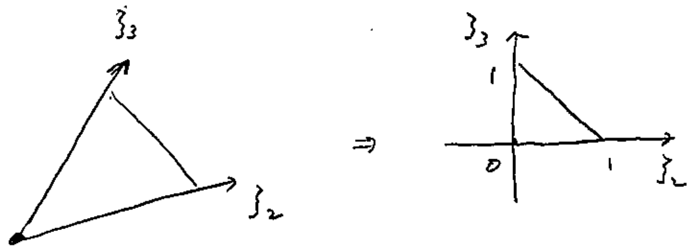
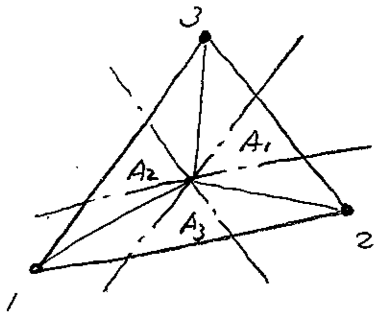
Triangular Element



Area coordinate



$$\xi_1 = \frac{A_1}{A}, \quad \xi_2 = \frac{A_2}{A}, \quad \xi_3 = \frac{A_3}{A}$$

$$A_1 + A_2 + A_3 = A$$

$$\xi_1 + \xi_2 + \xi_3 = 1$$

$$\begin{cases} u = f_1 u_1 + f_2 u_2 + f_3 u_3 \\ v = f_1 v_1 + f_2 v_2 + f_3 v_3 \end{cases}$$

$$\left.\begin{cases} \text{at node 1} \quad \xi_1 = 1 \quad \xi_2 = 0 \quad \xi_3 = 0 \\ \text{at node 2} \quad \xi_1 = 0 \quad \xi_2 = 1 \quad \xi_3 = 0 \\ \text{at node 3} \quad \xi_1 = 0 \quad \xi_2 = 0 \quad \xi_3 = 1 \end{cases}\right\} \Rightarrow f_i = \xi_i$$

$$\begin{cases} u = \xi_1 u_1 + \xi_2 u_2 + \xi_3 u_3 \\ v = \xi_1 v_1 + \xi_2 v_2 + \xi_3 v_3 \end{cases}$$

$$\begin{cases} x = \xi_1 x_1 + \xi_2 x_2 + \xi_3 x_3 \\ y = \xi_1 y_1 + \xi_2 y_2 + \xi_3 y_3 \\ 1 = \xi_1 + \xi_2 + \xi_3 \end{cases}$$

$$\begin{bmatrix} 1 \\ x \\ y \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 \\ x_1 & x_2 & x_3 \\ y_1 & y_2 & y_3 \end{bmatrix} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix}$$

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \left[\begin{array}{cc|cc|cc} f_{1,x} & 0 & f_{2,x} & 0 & f_{3,x} & 0 \\ 0 & f_{1,y} & 0 & f_{2,y} & 0 & f_{3,y} \\ f_{1,y} & f_{1,x} & f_{2,y} & f_{2,x} & f_{3,y} & f_{3,x} \end{array}\right] \begin{bmatrix} u_1 \\ v_1 \\ u_2 \\ v_2 \\ u_3 \\ v_3 \end{bmatrix}$$

$$f_{1,x} = \zeta_{1,x} = \frac{1}{2A}(y_2 - y_3)$$

$$f_{1,y} = \zeta_{1,y} = \frac{1}{2A}(x_3 - x_2)$$

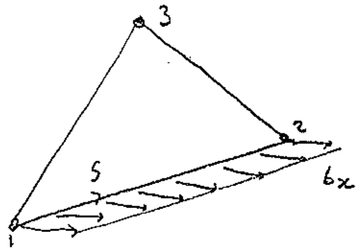
$$f_{2,x} = \zeta_{2,x} = \frac{1}{2A}(y_3 - y_1)$$

$$f_{2,y} = \zeta_{2,y} = \frac{1}{2A}(x_1 - x_3)$$

$$f_{3,x} = \zeta_{3,x} = \frac{1}{2A}(y_1 - y_2)$$

$$f_{3,y} = \zeta_{3,y} = \frac{1}{2A}(x_2 - x_1)$$

$$\underset{\sim}{K} = \int \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, dV = \underbrace{\underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B}\, t \cdot A}_{\Downarrow \ \text{constants}}$$



$$\underset{\sim}{b} = \begin{bmatrix} b_x \\ 0 \end{bmatrix}$$

$$\underset{\sim}{f} = \begin{bmatrix} f_1 & 0 & f_2 & 0 & f_3 & 0 \\ 0 & f_1 & 0 & f_2 & 0 & f_3 \end{bmatrix} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & \zeta_3 \end{bmatrix}$$

$$\overline{12} \Rightarrow \zeta_3 = 0 \qquad \underset{\sim}{f} = \begin{bmatrix} \zeta_1 & 0 & \zeta_2 & 0 & 0 & 0 \\ 0 & \zeta_1 & 0 & \zeta_2 & 0 & 0 \end{bmatrix}$$

