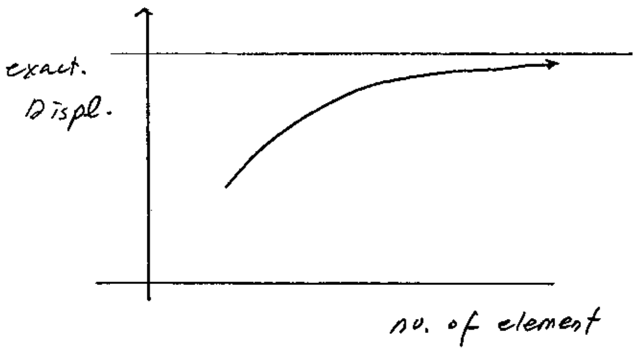
In case of non-conforming element, the continuity of the displacement which is a requirement for the principle of Total potential energy is not satisfied.

Accordingly, Π = minimum and $\Delta_{exact} > \Delta_{approx}$ is not always applicable.

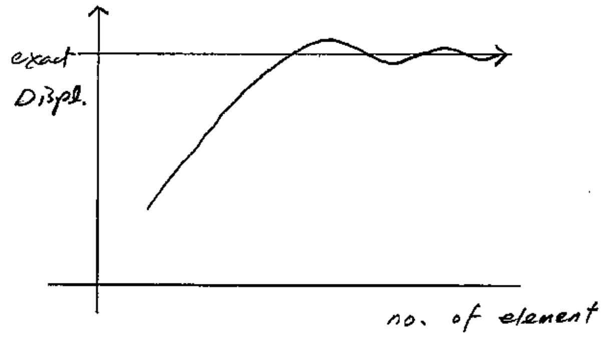


Convergence criteria

Convergence



monotonic convergence

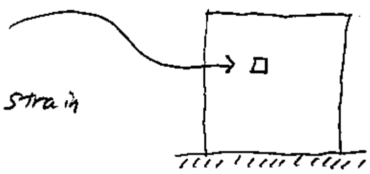


non-monotonic convergence

1. Essential Criterion for convergence (completeness)

1) Element should be able to describe states of constant strains which is the basic state of strain

when a element is very small, the strains are almost uniform over the element

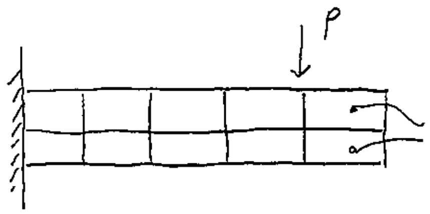


"0" strain is the special case of constant strain

$\frac{\partial u}{dx} = \varepsilon_x$ ... ⇒ Thus, polynomial displ. function should contain constant term and 1st order term.

2) Element should be able to describe rigid body motions.

that is the extreme case of the constant strain condition



(no stresses, no strains, only displacements)

$$\left.\begin{array}{l} \underset{\sim}{K}\,\underset{\sim}{q} = \underset{\sim}{P}\,(=\underset{\sim}{0}) \\ \underset{\sim}{\varepsilon} = \underset{\sim}{B}\,\underset{\sim}{q} = \underset{\sim}{0} \end{array}\right\} \quad \underset{\sim}{q} = \text{rigid body displacement}$$

Rigid body modes can be estimated using followings

$$\underset{\sim}{K}\,\underset{\sim}{q} = \lambda\,\underset{\sim}{q} \quad \text{or} \quad (\underset{\sim}{K} - \lambda\,\underset{\sim}{I})\,\underset{\sim}{q} = \underset{\sim}{0}$$

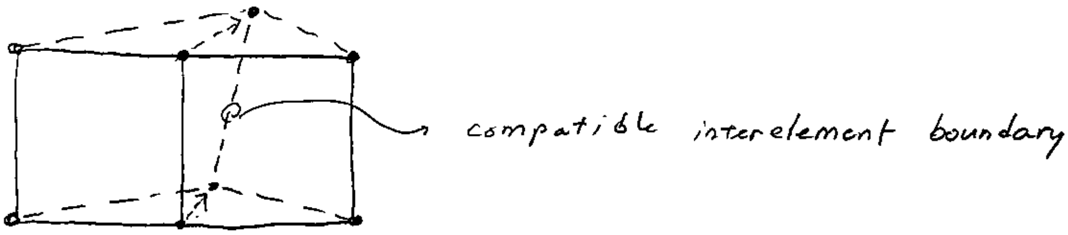
by solving the eigenvalue problem,

the rigid body modes $\underset{\sim}{q}$ correspoding to $\lambda = 0$

can be estimated.

