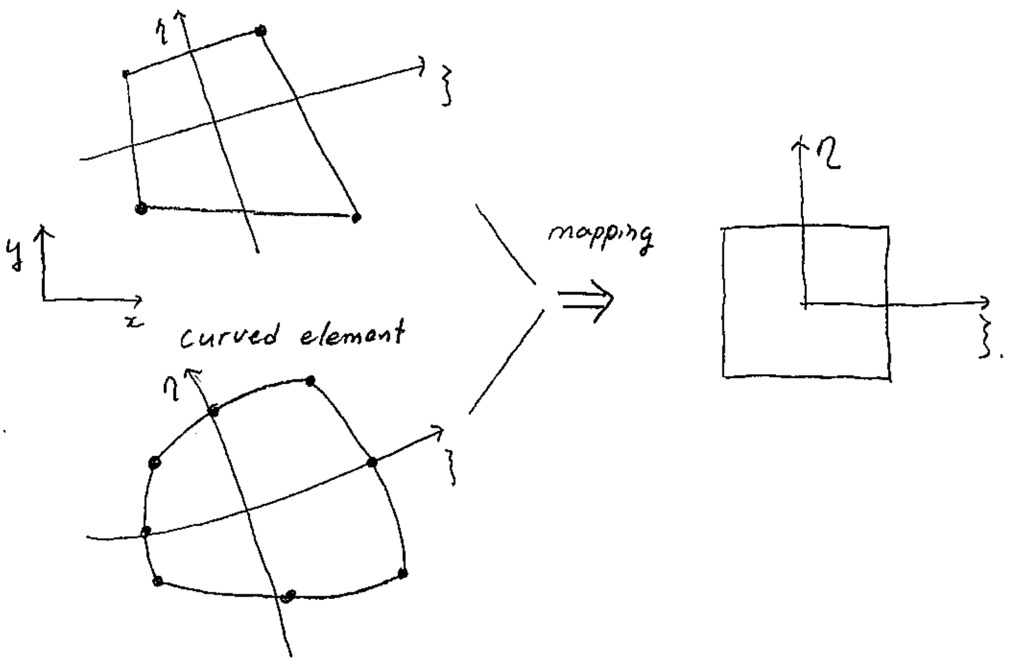
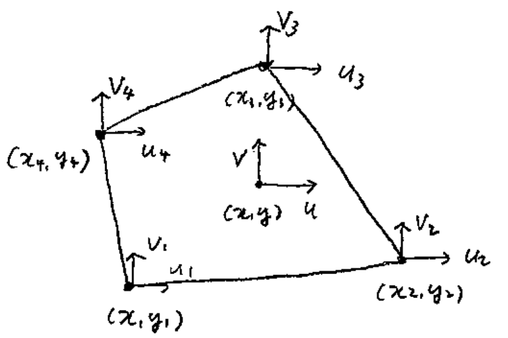
Isoparametric Formulation

Formulation for irregular shape elements

quadrilateral element





Mapped Element

uses shape functions

for both displacement field

and Geometry field.

Displ. field
$$\begin{cases} u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4 = \Sigma f_i u_i \\ v = \Sigma f_i v_i \end{cases}$$

Geometry field
$$\begin{cases} x = f_1 x_1 + f_2 x_2 + f_3 x_3 + f_4 x_4 = \Sigma f_i x_i \\ y = \Sigma f_i y_i \end{cases}$$

information between x-y coord. and ξ-η coord.

Classification of Element Types w.r.t order of shape function

- Isoparametric Element

  same interpolation function for both geometric) displacement) shape function

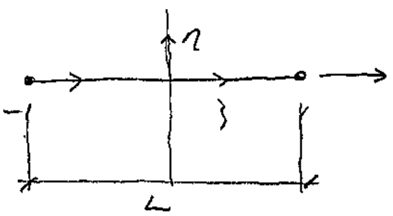
- Subparametric Element

  lower order function for geometric shape function

- Superparametric Element

  higher order function for geometric shape function.

2-node Truss



$$x = \Sigma f_i x_i = f_1 x_1 + f_2 x_2$$

$$u = \Sigma f_i u_i = f_1 u_1 + f_2 u_2$$

$$f_1 = \frac{1}{2}(1-\xi), \quad f_2 = \frac{1}{2}(1+\xi) \qquad f_i = \frac{1}{2}(1+\xi_i \xi)$$

$$\varepsilon = \frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx}$$

$$\frac{du}{d\xi} = \frac{df_1}{d\xi}u_1 + \frac{df_2}{d\xi}u_2 = \frac{1}{2}(u_2 - u_1)$$

$$\frac{dx}{d\xi} = \frac{df_1}{d\xi}x_1 + \frac{df_2}{d\xi}x_2 = \frac{1}{2}(x_2 - x_1) = \frac{L}{2}$$

$$\varepsilon = \frac{du}{d\xi}\frac{d\xi}{dx} = \frac{1}{L}(u_2 - u_1) = \underbrace{\frac{1}{L}\begin{bmatrix}-1 & 1\end{bmatrix}}_{\underset{\sim}{B}}\begin{bmatrix}u_1 \\ u_2\end{bmatrix}$$

$$\underset{\sim}{K} = \int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dV$$

