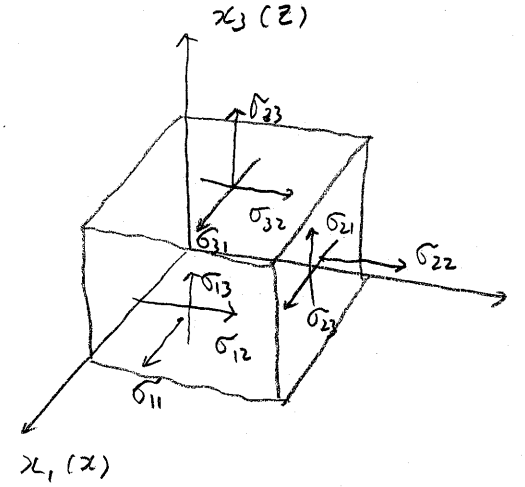
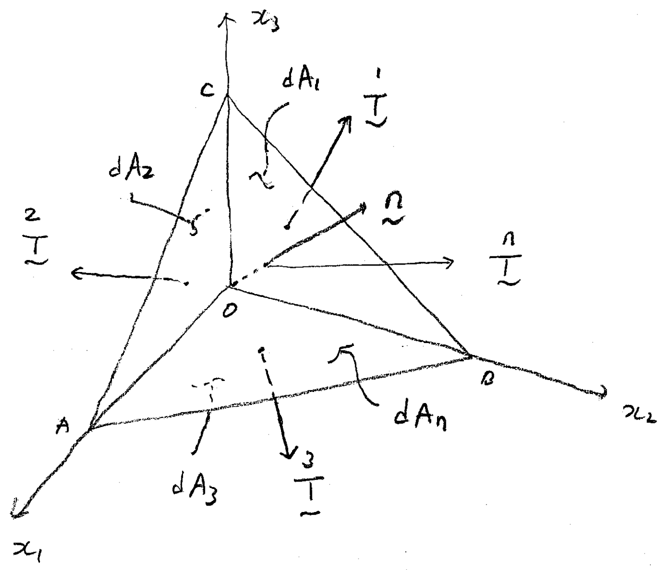
Chapter 4. General Solids



σ_{ij} — j: direction of stress; i: orientation of face



$$\underset{\sim}{n} = n_1 i + n_2 j + n_3 k$$

$$\underset{\sim}{e}_1 = i$$

$$\underset{\sim}{e}_2 = j$$

$$\underset{\sim}{e}_3 = k$$

$$\overset{1}{\underset{\sim}{T}} = \sigma_{11} i + \sigma_{12} j + \sigma_{13} k$$

$$\overset{2}{\underset{\sim}{T}} = \sigma_{21} i + \sigma_{22} j + \sigma_{23} k$$

$$\overset{3}{\underset{\sim}{T}} = \sigma_{31} i + \sigma_{32} j + \sigma_{33} k$$

$$\overset{i}{\underset{\sim}{T}} = \sigma_{ij}$$

$\underset{\sim}{T}$: surface traction

Area of ABC $= dA_n$

$$OBC = dA_1 = dA_n \cdot \underset{\sim}{n} \cdot \underset{\sim}{e}_1 = dA_n\, n_i$$

$$OCA = dA_2 = dA_n \cdot \underset{\sim}{n} \cdot \underset{\sim}{e}_2 = dA_n\, n_2$$

$$OAB = dA_3 = dA_n\, \underset{\sim}{n} \cdot \underset{\sim}{e}_3 = dA_n\, n_3$$

or

$$\frac{dA_1}{dA_n} = n_1, \quad \frac{dA_2}{dA_n} = n_2, \quad \frac{dA_3}{dA_n} = n_3$$

Force Equilibrium

$$\overset{n}{\underset{\sim}{T}}\, dA_n = \overset{1}{\underset{\sim}{T}}\, dA_1 + \overset{2}{\underset{\sim}{T}}\, dA_2 + \overset{3}{\underset{\sim}{T}}\, dA_3$$

$$\overset{n}{\underset{\sim}{T}} = \overset{1}{\underset{\sim}{T}} \frac{dA_1}{dA_n} + \overset{2}{\underset{\sim}{T}} \frac{dA_2}{dA_n} + \overset{3}{\underset{\sim}{T}} \frac{dA_3}{dA_n}$$

$$= \overset{1}{\underset{\sim}{T}}\, n_1 + \overset{2}{\underset{\sim}{T}}\, n_2 + \overset{3}{\underset{\sim}{T}}\, n_3$$