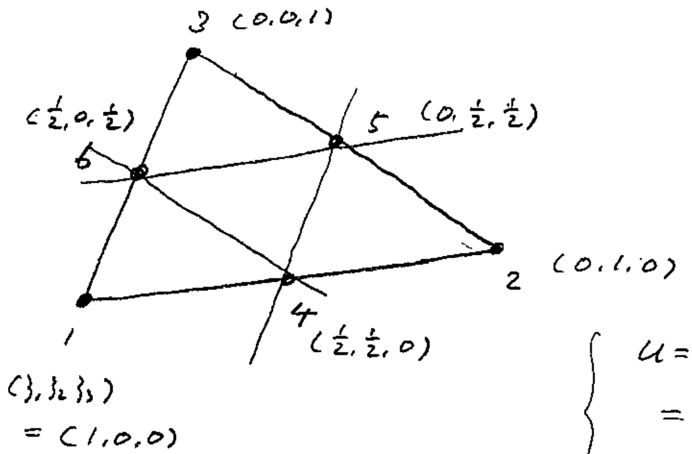
$$\underset{\sim}{P_b} = \int_0^l \underset{\sim}{f}^T \underset{\sim}{b} \, ds = \begin{bmatrix} b_x \int_0^l \zeta_1 \, ds \\ 0 \\ b_x \int_0^l \zeta_2 \, ds \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

$$s = l(1-\zeta_1) \qquad ds = -l \, d\zeta_1$$

$$s = l \zeta_2 \qquad ds = l \, d\zeta_2$$

$$= \begin{bmatrix} b_x \int_1^0 \zeta_1 (-l) d\zeta_1 \\ 0 \\ b_x \int_0^1 \zeta_2 l \, d\zeta_2 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} \frac{b_x l}{2} \\ 0 \\ \frac{b_x l}{2} \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

6-node subparametric triangular Element



$\zeta_1 = \frac{A_1}{A}, \quad \zeta_2 = \frac{A_2}{A}, \quad \zeta_3 = \frac{A_3}{A}$

$\zeta_1 + \zeta_2 + \zeta_3 = 1$

$$\begin{cases} u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4 + f_5 u_5 + f_6 u_6 \\ \quad = \sum_i^6 f_i u_i \\ v = \sum_i^6 f_i v_i \end{cases}$$

$$\left.\begin{array}{llll} \text{at nodes } 2,5,3 & \Rightarrow & \zeta_1 = 0, & f_1 = 0 \\ \text{at nodes } 4,6 & \Rightarrow & \zeta_1 = \frac{1}{2}, & f_1 = 0 \\ \text{at node } 1 & \Rightarrow & \zeta_1 = 1, & f_1 = 1 \end{array}\right\} \Rightarrow \begin{array}{l} f_1 = 2\zeta_1(\zeta_1 - \frac{1}{2}) \\ \quad = \zeta_1(2\zeta_1 - 1) \end{array}$$

$$\begin{cases} f_1 = \zeta_1(2\zeta_1 - 1) \\ f_2 = \zeta_2(2\zeta_2 - 1) \\ f_3 = \zeta_3(2\zeta_3 - 1) \end{cases}$$

$$\left.\begin{array}{llll} \text{at nodes } 2,5,3 & \Rightarrow & \zeta_1 = 0, & f_4 = 0 \\ \text{at nodes } 1,6,3 & \Rightarrow & \zeta_2 = 0, & f_4 = 0 \\ \text{at node } 4 & \Rightarrow & \zeta_1 = \frac{1}{2}, \ \zeta_2 = \frac{1}{2}, & f_4 = 1 \end{array}\right\} \Rightarrow f_4 = 4\zeta_1\zeta_2$$

$$\begin{cases} f_4 = 4\zeta_1\zeta_2 \\ f_5 = 4\zeta_2\zeta_3 \\ f_6 = 4\zeta_3\zeta_1 \end{cases}$$