$$= \int \underset{\sim}{B}^T E\, \underset{\sim}{B}\, A\, dx \qquad \left(dx = \frac{L}{2} d\xi\right) \qquad = \int_{-1}^{1} \underset{\sim}{B}^T E\, \underset{\sim}{B}\, A\left(\frac{L}{2}\right) d\xi$$

$$= \frac{EA}{L}\begin{bmatrix}1 & -1 \\ -1 & 1\end{bmatrix}$$

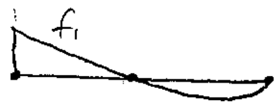
3-node Truss Element

$$x = \sum f_i x_i = f_1 x_1 + f_2 x_2 + f_3 x_3$$

$\xi = -1, \quad f_1 = 1$

$\xi = 0, 1 \quad f_1 = 0$

$\Rightarrow f_1 = -\frac{1}{2}\xi(1-\xi)$



$$f_2 = \frac{1}{2}\xi(1+\xi)$$

$$f_3 = (1-\xi)(1+\xi)$$

$$u = \sum f_i u_i = f_1 u_1 + f_2 u_2 + f_3 u_3$$

$$\varepsilon = \frac{du}{dx} = \frac{du}{d\xi}\frac{d\xi}{dx}$$

$$\frac{dx}{d\xi} = \frac{df_1}{d\xi}x_1 + \frac{df_2}{d\xi}x_2 + \frac{df_3}{d\xi}x_3 = J \qquad \text{Jacobian (Matrix)}$$

$$\frac{d\xi}{dx} = J^{-1}$$

$$\frac{du}{dx} = \frac{du}{d\xi}J^{-1} = J^{-1}\frac{du}{d\xi} \Rightarrow \frac{d}{dx} = J^{-1}\frac{d}{d\xi}$$

$$\varepsilon = J^{-1}\frac{du}{d\xi} = \underbrace{J^{-1}\begin{bmatrix} \frac{\partial f_1}{\partial \xi} & \frac{\partial f_2}{\partial \xi} & \frac{\partial f_3}{\partial \xi} \end{bmatrix}}_{\underset{\sim}{B}} \begin{bmatrix} u_1 \\ u_2 \\ u_3 \end{bmatrix}$$

$$\underset{\sim}{K} = \int \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B}\, dV$$

$$= \int \underset{\sim}{B}^T E \underset{\sim}{B} A\, dx$$

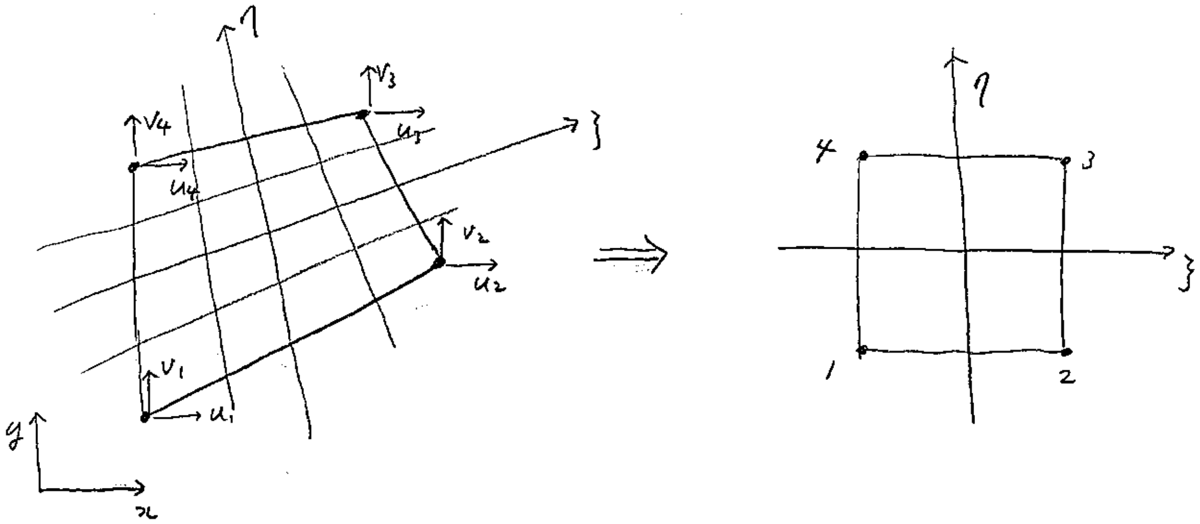
$$dx = \frac{dx}{d\xi}d\xi \qquad dx = J\,d\xi \qquad \text{Generally} \quad dx = |J|\,d\xi$$

$$\underset{\sim}{K} = \int_{-1}^{1} \underset{\sim}{B}^T E \underset{\sim}{B} A |J|\, d\xi$$

$\Rightarrow$ function of ξ

requires numerical integration (Gaussian Integration)