2. criterion for monotonic convergence

Compatibility criterion ---- not always necessary for non-monotonic convergence.



1) The incompatible boundary violates the assumption of continuity that is used for the energy principle. Thus, the calculated (assumed) total potential energy is not necessarily an upper bound to the exact total potential energy of the system, and consequently, monotonic convergence is not assured.

2) It is easy to keep the compatibility for plane stress, plane strain, and solid elements with only displacement dof (u, v, w) with low order displacement functions: the displacement at a boundary can be defined as the interpolation of the nodal displacements.

3) On the other hand, it is difficult to retain the compatibility for plate bending with rotation (derivative of displacement) as well as the displacement (w)

4) However, even for plate bending and shell element, if mindlin theory (rotational dof is separated from displacement dof, using low order equation) is applied, the inter-element compatibility can be easily achieved.

Kirchhoff theory

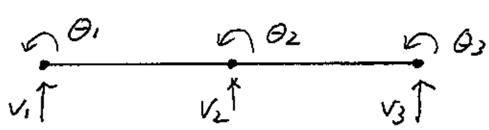
$$v = f_1 v_1 + f_2 \theta_1 + f_3 v_2 + f_4 \theta_2$$

$$\frac{\partial v}{\partial x}(\theta) = f_1' v_1 + f_2' \theta_1 + f_3' v_2 + f_4' \theta_2$$

Mindlin theory

$$v = f_1 v_1 + f_2 v_2 + f_3 v_3$$

$$\theta = f_1 \theta_1 + f_2 \theta_2 + f_3 \theta_3$$



$$\frac{\partial v}{\partial x} \neq \theta \qquad \theta = \theta_{bending} + \theta_{shear}$$

3. rate of convergence

Complete polynomials and spatially isotropic element

An example of spatially anisotropic element

$$u(x,y) = \alpha_1 + \alpha_2 x + \alpha_3 y + \alpha_4 xy$$

$$v(x,y) = \beta_1 + \beta_2 x + \beta_3 y + \beta_4 x^2$$

The element displacements are dependent on the orientation of the coordinate axes, because the polynomial terms associated with α_4 and β_4 are not the same.

The stiffness matrix of an element will depend on the orientation of the local element coordinate system.

Thus, it is better to use complete and balanced polynomials.

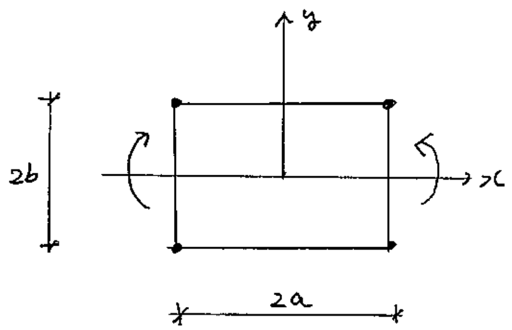
$$
\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & x & & y & & & \\
 & & x^2 & & xy & & y^2 & & \\
 & x^3 & & x^2y & & xy^2 & & y^3 & \\
x^4 & & x^3y & & x^2y^2 & & xy^2 & & y^4
\end{array}
$$

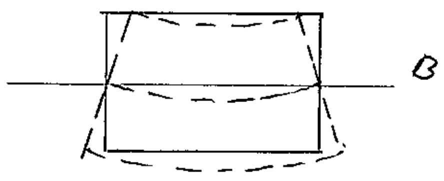
Calculation of stresses

$$\underline{\sigma} = \underline{\underline{E}}\,\underline{\varepsilon} = \underline{\underline{E}}\,\underline{\underline{B}}\,\underline{q}$$

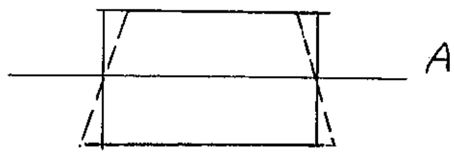
1) For compatible (or conforming) elements, the displacements are continuous across the element boundaries. However, this continuity does not mean that the stresses in the elements are continuous across the element boundaries.
2) This is because the force-equilibrium is not directly considered in the finite element mechanics: an approximate simplified displacement function is used, and the strains and stresses are defined with the lower order function. Thus, the accuracy of strains and stresses is worse than that of displacements. The force-equilibrium is indirectly considered using the energy principle.
3) Thus, generally, the stresses of the element are not continuous across the element boundaries. Likewise, the element stresses at the surface of the structure are not in equilibrium with the externally applied tractions (i.e. distributed loading).
4) But, the force equilibrium can be approximately satisfied in the limit as the number of elements increase.
5) Usually, element stresses are calculated at Gauss Integration points than at the nodal points of an element.