$$\left\{\begin{array}{l} x = f_1^u x_1 + f_2^u x_2 + f_3^u x_3 \\ \quad = \zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_3 \\ y = \zeta_1 y_1 + \zeta_2 y_2 + \zeta_3 y_3 \\ 1 = \zeta_1 + \zeta_2 + \zeta_3 \end{array}\right. \Rightarrow \text{subparametric}$$

$$\begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \end{bmatrix} = \frac{1}{2A} \begin{bmatrix} (x_2 y_3 - x_3 y_2) & (y_2 - y_3) & (x_3 - x_2) \\ (x_3 y_1 - x_1 y_3) & (y_3 - y_1) & (x_1 - x_3) \\ (x_1 y_2 - x_2 y_1) & (y_1 - y_2) & (x_2 - x_1) \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \end{bmatrix}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{bmatrix} = \underbrace{\left[\begin{array}{cc|c|cc} f_{1,x} & 0 & & f_{6,x} & 0 \\ 0 & f_{1,y} & \cdots & 0 & f_{6,y} \\ f_{1,y} & f_{1,x} & & f_{6,y} & f_{6,x} \end{array}\right]}_{\underset{\sim}{B}} \begin{bmatrix} u_1 \\ v_1 \\ \vdots \\ u_6 \\ v_6 \end{bmatrix}$$

$$f_{1,x} = \frac{\partial f_1}{\partial x} = \frac{\partial f_1}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial x} + \cancel{\frac{\partial f_1}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial x}}^{0} + \cancel{\frac{\partial f_1}{\partial \zeta_3}\frac{\partial \zeta_3}{\partial x}}^{0}$$

$$= \left[2(\zeta_1 - \tfrac{1}{2}) + 2\zeta_1\right]\left[\frac{1}{2A}(y_2 - y_3)\right]$$

$$= \frac{1}{2A}(4\zeta_1 - 1)(y_2 - y_3)$$

$$f_{1,y} = \cdots$$

$$\underset{\sim}{K} = t\int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dA$$

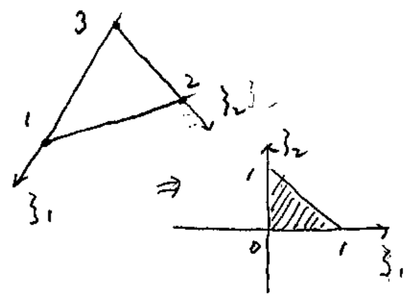
$$= \sum_{i=1} \int \zeta_1^{a_i} \zeta_2^{b_i} \zeta_3^{c_k}\, dA$$

Integrals of polynomial terms in the area coordinates ζ_1, ζ_2, ζ_3 can be obtained, using Γ function :

$$\int_A \zeta_1^a \zeta_2^b \zeta_3^c \, dA = \frac{a!\, b!\, c!}{(a+b+c+2)!} (2A)$$

But usually, also in the area coordinates, numerical integration is used.

$$\begin{cases} x = \zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_3 \\ y = \zeta_1 y_1 + \zeta_2 y_2 + \zeta_3 y_3 \end{cases}$$



$$\Rightarrow \quad x = \zeta_1 x_1 + \zeta_2 x_2 + (1-\zeta_1-\zeta_2) x_3$$

$$= (x_1 - x_3)\zeta_1 + (x_2 - x_3)\zeta_2 + x_3$$

$$y = (y_1 - y_3)\zeta_1 + (y_2 - y_3)\zeta_2 + y_3$$

$$\underset{\sim}{J} = \begin{bmatrix} \frac{\partial x}{\partial \zeta_1} & \frac{\partial y}{\partial \zeta_1} \\ \frac{\partial x}{\partial \zeta_2} & \frac{\partial y}{\partial \zeta_2} \end{bmatrix} = \begin{bmatrix} (x_1 - x_3) & (y_1 - y_3) \\ (x_2 - x_3) & (y_2 - y_3) \end{bmatrix}$$

$$|\underset{\sim}{J}| = 2A$$

$$\underset{\sim}{K} = t \int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dA = t \iint \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B} \tfrac{1}{2} |J| \, d\zeta_1 \, d\zeta_2$$

$$= \qquad (\zeta_3 = 1 - \zeta_1 - \zeta_2)$$

it is convenient to use alternative sampling points for the second integration by use of a special Gauss expression