4 node Quadrilateral Element



$$\begin{cases} u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4 = \Sigma f_i u_i \\ v = \Sigma f_i v_i \end{cases}$$

$$\begin{cases} x = \Sigma f_i x_i \\ y = \Sigma f_i y_i \end{cases}$$ → relationship between (x, y) and (ξ, η)

$$f_1 = \frac{1}{4}(1-\xi)(1-\eta) \qquad f_2 = \frac{1}{4}(1+\xi)(1-\eta)$$

$$f_3 = \frac{1}{4}(1+\xi)(1+\eta) \qquad f_4 = \frac{1}{4}(1-\xi)(1+\eta)$$

$$f_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta)$$

Strain - Generic Displacement

$$\varepsilon_x = \frac{\partial u}{\partial x} = \frac{\partial f_1}{\partial x}u_1 + \frac{\partial f_2}{\partial x}u_2 + \frac{\partial f_3}{\partial x}u_3 + \frac{\partial f_4}{\partial x}u_4 = \Sigma \frac{\partial f_i}{\partial x}u_i$$

$$\varepsilon_y = \frac{\partial v}{\partial y} = \Sigma \frac{\partial f_i}{\partial y} v_i$$

$$\gamma_{xy} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x} = \Sigma \frac{\partial f_i}{\partial y}u_i + \Sigma \frac{\partial f_i}{\partial x} v_i$$

$$\frac{\partial f_i}{\partial x} = \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$\frac{\partial f_i}{\partial y} = \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

$$\begin{bmatrix} \frac{\partial f_i}{\partial x} \\ \frac{\partial f_i}{\partial y} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}}_{\rightarrow \underset{\sim}{J}^{-1}} \begin{bmatrix} \frac{\partial f_i}{\partial \xi} \\ \frac{\partial f_i}{\partial \eta} \end{bmatrix}$$

Let's think about

$$\begin{bmatrix} \frac{\partial f_i}{\partial \xi} \\ \frac{\partial f_i}{\partial \eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\text{Jacobian Matrix} = \underset{\sim}{J}} \begin{bmatrix} \frac{\partial f_i}{\partial x} \\ \frac{\partial f_i}{\partial y} \end{bmatrix}$$

$$\underset{\sim}{J} = \begin{bmatrix} J_{11} & J_{12} \\ J_{21} & J_{22} \end{bmatrix} = \begin{bmatrix} x_{,\xi} & y_{,\xi} \\ x_{,\eta} & y_{,\eta} \end{bmatrix}$$

$$x = \Sigma f_i x_i, \qquad y = \Sigma f_i y_i$$

$$J_{11} = x_{,\xi} = \Sigma \frac{\partial f_i}{\partial \xi} x_i$$

$$J_{12} = y_{,\xi} = \Sigma \frac{\partial f_i}{\partial \xi} y_i$$

$$J_{21} = x_{,\eta} = \Sigma \frac{\partial f_i}{\partial \eta} x_i$$

$$J_{22} = y_{,\eta} = \Sigma \frac{\partial f_i}{\partial \eta} y_i$$

$$\underset{\sim}{J}^{-1} = \underset{\sim}{J}^{*} = \begin{bmatrix} J_{11}^{*} & J_{12}^{*} \\ J_{21}^{*} & J_{22}^{*} \end{bmatrix}$$

since $\underset{\sim}{J}^{*}$ is complicated functions, the integral of $\underset{\sim}{J}^{*}$ cannot be explicitly calculated. Therefore, numerical integration should be used to obtain the stiffness matrix $\underset{\sim}{k}$

Strain - nodal displacement

$$\underset{\sim}{\varepsilon} = \underset{\sim}{B}\,\underset{\sim}{q}$$

$$\begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \end{bmatrix}$$

$$= \begin{bmatrix} \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} f_1 & 0 & f_2 & 0 & f_3 & 0 & f_4 & 0 \\ 0 & f_1 & 0 & f_2 & 0 & f_3 & 0 & f_4 \end{bmatrix} \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_4 \\ v_4 \end{bmatrix}$$

$$= \underbrace{\left[\begin{array}{cc|cc|cc|cc} f_{1,x} & 0 & f_{2,x} & 0 & f_{3,x} & 0 & f_{4,x} & 0 \\ 0 & f_{1,y} & 0 & f_{2,y} & 0 & f_{3,y} & 0 & f_{4,y} \\ f_{1,y} & f_{1,x} & f_{2,y} & f_{2,x} & f_{3,y} & f_{3,x} & f_{4,y} & f_{4,x} \end{array}\right]}_{\underset{\sim}{B}} \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_4 \\ v_4 \end{bmatrix}$$

$$\frac{\partial f_i}{\partial x} = \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial x}$$

$$= J_{11}^{*}\frac{\partial f_i}{\partial \xi} + J_{12}^{*}\frac{\partial f_i}{\partial \eta}$$

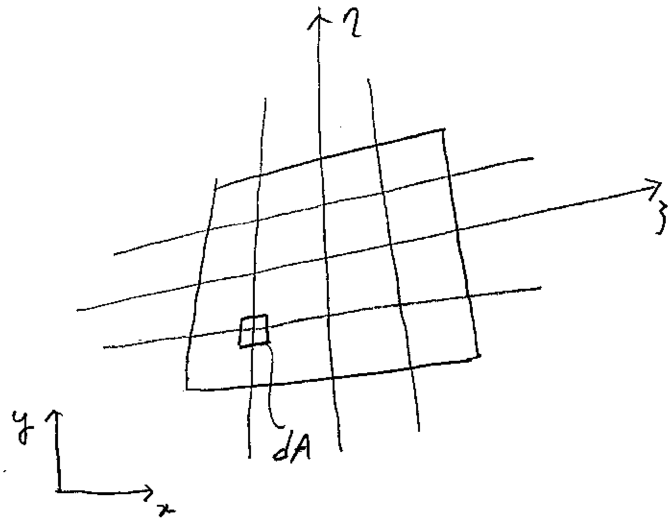
$$\frac{\partial f_i}{\partial y} = \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial y}$$