$$= \overset{j}{\underset{\sim}{T}}\, n_j$$

or

$$\overset{n}{T}_1\, dA_n = \overset{1}{T}_1\, dA_1 + \overset{2}{T}_1\, dA_2 + \overset{3}{T}_1\, dA_3$$

$$= \sigma_{11}\, dA_1 + \sigma_{21}\, dA_2 + \sigma_{31}\, dA_3$$

$$\overset{n}{T}_1 = \sigma_{11}\, n_1 + \sigma_{21}\, n_2 + \sigma_{31}\, n_3$$

$$= \sigma_{j1}\, n_j$$

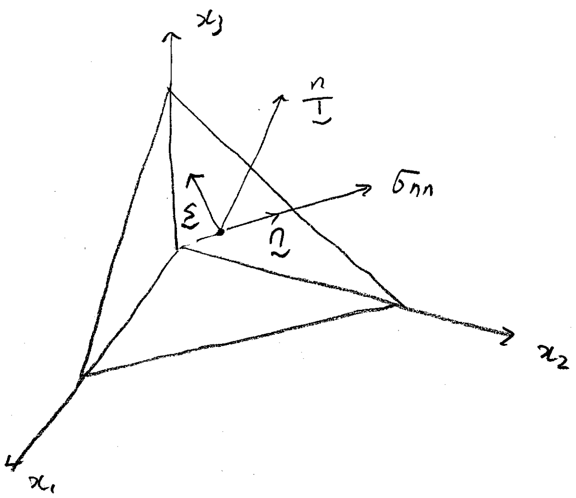
Generally $\overset{n}{T}_i = \sigma_{ji}\, n_j$ (Cauchy's formula)

$$= \sigma_{ij}\, n_j \quad (\sigma_{ij} = \sigma_{ji})$$

$$
\left\{
\begin{aligned}
\overset{n}{T}_1 &= \sigma_{11} n_1 + \sigma_{21} n_2 + \sigma_{31} n_3 \\
\overset{n}{T}_2 &= \sigma_{12} n_1 + \sigma_{22} n_2 + \sigma_{32} n_3 \\
\overset{n}{T}_3 &= \sigma_{13} n_1 + \sigma_{23} n_2 + \sigma_{33} n_3
\end{aligned}
\right.
$$

$$
\underset{\sim}{\overset{n}{T}} = \begin{bmatrix} \overset{n}{T}_1 \\ \overset{n}{T}_2 \\ \overset{n}{T}_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{21} & \sigma_{31} \\ \sigma_{12} & \sigma_{22} & \sigma_{23} \\ \sigma_{13} & \sigma_{23} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}
$$

$$
= \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}
$$



normal stress

$$\sigma_{nn} = \overset{n}{\underset{\sim}{T}} \cdot \underset{\sim}{n} = \underset{\sim}{n} \cdot \overset{n}{\underset{\sim}{T}} = \underset{\sim}{n} \cdot (\sigma_{ij} n_j)$$

$$= [n_1 \quad n_2 \quad n_3] \begin{bmatrix} \overset{n}{T_1} \\ \overset{n}{T_2} \\ \overset{n}{T_3} \end{bmatrix}$$

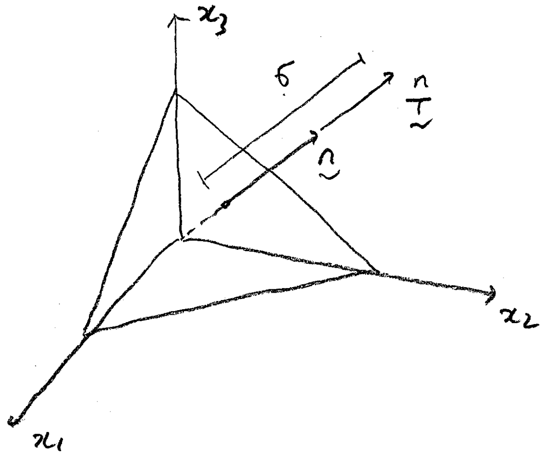
$$= [n_1 \quad n_2 \quad n_3] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

shear stress

$$\sigma_{ns} = \overset{n}{\underset{\sim}{T}} \cdot \underset{\sim}{s}$$

$$= [s_1 \quad s_2 \quad s_3] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

principal stress



$$\left.\begin{aligned} \overset{n}{T_i} &= \sigma_{ij} n_j \\ \overset{n}{T_i} &= \sigma n_i \end{aligned}\right] \rightarrow \sigma n_i = \sigma_{ij} n_j$$