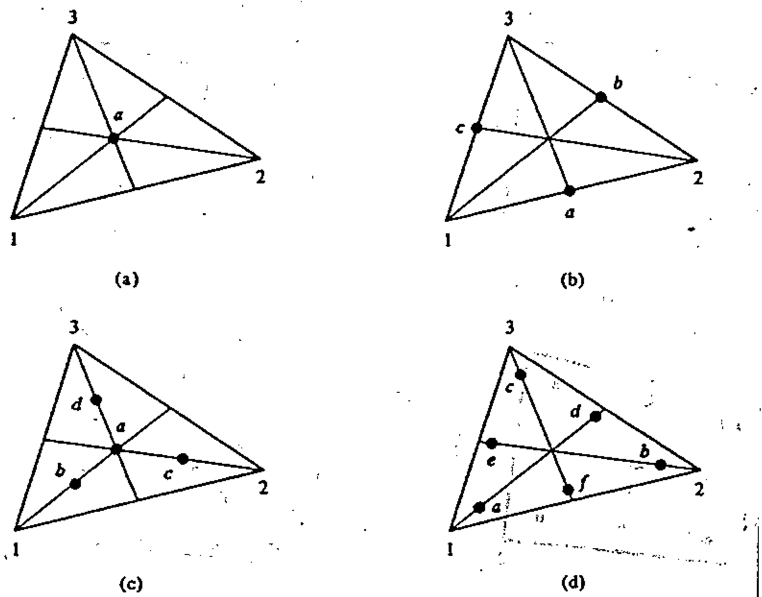


Figure 3.6 Integration Points for Triangle: (a) $n = 1$ (b) $n = 3$ (c) $n = 4$ (d) $n = 6$

TABLE 3.1 Numerical Integration Constants for Triangles

Figure	n	Order	Points	ξ_1	ξ_2	ξ_3	W_j
3.6(a)	1	Linear	a	$\frac{1}{3}$	$\frac{1}{3}$	$\frac{1}{3}$	1
3.6(b)	3	Quadratic	a b c	$\frac{1}{2}$ 0 $\frac{1}{2}$	$\frac{1}{2}$ $\frac{1}{2}$ 0	0 $\frac{1}{2}$ $\frac{1}{2}$	$\frac{1}{3}$ $\frac{1}{3}$ $\frac{1}{3}$
3.6(c)	4	Cubic	a b c d	$\frac{1}{3}$ 0.6 0.2 0.2	$\frac{1}{3}$ 0.2 0.6 0.2	$\frac{1}{3}$ 0.2 0.2 0.6	γ_1 γ_2 γ_2 γ_2
3.6(d)	6	Quartic	a b c d e f	α_1 β_1 β_1 α_2 β_2 β_2	β_1 α_1 β_1 β_2 α_2 β_2	β_1 β_1 α_1 β_2 β_2 α_2	γ_3 γ_3 γ_3 γ_4 γ_4 γ_4

$\alpha_1 = 0.8168475730$ $\quad \gamma_1 = -\frac{27}{48}$

$\beta_1 = 0.0915762135$ $\quad \gamma_2 = \frac{25}{48}$

$\alpha_2 = 0.1081030182$ $\quad \gamma_3 = 0.1099517437$

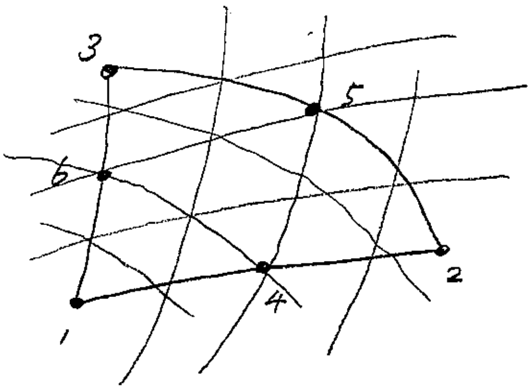
$\beta_2 = 0.4459484909$ $\quad \gamma_4 = 0.2233815897$

$$\underset{\sim}{K} = t \iint \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, dA$$

$$= \frac{1}{2} t \sum_{i=1}^{n} R_i \, g(\xi_1, \xi_2, \xi_3)_i \, |\underset{\sim}{J}|$$

$$g(\xi_1 \xi_2 \xi_3) = \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B}$$

6-node isoparametric Element.