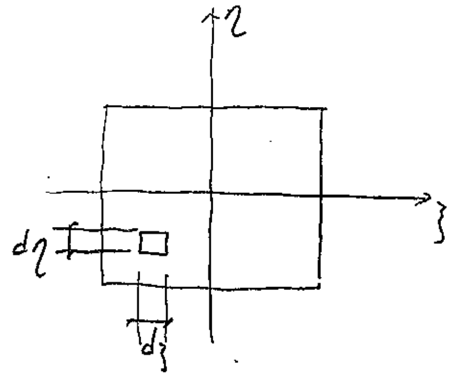
$$= J_{21}^{*}\frac{\partial f_i}{\partial \xi} + J_{22}^{*}\frac{\partial f_i}{\partial \eta}$$

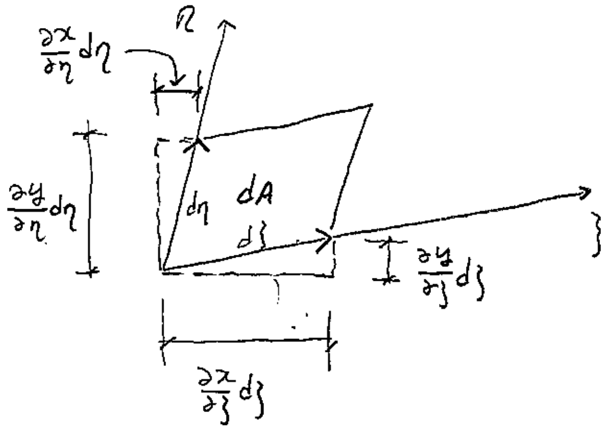
$$\underset{\sim}{K} = \int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dV = t\iint \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dA$$

$\underset{\sim}{B}$ = function of ξ, η

Surface Integral



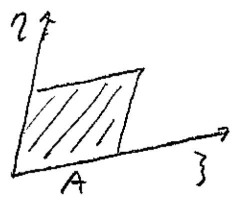


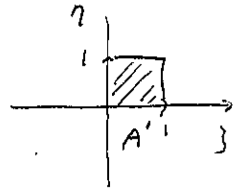


$$dA = \left(\frac{\partial x}{\partial \xi}d\xi\, i + \frac{\partial y}{\partial \xi}d\xi\, j\right) \times \left(\frac{\partial x}{\partial \eta}d\eta\, i + \frac{\partial y}{\partial \eta}d\eta\, j\right)$$

$$= \left[\left(\frac{\partial x}{\partial \xi}d\xi\right)\left(\frac{\partial y}{\partial \eta}d\eta\right) - \left(\frac{\partial y}{\partial \xi}d\xi\right)\left(\frac{\partial x}{\partial \eta}d\eta\right)\right] k$$

$$dA = \begin{vmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{vmatrix} d\xi d\eta = |\underset{\sim}{J}|\, d\xi d\eta$$





$|\underset{\sim}{J}| = A/A'$

$$\underset{\sim}{K} = t\iint \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dA$$

$$= t\iint \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, |J|\, d\xi d\eta$$

$$= \iint f(\xi,\eta)\, d\xi d\eta$$

Since $f(\xi,\eta)$ is a complicated function, generally it is not possible to calculate explicitly the integration.

$\Rightarrow$ we need numerical integration.

Numerical Integration

Newton - Cotes quadrature (numerical integration)

n values of the function can define a polynomial of degree n-1.

$$I = \int_{-1}^{1} f(\xi)\,d\xi = \sum_{1}^{n} R_i f(\xi_i)$$

$n=2$, trapezoidal rule, $I = f(-1) + f(1)$

$n=3$, simpson rule, $I = \frac{1}{3}[f(-1) + 4f(0) + f(1)]$

$n=4$, $I = \frac{1}{4}[f(-1) + 3f(-\frac{1}{3}) + 3f(\frac{1}{3}) + f(1)]$

⋮

⇒ Generalized method
- equal spaced nodes
- no specified interval

for example $n=3$ 3 points

$$f = a_0 + a_1\xi + a_2\xi^2$$

$$\int_{-1}^{1} f\,d\xi = [a_0\xi + \frac{1}{2}a_1\xi^2 + \frac{1}{3}a_2\xi^3]_{-1}^{1}$$

$$= 2a_0 + \frac{2}{3}a_2$$

$$c_1 f(-1) + c_2 f(0) + c_3 f(1)$$

$$= a_0(c_1 + c_2 + c_3) + a_1(-c_1 + c_3) + a_2(c_1 + c_3)$$

$$\left.\begin{aligned} c_1 + c_2 + c_3 &= 2 \\ -c_1 + c_3 &= 0 \\ c_1 + c_3 &= \tfrac{2}{3} \end{aligned}\right\} \qquad \begin{aligned} c_1 = c_3 &= \tfrac{1}{3} \\ c_2 &= \tfrac{4}{3} \end{aligned}$$