$$\left[\begin{aligned} \sigma n_1 &= \sigma_{11} n_1 + \sigma_{12} n_2 + \sigma_{13} n_3 \\ \sigma n_2 &= \sigma_{21} n_1 + \sigma_{22} n_2 + \sigma_{23} n_3 \\ \sigma n_3 &= \sigma_{31} n_1 + \sigma_{32} n_2 + \sigma_{33} n_3 \end{aligned}\right.$$

$$\begin{bmatrix} \sigma & 0 & 0 \\ 0 & \sigma & 0 \\ 0 & 0 & \sigma \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix} = \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_1 \\ n_2 \\ n_3 \end{bmatrix}$$

$$\sigma \underset{\sim}{I} \underset{\sim}{n} = \underset{\sim}{\sigma} \underset{\sim}{n}$$

$$[\underset{\sim}{\sigma} - \sigma \underset{\sim}{I}] \underset{\sim}{n} = \underset{\sim}{0}$$

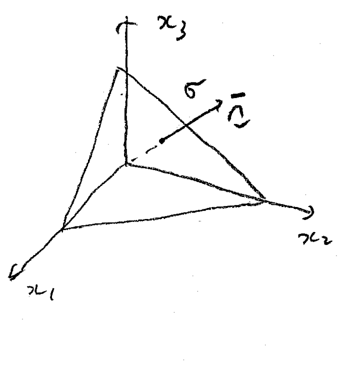
or $(\sigma_{ij} - \sigma \delta_{ij}) n_j = 0$

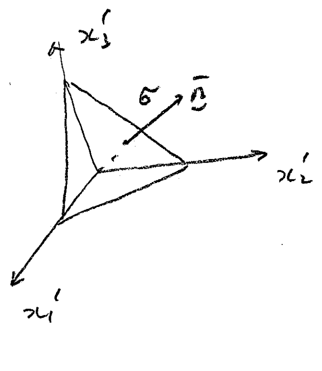
for non-trivial solution

$$|\underset{\sim}{\sigma} - \sigma \underset{\sim}{I}| = 0$$

$$\sigma^3 + I_1 \sigma^2 + I_2 \sigma + I_3 = 0$$

I_1, I_2, I_3 = stress invariants





same principal stresses

$\Rightarrow$ I_1, I_2, I_3 = invariants

$$I_1 = \sigma_{11} + \sigma_{22} + \sigma_{33} = \sigma_{kk}$$

$$I_2 = \begin{vmatrix} \sigma_{22} & \sigma_{23} \\ \sigma_{32} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{13} \\ \sigma_{31} & \sigma_{33} \end{vmatrix} + \begin{vmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{vmatrix}$$

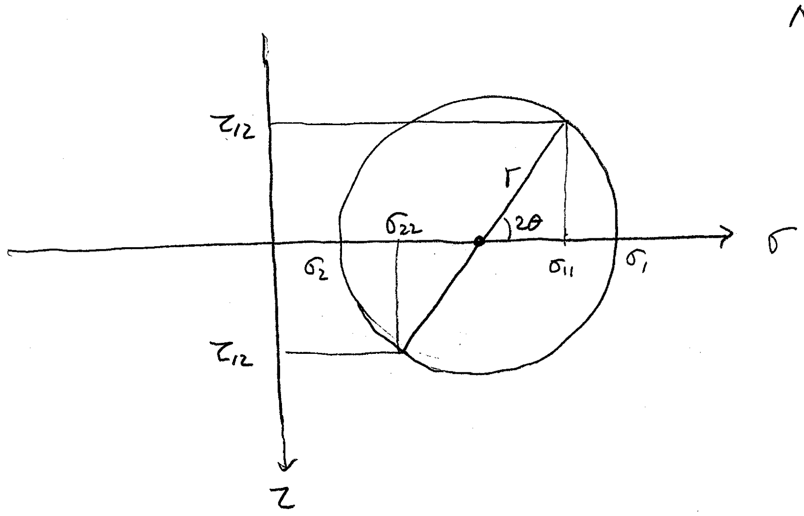
$$= \sigma_{11}\sigma_{22} + \sigma_{22}\sigma_{33} + \sigma_{11}\sigma_{33} - \sigma_{12}^2 - \sigma_{23}^2 - \sigma_{31}^2$$

$$= \frac{1}{2}(\sigma_{ii}\sigma_{jj} - \sigma_{ij}\sigma_{ji})$$

$$I_3 = \det(\sigma_{ij})$$

$$= \sigma_{11}\sigma_{22}\sigma_{33} + 2\sigma_{12}\sigma_{23}\sigma_{31} - \sigma_{12}^2\sigma_{33} - \sigma_{23}^2\sigma_{11} - \sigma_{31}^2\sigma_{22}$$