Incompatible mode element





pure bending mode



bending mode of 4-node rectangular element

The displacement function for pure bending is described by

$$u = c_1 xy \qquad \text{(a)}$$

$$v = \frac{1}{2} c_1 \left(a^2 - x^2\right) \qquad \text{(b)}$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = 0 \qquad \text{----- pure bending}$$

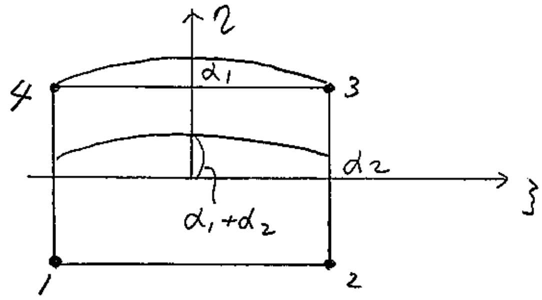
When 4-node rectangular is used, the displacement field contains only Eq. (a) (or Mode A) because the displacement function is defined with 1^{st} order polynomial equation.

To describe bending deformation shown Eq. (b) (or Mode B), the number of nodes can be increased to use higher order displacement equation. However, the increase of nodes and corresponding DOFs causes significant increase of computational time.

Alternatively, the displacement of pure bending mode (Eq. (b)) can be added to the displacement function rather than increasing the number of nodes.

The displacement function works only inside the element, and cause incompatibility at the element boundaries. Thus, such element belongs to incompatible element

2-D 4 nodes incompatible element



$$u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4 + \alpha_1 \left(1-\xi^2\right) + \alpha_2 \left(1-\eta^2\right)$$

$$v = f_1 v_1 + f_2 v_2 + f_3 v_3 + f_4 v_4 + \beta_1 \left(1-\xi^2\right) + \beta_2 \left(1-\eta^2\right)$$

1) α_1, α_2 and β_1, β_2 are additional DOF with a higher order displacement interpolation.

2) The additional dof allow the element to represent a constant bending moment.

3) The functions $\left(1-\xi^2\right), \left(1-\eta^2\right)$ satisfy the nodal displacement boundary conditions.

4) Total number of DOF is increased to 12. However, the stiffness matrix can be reduced by using static condensation.

$$\underset{\sim}{K}\ (12 \times 12) \Longrightarrow \underset{\sim}{K}\ (8 \times 8) \quad \text{static condensation}$$

$$\begin{bmatrix} \underset{\sim}{K}_{aa} & \underset{\sim}{K}_{ac} \\ \underset{\sim}{K}_{ca} & \underset{\sim}{K}_{cc} \end{bmatrix} \cdot \begin{bmatrix} \underset{\sim}{U}_a \\ \underset{\sim}{U}_c \end{bmatrix} = \begin{bmatrix} \underset{\sim}{R}_a \\ \underset{\sim}{R}_c \end{bmatrix}$$

$$\underset{\sim}{U}_c = \underset{\sim}{K}_{cc}^{-1} \left(\underset{\sim}{R}_c - \underset{\sim}{K}_{ca} \underset{\sim}{U}_a \right)$$

$$\left(\underset{\sim}{K}_{aa} - \underset{\sim}{K}_{ac} \underset{\sim}{K}_{cc}^{-1} \underset{\sim}{K}_{ca} \right) \underset{\sim}{U}_a = \underset{\sim}{R}_a - \underset{\sim}{K}_{ac} \underset{\sim}{K}_{cc}^{-1} \underset{\sim}{R}_c$$

5) Since α_1, α_2 and β_1, β_2 are not associated with the nodal dofs, displacement incompatibilities arise between elements. Thus, not monotonic convergence.

6) Convergence can be accomplished only for regular element shapes such as rectangle and parallelogram. Convergence is not accomplished for irregular element shapes.