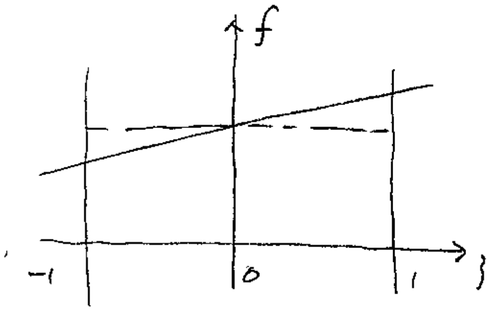
Gauss Quadrature

If in place of specifying the position of sampling points a priori, we allow these to be located at points to be determined so as to aim for best accuracy, then for a given number of sampling points increased accuracy can be obtained

$$I = \int_{-1}^{1} f(\xi)\,d\xi = \sum_{1}^{n} R_i f(\xi_i)$$

$$\begin{cases} R_i = \text{weighting factors} \\ \xi_i = \text{integration points} \\ n = \text{number of integration points} \end{cases}$$

if $f(\xi) = a_0 + a_1\xi$, $\quad n = 1$



$$I = \int_{-1}^{1} f(\xi)\,d\xi = 2a_0$$

$$= R_i f(\xi_i) = R_1(a_0 + a_1\xi_1)$$

$$\xi_1 = 0, \quad R_i = 2$$

if $f(\xi) = a_0 + a_1\xi + a_2\xi^2 + a_3\xi^3$,

two points of the function, and the corresponding two weighting factors can exactly define the coefficients of a polynomials of degree 3, and vice-versa.

$$I = \int_{-1}^{1} f(\xi)\,d\xi = 2a_0 + \frac{2}{3}a_2$$

$$= R_1 f(\xi_1) + R_2 f(\xi_2)$$

$$f(\xi_1) = a_0 + a_1\xi_1 + a_2\xi_1^2 + a_3\xi_1^3$$

$$f(\xi_2) = a_0 + a_1\xi_2 + a_2\xi_2^2 + a_3\xi_2^3$$

$$R_1 f(\xi_1) + R_2 f(\xi_2)$$

$$= a_0 (R_1 + R_2) + a_1 (R_1 \xi_1 + R_2 \xi_2) + a_2 (R_1 \xi_1^2 + R_2 \xi_2^2)$$

$$+ a_3 (R_1 \xi_1^3 + R_2 \xi_2^3)$$

$$= 2a_0 + \frac{2}{3} a_2$$

$$\left\{ \begin{aligned} R_1 + R_2 &= 2 \\ R_1 \xi_1 + R_2 \xi_2 &= 0 \\ R_1 \xi_1^2 + R_2 \xi_2^2 &= \frac{2}{3} \\ R_1 \xi_1^3 + R_2 \xi_2^3 &= 0 \end{aligned} \right\} \Rightarrow \begin{aligned} &\xi_1 = -\xi_2 \\ &R_1 = R_2 = 1 \\ &\xi_1^2 = \frac{1}{3} \Rightarrow \xi_1 = -\frac{1}{\sqrt{3}}, \quad \xi_2 = \frac{1}{\sqrt{3}} \end{aligned}$$

for n sampling points, we have $2n$ unknowns (R_i and ξ_i) and hence a polynomial of degree $2n-1$ could be constructed and exactly integrated.

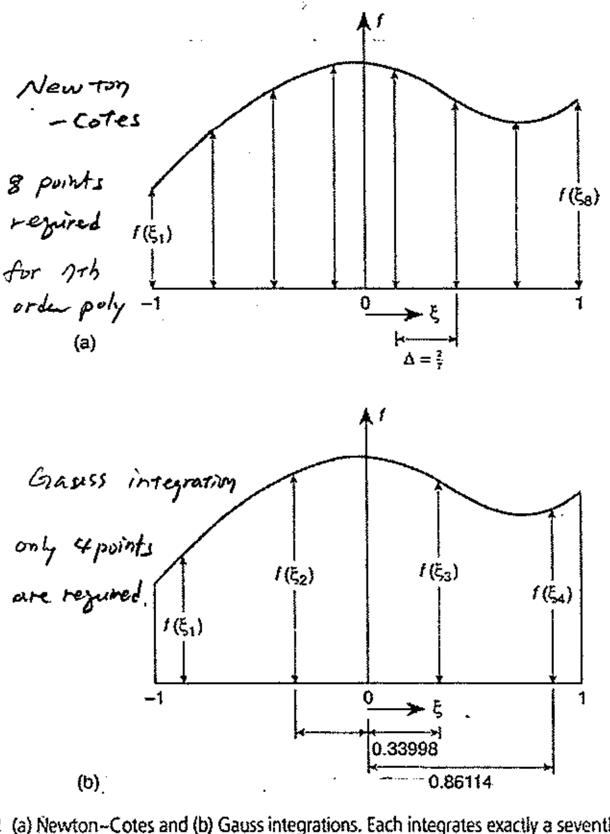


Fig. 9.12 (a) Newton–Cotes and (b) Gauss integrations. Each integrates exactly a seventh-order polynomial [i.e., error $O(h^8)$].