Mohr's circle



stress invariants : $\frac{1}{2}(\sigma_{11} + \sigma_{22})$ or $\frac{1}{2}(\sigma_1 + \sigma_2)$

$$r^2 = \left[\frac{1}{2}(\sigma_{11} - \sigma_{22})\right]^2 + \tau_{12}^2$$

$$= \frac{1}{4}\left[\sigma_{11}^2 - 2\sigma_{11}\sigma_{22} + \sigma_{22}^2\right] + \tau_{12}^2$$

$$= \frac{1}{4}(\sigma_{11} + \sigma_{22})^2 - \sigma_{11}\sigma_{22} + \tau_{12}^2$$

$$= \frac{1}{4} I_1^2 - I_2$$

Material Strength Criteria

★ J_2 theory - von Mises criterion (for metals)

$$s^3 + J_1 s^2 + J_2 s + J_3 = 0 \qquad s = \sigma - \frac{1}{3} I_1$$

$$J_1 = 0$$

$$\sigma_y = \sqrt{3 J_2}$$

$$\sqrt{3J_2} = \sqrt{\frac{1}{2}[(\sigma_{11}-\sigma_{22})^2 + (\sigma_{22}-\sigma_{33})^2 + (\sigma_{33}-\sigma_{11})^2 + 6(\tau_{12}^2 + \tau_{23}^2 + \tau_{31}^2)}$$

$$= \sqrt{\frac{1}{2}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2}$$

★ Elastic - hardening plasticity model (for concrete)

$$A_1 \sigma + A_2 \sqrt{3 J_2} + A_3 = \bar{\sigma} \qquad \sigma = \frac{1}{3}\underbrace{(\sigma_1 + \sigma_2 + \sigma_3)}_{I_1}$$

Why are the existing material failure criteria defined with the stress - invariants?

⇒ For isotropic materials and similarities,

the material failure criteria should not be dependent on the definition of coordinate system.

$|\underset{\sim}{\sigma} - \sigma \underset{\sim}{I}| = 0$

solutions $= \sigma_1, \sigma_2, \sigma_3$ Eigen values $\Rightarrow$ principal stresses

corresponding vectors $\bar{\underset{\sim}{n}}_1, \bar{\underset{\sim}{n}}_2, \bar{\underset{\sim}{n}}_3$ Eigen values

$\Rightarrow$ orientations of principal axes

$$\underset{\sim}{N} = [\bar{\underset{\sim}{n}}_1 \ \bar{\underset{\sim}{n}}_2 \ \bar{\underset{\sim}{n}}_3]$$

$$= \begin{bmatrix} \bar{n}_{11} & \bar{n}_{21} & \bar{n}_{31} \\ \bar{n}_{12} & \bar{n}_{22} & \bar{n}_{32} \\ \bar{n}_{13} & \bar{n}_{23} & \bar{n}_{33} \end{bmatrix}$$

$$\left.\begin{aligned} \sigma_1 \underset{\sim}{I} \bar{\underset{\sim}{n}}_1 &= \underset{\sim}{\sigma} \bar{\underset{\sim}{n}}_1 \\ \sigma_2 \underset{\sim}{I} \bar{\underset{\sim}{n}}_2 &= \underset{\sim}{\sigma} \bar{\underset{\sim}{n}}_2 \\ \sigma_3 \underset{\sim}{I} \bar{\underset{\sim}{n}}_3 &= \underset{\sim}{\sigma} \bar{\underset{\sim}{n}}_3 \end{aligned}\right] \Rightarrow$$

$$\sigma_1 \bar{\underset{\sim}{n}}_1^T \bar{\underset{\sim}{n}}_1 = \bar{\underset{\sim}{n}}_1^T \underset{\sim}{\sigma} \bar{\underset{\sim}{n}}_1$$

$$\bar{\underset{\sim}{n}}_1^T \bar{\underset{\sim}{n}}_1 = 1$$

$$\begin{cases} \sigma_1 = \bar{\underset{\sim}{n}}_1^T \underset{\sim}{\sigma} \underset{\sim}{N} \\ \sigma_2 = \bar{\underset{\sim}{n}}_2^T \underset{\sim}{\sigma} \underset{\sim}{N} \\ \sigma_3 = \bar{\underset{\sim}{n}}_3^T \underset{\sim}{\sigma} \underset{\sim}{N} \end{cases} \qquad \bar{\underset{\sim}{n}}_i^T \underset{\sim}{\sigma} \bar{\underset{\sim}{n}}_j = 0$$

$$\bar{\underset{\sim}{\sigma}} = \underset{\sim}{N}^T \underset{\sim}{\sigma} \underset{\sim}{N}$$