$$\zeta_1 = \frac{A_1}{A}, \quad \zeta_2 = \frac{A_2}{A}, \quad \zeta_3 = \frac{A_3}{A}$$

$$A_1 + A_2 + A_3 = A$$

$$\zeta_1 + \zeta_2 + \zeta_3 = 1$$

$$\begin{cases} u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4 + f_5 u_5 + f_6 u_6 = \sum_i^6 f_i u_i \\ v = \sum_i^6 f_i v_i \end{cases}$$

$$\begin{cases} x = f_1 x_1 + f_2 x_2 + f_3 x_3 + f_4 x_4 + f_5 x_5 + f_6 x_6 = \sum_i^6 f_i x_i \\ y = \sum_i^6 f_i y_i \end{cases}$$

$$f_1 = \zeta_1 (2\zeta_1 - 1) \qquad f_4 = 4\zeta_1 \zeta_2$$

$$f_2 = \zeta_2 (2\zeta_2 - 1) \qquad f_5 = 4\zeta_2 \zeta_3$$

$$f_3 = \zeta_3 (2\zeta_3 - 1) \qquad f_6 = 4\zeta_3 \zeta_1$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{B} \, \underset{\sim}{q}$$

$$\underset{\sim}{B} = \left[\begin{array}{cc|c|cc} f_{1,x} & 0 & & f_{6,x} & 0 \\ 0 & f_{1,y} & \cdots\cdots & 0 & f_{6,y} \\ f_{1,y} & f_{1,x} & & f_{6,y} & f_{6,x} \end{array} \right]$$

$$f_i(\zeta_1, \zeta_2, \zeta_3) \Rightarrow f_i(\zeta_1, \zeta_2) \quad \text{using} \quad \zeta_3 = 1 - \zeta_1 - \zeta_2$$

$$\begin{bmatrix} \dfrac{\partial}{\partial \zeta_1} \\ \\ \dfrac{\partial}{\partial \zeta_2} \end{bmatrix} = \underbrace{\begin{bmatrix} \dfrac{\partial x}{\partial \zeta_1} & \dfrac{\partial y}{\partial \zeta_1} \\ \\ \dfrac{\partial x}{\partial \zeta_2} & \dfrac{\partial y}{\partial \zeta_2} \end{bmatrix}}_{\underset{\sim}{J}} \begin{bmatrix} \dfrac{\partial}{\partial x} \\ \\ \dfrac{\partial}{\partial y} \end{bmatrix}$$

$$\frac{\partial x}{\partial \zeta_1} = \sum \frac{\partial f_i}{\partial \zeta_1} x_i \qquad \frac{\partial y}{\partial \zeta_1} = \sum \frac{\partial f_i}{\partial \zeta_1} y_i$$

$$\frac{\partial x}{\partial \zeta_2} = \sum \frac{\partial f_i}{\partial \zeta_2} x_i \qquad \frac{\partial y}{\partial \zeta_2} = \sum \frac{\partial f_i}{\partial \zeta_2} y_i$$

$$\underset{\sim}{J}^{-1} = \underset{\sim}{J}^{*} = \begin{bmatrix} \dfrac{\partial \zeta_1}{\partial x} & \dfrac{\partial \zeta_2}{\partial x} \\ \\ \dfrac{\partial \zeta_1}{\partial y} & \dfrac{\partial \zeta_2}{\partial y} \end{bmatrix}$$

from $\underset{\sim}{B}$

$$\frac{\partial f_i}{\partial x} = \frac{\partial f_i}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial x} + \frac{\partial f_i}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial x} = J_{11}^{*}\frac{\partial f_i}{\partial \zeta_1} + J_{12}^{*}\frac{\partial f_i}{\partial \zeta_2}$$

$$\frac{\partial f_i}{\partial y} = \frac{\partial f_i}{\partial \zeta_1}\frac{\partial \zeta_1}{\partial y} + \frac{\partial f_i}{\partial \zeta_2}\frac{\partial \zeta_2}{\partial y} = J_{21}^{*}\frac{\partial f_i}{\partial \zeta_1} + J_{22}^{*}\frac{\partial f_i}{\partial \zeta_2}$$

$$\underset{\sim}{K} = t \int \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, dA$$

$$= \frac{t}{2} \int_{\Delta}^{1} \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, |J| \, d\zeta_1 d\zeta_2 d\zeta_3$$

$$= \frac{t}{2} \sum_{i=1}^{n} \omega_i f(\zeta_1, \zeta_2)_i \, |J(\zeta_1, \zeta_2)_i| \quad \text{(Numerical Integration)}$$