7-14

Table 9.1 Abscissae and weight coefficients of the gaussian quadrature formula $\int_{-1}^{1} f(x)\,dx = \sum_{j=1}^{n} H_i f(a_j)$

$\pm a$		H
	$n = 1$	
0		2.000 000 000 000 000
	$n = 2$	
$1/\sqrt{3}$		1.000 000 000 000 000
	$n = 3$	
$\sqrt{0.6}$		5/9
0.000 000 000 000 000		8/9
	$n = 4$	
0.861 136 311 594 953		0.347 854 845 137 454
0.339 981 043 584 856		0.652 145 154 862 546
	$n = 5$	
0.906 179 845 938 664		0.236 926 885 056 189
0.538 469 310 105 683		0.478 628 670 499 366
0.000 000 000 000 000		0.568 888 888 888 889
	$n = 6$	
0.932 469 514 203 152		0.171 324 492 379 170
0.661 209 386 466 265		0.360 761 573 048 139
0.238 619 186 083 197		0.467 913 934 572 691
	$n = 7$	
0.949 107 912 342 759		0.129 484 966 168 870
0.741 531 185 599 394		0.279 705 391 489 277
0.405 845 151 377 397		0.381 830 050 505 119
0.000 000 000 000 000		0.417 959 183 673 469
	$n = 8$	
0.960 289 856 497 536		0.101 228 536 290 376
0.796 666 477 413 627		0.222 381 034 453 374
0.525 532 409 916 329		0.313 706 645 877 887
0.183 434 642 495 650		0.362 683 783 378 362
	$n = 9$	
0.968 160 239 507 626		0.081 274 388 361 574
0.836 031 107 326 636		0.180 648 160 694 857
0.613 371 432 700 590		0.260 610 696 402 935
0.324 253 423 403 809		0.312 347 077 040 003
0.000 000 000 000 000		0.330 239 355 001 260
	$n = 10$	
0.973 906 528 517 172		0.066 671 344 308 688
0.865 063 366 688 985		0.149 451 349 150 581
0.679 409 568 299 024		0.219 086 362 515 982
0.433 395 394 129 247		0.269 266 719 309 996
0.148 874 338 981 631		0.295 524 224 714 753

Numerical Integration in 2 dimensions

One dimension

$$I = \int_{-1}^{1} f(\xi)\, d\xi = \sum_{i=1}^{n} R_i f(\xi_i)$$

$$= R_1 \phi(\xi_1) + R_2 \phi(\xi_2) + \cdots + R_n \phi(\xi_n)$$

two dimension

$$I = \int_{-1}^{1}\int_{-1}^{1} f(\xi, \eta)\, d\xi\, d\eta = \sum_{i=1}^{n}\sum_{j=1}^{m} R_i R_j f(\xi_i \eta_j)$$

$$= R_1 \left[R_1 f(\xi_1, \eta_1) + R_2 f(\xi_1, \eta_2) + \cdots + R_m f(\xi_1, \eta_m) \right]$$

$$+ R_2 \left[R_1 f(\xi_2, \eta_1) + R_2 f(\xi_2 \eta_2) + \cdots + R_m f(\xi_2, \eta_m) \right]$$

$$+ \cdots$$

$$+ R_n \left[R_1 f(\xi_n \eta_1) + R_2 f(\xi_n \eta_2) + \cdots + R_m f(\xi_n, \eta_m) \right]$$

three dimension

$$I = \iiint f(\xi, \eta, \zeta)\, d\xi\, d\eta\, d\zeta$$

$$= \sum_{i=1}^{n}\sum_{j=1}^{m}\sum_{k=1}^{l} R_i R_j R_k f(\xi_i \eta_j \zeta_k)$$

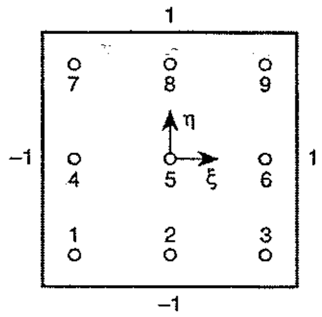


Fig. 9.13 Integrating points for $n = 3$ in a square region. (Exact for polynomial of fifth order in each direction).

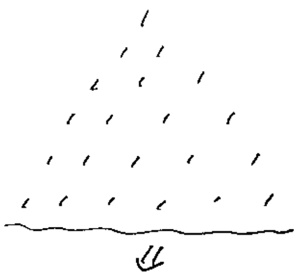
Using Gauss Quadrature, nine sampling points result in exact integrals of order 5 in each direction.

However, we could approach the problem directly and require an exact integration of a fifth-order polynomial in two dimensions.

$$I = \int_{-1}^{1}\int_{-1}^{1} f(\xi,\eta)\, d\xi d\eta = \sum_{1}^{m} w_i f(\xi_i,\eta_i)$$

Only seven points would suffice to obtain the same order of accuracy if the number of terms are 21.