$\underset{\sim}{N}$: orthogonal matrix

$\underset{\sim}{N}^T \underset{\sim}{N} = \underset{\sim}{I}$

Orthogonality

$$\underset{\sim}{\sigma}\, \underset{\sim}{\bar{n}_i} = \sigma_i \underset{\sim}{\bar{n}_i} \quad -① \qquad (i \neq j)$$

$$\underset{\sim}{\sigma}\, \underset{\sim}{\bar{n}_j} = \sigma_j \underset{\sim}{\bar{n}_j} \quad -②$$

$$① - \quad \underset{\sim}{\bar{n}_j}^T \underset{\sim}{\sigma}\, \underset{\sim}{\bar{n}_i} = \sigma_i \underset{\sim}{\bar{n}_j}^T \underset{\sim}{\bar{n}_i}$$

$$② - \quad \underset{\sim}{\bar{n}_i}^T \underset{\sim}{\sigma}\, \underset{\sim}{\bar{n}_j} = \sigma_j \underset{\sim}{\bar{n}_i}^T \underset{\sim}{\bar{n}_j} \quad -②'$$

$$②' - \quad \underset{\sim}{\bar{n}_j}^T \underset{\sim}{\sigma}\, \underset{\sim}{\bar{n}_i} = \sigma_j \underset{\sim}{\bar{n}_j}^T \underset{\sim}{\bar{n}_i}$$

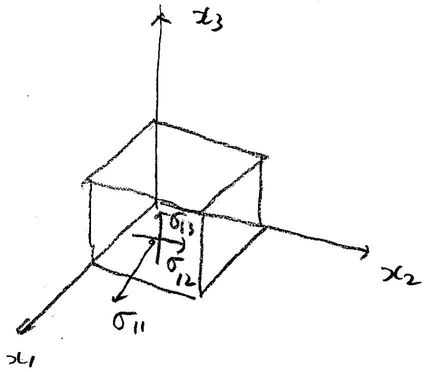
from ① and ②', $\quad \sigma_i \underset{\sim}{\bar{n}_j}^T \underset{\sim}{\bar{n}_i} = \sigma_j \underset{\sim}{\bar{n}_j}^T \underset{\sim}{\bar{n}_i}$

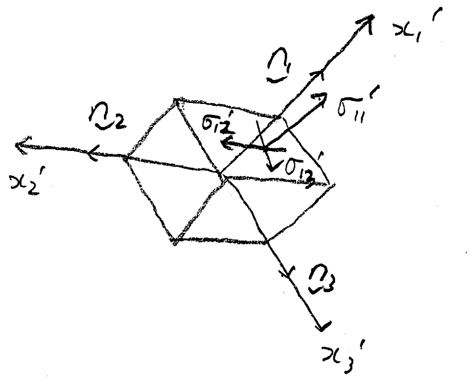
$$(\sigma_i - \sigma_j)\, \underset{\sim}{\bar{n}_j}^T \underset{\sim}{\bar{n}_i} = 0$$

since $\sigma_i \neq \sigma_j$, $\quad \underset{\sim}{\bar{n}_j}^T \underset{\sim}{\bar{n}_i} = 0 \quad (i \neq j)$

$$\Rightarrow \quad \underset{\sim}{\bar{n}_j} \perp \underset{\sim}{\bar{n}_i} \qquad \text{(orthogonality)}$$

Axis - transformation of stresses





$$\underset{\sim}{n}_1 = \begin{bmatrix} n_{11} \\ n_{12} \\ n_{13} \end{bmatrix} \qquad \underset{\sim}{n}_2 = \begin{bmatrix} n_{21} \\ n_{22} \\ n_{23} \end{bmatrix} \qquad \underset{\sim}{n}_3 = \begin{bmatrix} n_{31} \\ n_{32} \\ n_{33} \end{bmatrix}$$

$$\sigma_{11}' = \sigma_{n_1 n_1} = [n_{11}\ n_{12}\ n_{13}] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{12} \\ n_{13} \end{bmatrix}$$

$$= n_{1l}\, \sigma_{kl}\, n_{1k}$$

$$\sigma_{21}' = \sigma_{n_2 n_1} = [n_{21}\ n_{22}\ n_{23}] \begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix} \begin{bmatrix} n_{11} \\ n_{12} \\ n_{13} \end{bmatrix}$$