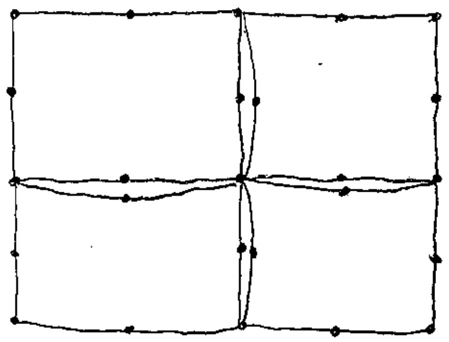
7) 3-D incompatible mode element (brick element)

$$u = \sum f_i u_i + \alpha_1 \left(1-\xi^2\right) + \alpha_2 \left(1-\eta^2\right) + \alpha_3 \left(1-\zeta^2\right)$$

$$v = \sum f_i v_i + \beta_1 \left(1-\xi^2\right) + \beta_2 \left(1-\eta^2\right) + \beta_3 \left(1-\zeta^2\right)$$

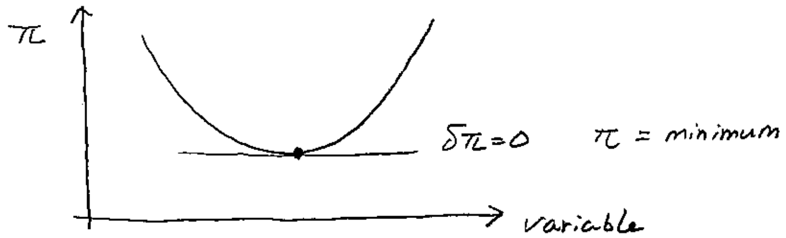
$$w = \sum f_i w_i + \gamma_1 \left(1-\xi^2\right) + \gamma_2 \left(1-\eta^2\right) + \gamma_3 \left(1-\zeta^2\right)$$

The use of the four node element with incompatible modes corresponds to the use of the eight-node rectangular element when the adjacent elements are not connected at the nodes at the middle of the edges.



Non-monotonic convergence of incompatible element

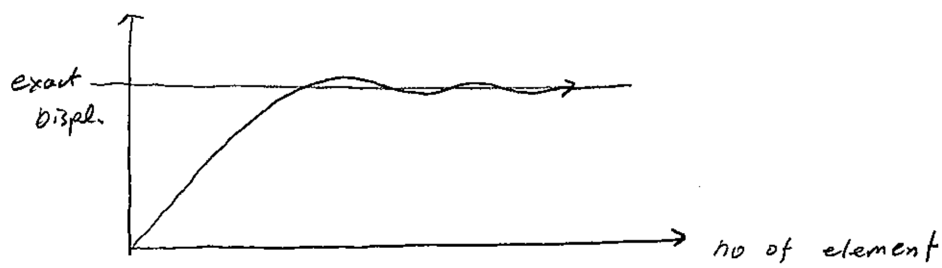
1) Since in finite analysis using incompatible (nonconforming) elements the Ritz analysis requirements (continuity requirement) are not satisfied, the calculated (assumed) total potential energy is not necessarily an upper bound to the exact total potential energy of the system, and consequently, monotonic convergence is not ~~assumed~~. assured



2) However, having relaxed the objective of monotonic convergence in the analysis, we still need to establish conditions that will assure at least a non-monotonic convergence.

3) For incompatible elements, although an individual element maybe able to represent all constant strain states (i.e. be complete), when the element is used in an assemblage, the incompatibilities between elements may prohibit constant strain states from being represented. ---- completeness condition on an element subassemblage.

4) As a test to investigate whether an assemblage of noncomforming element is complete, the patch test has been proposed.



5) The success or failure of the patch test may depend strongly on the geometry of the element used, i.e., on the topology of the element layout.

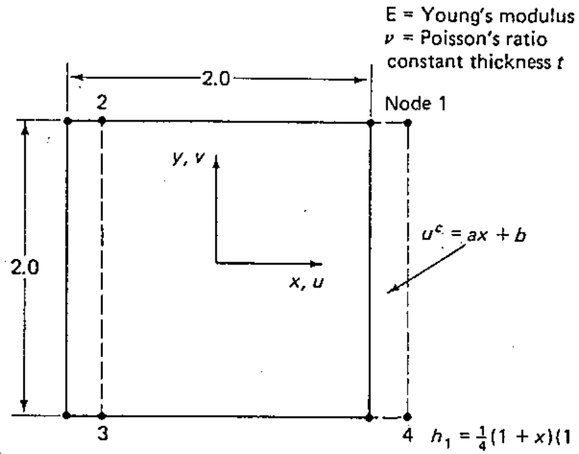
aim we need to subject an individual element of the element assemblage to the nodal point displacements, which by an exact analysis correspond to element constant strain conditions, and check whether constant strain conditions are indeed represented.

EXAMPLE 4.20: Consider the four-node two-dimensional element with incompatible modes in Fig. 4.29 and check whether the patch test is passed. The element under consideration is discussed in Section 5.6.2.