5th order →



21

$\Rightarrow$ we needs 7 points $\left\{ \begin{array}{l} R_i \\ \xi_i \\ \eta_i \end{array} \right.$

$7 \times 3 = 21$

Numerical Integration

$$\mathbf{K} = t\int_A \mathbf{B}^T\mathbf{E}\mathbf{B}\, dA$$

$$= t\iint \mathbf{B}^T\mathbf{E}\mathbf{B}\ |\mathbf{J}|\, d\xi\, d\eta = \sum_i^n \sum_j^m R_i R_j\ \mathbf{f}(\xi_i \eta_i)$$

$$\mathbf{f}(\xi,\eta) = t\,\mathbf{B}^T\mathbf{E}\mathbf{B}\,|\mathbf{J}|$$

$$\mathbf{P}_b = t\iint \mathbf{f}^T \mathbf{b}\ |\mathbf{J}|\, d\xi\, d\eta = \sum_i^n \sum_j^m R_i R_j\ \mathbf{g}(\xi_i \eta_j)$$

$$\mathbf{g}(\xi,\eta) = t\,\mathbf{f}^T\mathbf{b}\,|\mathbf{J}|$$

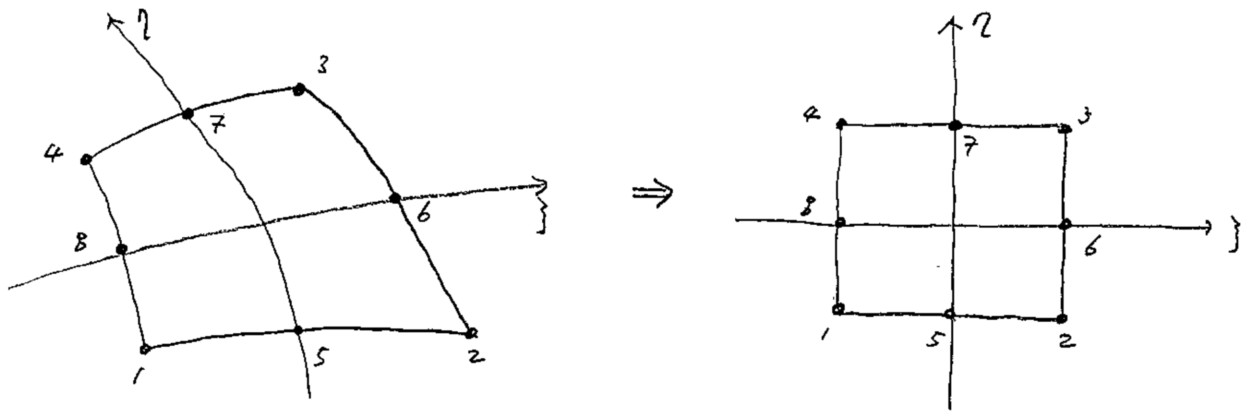
$$\mathbf{P}_0 = t\iint \mathbf{B}^T\mathbf{E}\,\boldsymbol{\varepsilon}_0\ |\mathbf{J}|\, d\xi\, d\eta = \sum_i^n \sum_j^m R_i R_j\ \mathbf{h}(\xi_i \eta_j)$$

$$\mathbf{h}(\xi,\eta) = t\,\mathbf{B}^T\mathbf{E}\,\boldsymbol{\varepsilon}_0\,|\mathbf{J}|$$

If $\boldsymbol{\varepsilon}_0$ and $\mathbf{b}$ are functions of x, y,

x and y are replaced by $\Sigma f_i x_i$ and $\Sigma f_i y_i$.

Isoparametric Quadrilateral 8-node Element



$$\begin{cases} u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4 + f_5 u_5 + f_6 u_6 + f_7 u_7 + f_8 u_8 \\ \quad = \sum_{i=1}^{8} f_i u_i \\ v = \sum f_i v_i \end{cases}$$

$$\begin{cases} x = \sum f_i x_i \\ y = \sum f_i y_i \end{cases}$$

at nodes 2, 6, 3, 7, 4, $f_1 = 0$

$\Rightarrow \quad \xi = 1, \quad \eta = 1, \quad f_1 = 0$

at nodes 5, 8, $\quad f_1 = 0$

$\Rightarrow \quad \eta = -\xi - 1, \quad f_1 = 0$

$\Rightarrow f_1 = \alpha_1 (1-\xi)(1-\eta)(-\xi-\eta-1)$

at node 1, $f_1 = 1 \quad \Rightarrow \alpha_1 = \frac{1}{4}$

$$f_1 = \frac{1}{4}(1-\xi)(1-\eta)(-\xi-\eta-1)$$

$$f_2 = \frac{1}{4}(1+\xi)(1-\eta)(\xi-\eta-1)$$

$$\vdots$$

$$\Rightarrow f_i = \frac{1}{4}(1+\xi_i\xi)(1+\eta_i\eta) \times (\xi_i\xi + \eta_i\eta - 1)$$