$(= \sigma_{sn})$

$$= n_{2l}\, \sigma_{kl}\, n_{1k}$$

Generally $\quad \sigma_{ij}' = n_{il} \cdot \sigma_{kl}\, n_{jk}$

$$\underbrace{\begin{bmatrix} \sigma_{11}' & \sigma_{12}' & \sigma_{13}' \\ \sigma_{21}' & \sigma_{22}' & \sigma_{23}' \\ \sigma_{31}' & \sigma_{32}' & \sigma_{33}' \end{bmatrix}}_{\underset{\sim}{\sigma}'} = \underbrace{\begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix}}_{\underset{\sim}{R}^T} \underbrace{\begin{bmatrix} \sigma_{11} & \sigma_{12} & \sigma_{13} \\ \sigma_{21} & \sigma_{22} & \sigma_{23} \\ \sigma_{31} & \sigma_{32} & \sigma_{33} \end{bmatrix}}_{\underset{\sim}{\sigma}} \underbrace{\begin{bmatrix} n_{11} & n_{21} & n_{31} \\ n_{12} & n_{22} & n_{32} \\ n_{13} & n_{23} & n_{33} \end{bmatrix}}_{\underset{\sim}{R}}$$

$$\underset{\sim}{\sigma}' = \underset{\sim}{R}^T \underset{\sim}{\sigma} \underset{\sim}{R}$$

In text book, $\underset{\sim}{R} = \begin{bmatrix} l_1 & l_2 & l_3 \\ m_1 & m_2 & m_3 \\ n_1 & n_2 & n_3 \end{bmatrix}$

$$\underset{\sim}{\sigma}' = \underset{\sim}{T_\sigma} \underset{\sim}{\sigma}$$

$$\underbrace{\begin{bmatrix} \sigma_{11}' \\ \sigma_{22}' \\ \sigma_{33}' \\ \sigma_{12}' \\ \sigma_{23}' \\ \sigma_{31}' \end{bmatrix}}_{\underset{\sim}{\sigma}'} = \underbrace{\left[\begin{array}{ccc|ccc} l_1^2 & m_1^2 & n_1^2 & 2l_1m_1 & 2m_1n_1 & 2n_1l_1 \\ l_2^2 & m_2^2 & n_2^2 & 2l_2m_2 & 2m_2n_2 & 2n_2l_2 \\ l_3^2 & m_3^2 & n_3^2 & 2l_3m_3 & 2m_3n_3 & 2m_3l_3 \\ \hline l_1l_2 & m_1m_2 & n_1n_2 & (l_1m_2+l_2m_1) & (m_1n_2+m_2n_1) & (n_1l_2+n_2l_1) \\ l_2l_3 & m_2m_3 & n_2n_3 & (l_2m_3+l_3m_2) & (m_2n_3+m_3n_2) & (n_2l_3+n_3l_2) \\ l_3l_1 & m_3m_1 & n_3n_1 & (l_3m_1+l_1m_3) & (m_3n_1+m_1n_3) & (n_3l_1+n_1l_3) \end{array}\right]}_{\underset{\sim}{T_\sigma}} \underbrace{\begin{bmatrix} \sigma_{11} \\ \sigma_{22} \\ \sigma_{33} \\ \sigma_{12} \\ \sigma_{23} \\ \sigma_{31} \end{bmatrix}}_{\underset{\sim}{\sigma}}$$

$$\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix}$$ where $\varepsilon_{12} = \frac{1}{2}\gamma_{12}$, $\varepsilon_{13} = \frac{1}{2}\gamma_{13}$ ----

$$\underset{3\times 3}{\underset{\sim}{\varepsilon}'} = \underset{\sim}{R}^T \underset{\sim}{\varepsilon} \underset{\sim}{R}$$

also, $\underset{6\times 1}{\underset{\sim}{\varepsilon}'} = \underset{\sim}{T}_\sigma \underset{\sim}{\varepsilon}$

if we use, $\underset{\sim}{\varepsilon} = [\varepsilon_{11} \ \varepsilon_{22} \ \varepsilon_{33} \ \gamma_{12} \ \gamma_{23} \ \gamma_{31}]^T$

$$\underset{\sim}{\varepsilon}' = \underset{\sim}{T}_\varepsilon \underset{\sim}{\varepsilon} \qquad \underset{\sim}{T}_\varepsilon \neq \underset{\sim}{T}_\sigma$$

using virtual strain energy density

$$(\delta \underset{\sim}{\varepsilon}')^T \underset{\sim}{\sigma}' = \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}$$

$$\delta \underset{\sim}{\varepsilon}^T \underset{\sim}{T}_\varepsilon^T \underset{\sim}{\sigma}' = \delta \underset{\sim}{\varepsilon}^T \underset{\sim}{\sigma}$$