As shown in Fig. 4.29 assume that the displacement field to which the element is subjected in the element assemblage is

$$u^c = ax + b$$

where u^c corresponds to constant strain conditions $\epsilon^c_{xx} = a$, $\epsilon^c_{yy} = 0$, and $\gamma^c_{xy} = 0$. In order to complete the patch test we would afterwards also consider nonzero constant strain conditions for ϵ^c_{yy} and γ^c_{xy}. To investigate whether on imposing u^c onto the finite element, an incompatible mode ϕ_i is activated, we consider the integral $\int_V (\mathcal{L}\phi_i)(\mathcal{S}u^c)\, dV$ with $\mathcal{L}$ and $\mathcal{S}$ being the appropriate strain and stress differential operators (see (4.62)). This is equivalent to evaluating whether constraining forces that correspond to the incompatible mode are required when u_c is imposed onto the element. If no constraining forces corresponding to any one of the possible incompatible modes are present, it follows that the incompatible modes have not been activated and the patch test is passed.



$h_1 = \frac{1}{4}(1 + x)(1 + y)$
$h_2 = \frac{1}{4}(1 - x)(1 + y)$
$h_3 = \frac{1}{4}(1 - x)(1 - y)$
$h_4 = \frac{1}{4}(1 + x)(1 - y)$

Displacement interpolation functions

$$u = \sum_{i=1}^{4} h_i u_i + \alpha_1 \phi_1 + \alpha_2 \phi_2$$

$$v = \sum_{i=1}^{4} h_i v_i + \alpha_3 \phi_3 + \alpha_4 \phi_4$$

$$\phi_1 = \phi_3 = (1 - x^2);\ \phi_2 = \phi_4 = (1 - y^2)$$

FIGURE 4.29 *Four-node rectangular element with incompatible modes (plane stress conditions).*

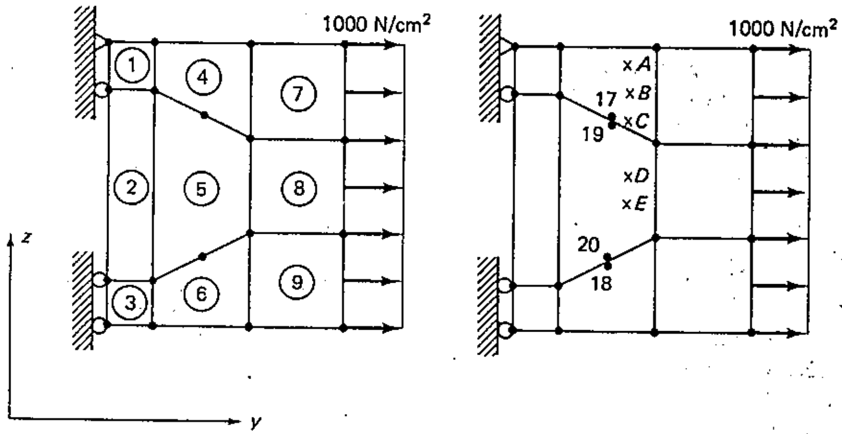
For plane stress conditions, we have the material law given in Table 4.3. The evaluation of the integral for the incompatible mode ϕ_1 in Fig. 4.29 gives

$$\frac{Et}{1 - \nu^2} \int_A (a)(-2x)dxdy = 0$$

But this is the requirement of the patch test.

Considering ϕ_2, ϕ_3, and ϕ_4 and then displacement fields that correspond to constant strain conditions ϵ^c_{yy} and γ^c_{xy} in a similar manner would complete the patch test. It is found that since the incompatible modes are of the form $(1 - x^2)$ and $(1 - y^2)$, all integrals of the form $\int_V (\mathcal{L}\phi_i)(\mathcal{S}u^c)\, dV$ are zero, and thus the patch test is passed. However, it should be observed that the success of the patch test depends on the geometry of the element under consideration, and indeed, the test is not passed when the element in Fig. 4.29 is not rectangular (or a parallelogram).

The use of the four-node element with incompatible modes corresponds to the use of the eight-node isoparametric element discussed in Section 5.3.1, when the midside nodes of adjacent elements are assigned individual degrees of freedom. Figure 4.30 shows the results obtained in the analysis of a constant



(a) Compatible element mesh; constant stress σ_{yy} = 1000 N/cm² in each element

(b) Incompatible element mesh; node 17 belongs to element 4, nodes 19 and 20 belong to element 5, and node 18 belongs to element 6.

σ_{yy} stress predicted by the incompatible element mesh:

Point	σ_{yy} (N/cm²)
A	1066
B	716
C	359
D	1303
E	1303

FIGURE 4.30 *Effect of displacement incompatibility in stress prediction (3 × 3 Gauss integration).*

6-12