$i = 1 \sim 4$

at nodes 1, 2, 6, 3, 7, 4, 8, $f_5 = 0$

$\Rightarrow \quad \xi = -1, \quad \xi = 1, \quad \eta = 1, \quad f_5 = 0 \quad \Rightarrow \quad f_5 = \alpha_5 (1-\xi)(1+\xi)(1-\eta)$

at node 5, $f_5 = 1 \quad \Rightarrow \quad \alpha_5 = \frac{1}{2}$

$$f_5 = \frac{1}{2}(1-\xi)(1+\xi)(1-\eta) = \frac{1}{2}(1-\xi^2)(1-\eta)$$

$$\begin{cases} f_i = \frac{1}{2}(1-\xi^2)(1+\eta_i\eta) & i = 5, 7 \\ f_i = \frac{1}{2}(1+\xi_i\xi)(1-\eta^2) & i = 6, 8 \end{cases}$$

strain - nodal displacements

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \left[\begin{array}{cc|c|cc} f_{1,x} & 0 & & f_{8,x} & 0 \\ 0 & f_{1,y} & \cdots\cdots & 0 & f_{8,y} \\ f_{1,y} & f_{1,x} & & f_{8,y} & f_{8,x} \end{array}\right] \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_8 \\ v_8 \end{bmatrix}$$

$$\begin{bmatrix} f_{i,\xi} \\ f_{i,\eta} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} \\ \frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} \end{bmatrix}}_{\underset{\sim}{J}} \begin{bmatrix} f_{i,x} \\ f_{i,\eta} \end{bmatrix}$$

$$\begin{cases} J_{11} = \frac{\partial x}{\partial \xi} = \Sigma f_{i,\xi}\, x_i \\ J_{12} = \frac{\partial y}{\partial \xi} = \Sigma f_{i,\xi}\, y_i \\ J_{21} = \frac{\partial x}{\partial \eta} = \Sigma f_{i,\eta}\, x_i \\ J_{22} = \frac{\partial y}{\partial \eta} = \Sigma f_{i,\eta}\, y_i \end{cases}$$

$$\begin{bmatrix} f_{i,x} \\ f_{i,y} \end{bmatrix} = \underbrace{\begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} \end{bmatrix}}_{\underset{\sim}{J}^{-1} = \underset{\sim}{J}^*} \begin{bmatrix} f_{i,\xi} \\ f_{i,\eta} \end{bmatrix}$$

$$f_{i,x} = J_{11}^* f_{i,\xi} + J_{12}^* f_{i,\eta}$$

$$f_{i,y} = J_{21}^* f_{i,\xi} + J_{22}^* f_{i,\eta}$$

$$\underset{\sim}{K} = t \iint \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, |J| \, d\xi \, d\eta$$

$$= \sum_i^n \sum_j^m R_i R_j \, g(\xi_i \eta_j)$$

- more integration points than 4-node elements because $g(\xi, \eta)$ is higher order
- usually 3×3 points are used.

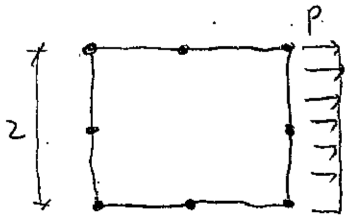
Lumping of Structure Properties and Loads

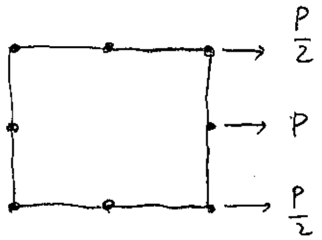
consistent load

$$\int \underset{\sim}{f}^T \underset{\sim}{b} \, dV$$

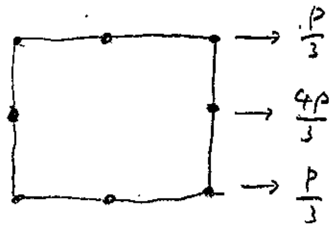
consistent mass

$$\int \rho \underset{\sim}{f}^T \underset{\sim}{f} \, dV$$





: lumped load

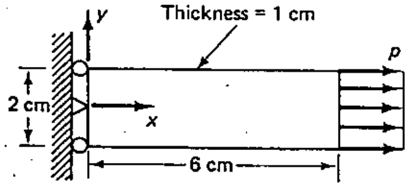


: consistent load

- When using the load lumping procedure it should be recognized that the nodal point loads are, in general, only calculated approximately, and if a course f.E mesh is employed, the resulting solution may be very inaccurate. for higher-order finite elements, surprising results are obtained.

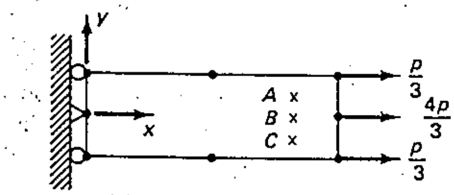
- An important advantage of using a lumped mass matrix is that the matrix is diagonal, and, the numerical operations are reduced very significantly.

Lumping of Structure Properties and Loads



$p = 300\ N/cm^2$
$E = 3 \times 10^7\ N/cm^2$
$\nu = 0.3$

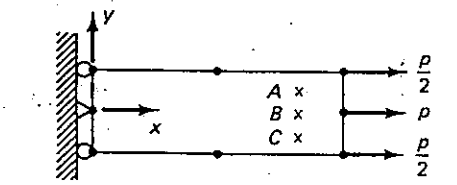
(a) Problem



(b) Finite element model with consistent loading

Integration point	τ_{xx}	τ_{yy}	τ_{xy}
A	300.00	0.0	0.0
B	300.00	0.0	0.0
C	300.00	0.0	0.0

(All stresses have units of N/cm^2)



(c) Finite element model with lumped loading

Integration point	τ_{xx}	τ_{yy}	τ_{xy}
A	301.41	−7.85	−24.72
B	295.74	−9.55	0.0
C	301.41	−7.85	24.72

(All stresses have units of N/cm^2)

FIGURE 4.22 *Some sample analysis results with and without consistent loading.*