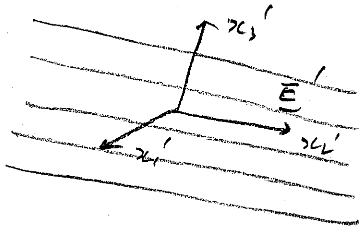
$$\underset{\sim}{T}_\varepsilon^T \cdot \underset{\sim}{T}_\sigma \underset{\sim}{\sigma} = \underset{\sim}{\sigma}$$

$$\underline{\underset{\sim}{T}_\sigma = \underset{\sim}{T}_\varepsilon^{-T}}$$

Ax3 - transformation of constitutive matrix $\underset{\sim}{E}$

$$\underset{\sim}{\sigma}' = \underset{\sim}{E}' \underset{\sim}{\varepsilon}'$$

$$\underset{\sim}{T_\sigma} \underset{\sim}{\sigma} = \underset{\sim}{E}' \underset{\sim}{T_\varepsilon} \underset{\sim}{\varepsilon}$$



$$\underset{\sim}{\sigma} = \underset{\sim}{T_\sigma}^{-1} \underset{\sim}{E}' \underset{\sim}{T_\varepsilon} \underset{\sim}{\varepsilon}$$

$$= \underset{\sim}{T_\varepsilon}^{T} \underset{\sim}{E}' \underset{\sim}{T_\varepsilon} \underset{\sim}{\varepsilon}$$

$$= \underset{\sim}{E} \underset{\sim}{\varepsilon}$$

Therefore, $\underset{\sim}{E} = \underset{\sim}{T_\varepsilon}^{T} \underset{\sim}{E}' \underset{\sim}{T_\varepsilon}$

However, for isotropic material, $\underset{\sim}{E} = \underset{\sim}{E}'$

Stress - Strain Relationship

$$\underset{\sim}{\sigma} = \underset{\sim}{E} \underset{\sim}{\varepsilon}$$

for isotropic material

$$\underset{\sim}{E} = \frac{E}{(1+\nu) e_2} \begin{bmatrix} e_1 & \nu & \nu & & & \\ & e_1 & \nu & & \underset{\sim}{0} & \\ & & e_1 & & & \\ & & & e_3 & & \\ & \text{sym.} & & & e_3 & \\ & & & & & e_3 \end{bmatrix}$$

$$e_1 = 1-\nu, \quad e_2 = 1-2\nu, \quad e_3 = \frac{e_2}{2}$$

$\underset{\sim}{E}$ can be defined with 2 independent constants E and ν

why?

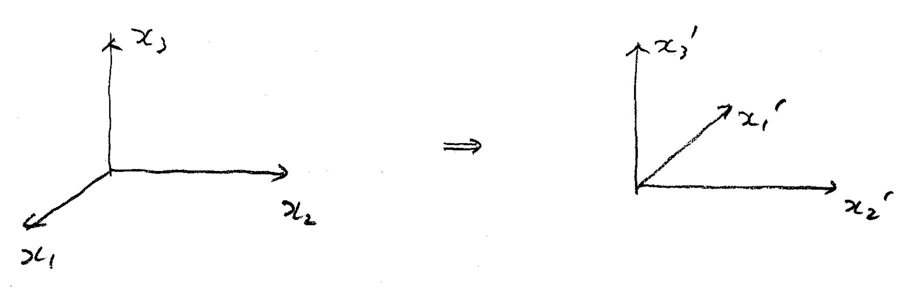
Anisotropic material

$$\sigma_i = C_{ij}\,\varepsilon_j$$

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16} \\ & C_{22} & C_{23} & C_{24} & C_{25} & C_{26} \\ & & C_{33} & C_{34} & C_{35} & C_{36} \\ & & & C_{44} & C_{45} & C_{46} \\ & \text{sym.} & & & C_{55} & C_{56} \\ & & & & & C_{66} \end{bmatrix}$$

21 constants

Elastic symmetry with respect to one plane (Monoclinic Material)



Transformation matrix

$$\underset{\sim}{R} = \begin{bmatrix} n_{11} & n_{12} & n_{13} \\ n_{21} & n_{22} & n_{23} \\ n_{31} & n_{32} & n_{33} \end{bmatrix} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} x_1' \\ x_2' \\ x_3' \end{array}$$

$\Big\langle$ seek condition of $\underset{\sim}{C}$ to satisfy $\underset{\sim}{\sigma} = \underset{\sim}{C}\,\underset{\sim}{\varepsilon}$ and $\underset{\sim}{\sigma}' = \underset{\sim}{C}\,\underset{\sim}{\varepsilon}'$,
where $\underset{\sim}{\sigma}' = \underset{\sim}{R}^T \underset{\sim}{\sigma}\,\underset{\sim}{R}$ and $\underset{\sim}{\varepsilon}' = \underset{\sim}{R}^T \underset{\sim}{\varepsilon}\,\underset{\sim}{R}$. $\Big\rangle$

$$\underset{\sim}{\sigma}' = \underset{\sim}{R}^T \underset{\sim}{\sigma} \underset{\sim}{R}$$

$$\Rightarrow \quad \sigma_{11}' = \sigma_{11}, \quad \sigma_{22}' = \sigma_{22}, \quad \sigma_{33}' = \sigma_{33}$$

$$\sigma_{12}' = -\sigma_{12}, \quad \sigma_{23}' = \sigma_{23}, \quad \sigma_{31}' = -\sigma_{31}$$

Also $\quad \underset{\sim}{\varepsilon}' = \underset{\sim}{R}^T \underset{\sim}{\varepsilon} \underset{\sim}{R}$

$$\Rightarrow \quad \varepsilon_{11}' = \varepsilon_{11}, \quad \varepsilon_{22}' = \varepsilon_{22}, \quad \varepsilon_{33}' = \varepsilon_{33}$$

$$\varepsilon_{12}' = -\varepsilon_{12}, \quad \varepsilon_{23}' = \varepsilon_{23}, \quad \varepsilon_{31}' = -\varepsilon_{31}$$

from $\underset{\sim}{\sigma} = \underset{\sim}{C} \underset{\sim}{\varepsilon}$

$$\sigma_{11} = C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} + C_{14}\varepsilon_{12} + C_{15}\varepsilon_{23} + C_{16}\varepsilon_{31}$$

from $\underset{\sim}{\sigma}' = \underset{\sim}{C} \underset{\sim}{\varepsilon}'$

$$\sigma_{11}' = C_{11}\varepsilon_{11}' + C_{12}\varepsilon_{22}' + C_{13}\varepsilon_{33}' + C_{14}\varepsilon_{12}' + C_{15}\varepsilon_{23}' + C_{16}\varepsilon_{31}'$$

$$= C_{11}\varepsilon_{11} + C_{12}\varepsilon_{22} + C_{13}\varepsilon_{33} - C_{14}\varepsilon_{12} + C_{15}\varepsilon_{23} - C_{16}\varepsilon_{31}$$

from $\sigma_{11} = \sigma_{11}'$, $\quad C_{14} = C_{16} = 0$

Similarly, $\quad C_{24} = C_{26} = C_{34} = C_{36} = C_{45} = C_{36} = 0$

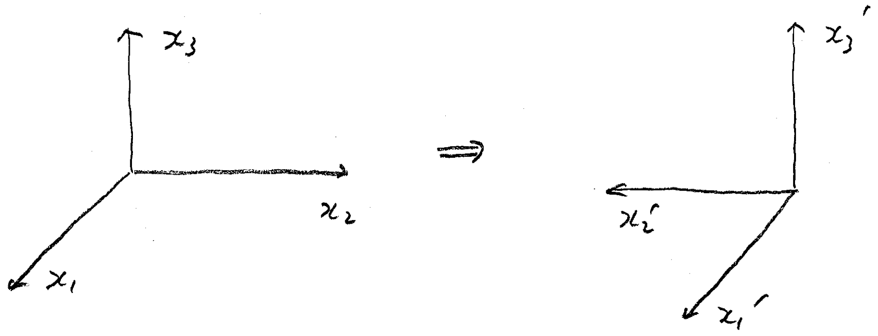
Hence,

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & C_{15} & 0 \\ & C_{22} & C_{23} & 0 & C_{25} & 0 \\ & & C_{33} & 0 & C_{35} & 0 \\ & & & C_{44} & 0 & C_{45} \\ & \text{sym.} & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}$$

13 independent constants for monoclinic material

Elastic symmetry w.r.t two orthogonal planes

(Orthotropic material)



$$\underset{\sim}{R} = \begin{array}{cc} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} & \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} & \begin{array}{c} x_1' \\ x_2' \\ x_3' \end{array} \end{array}$$

from $\underset{\sim}{C}' = \underset{\sim}{C}$, $\sigma_{ii}' = \sigma_{ii}$, $\sigma_{12}' = -\sigma_{12}$,

$\sigma_{23}' = -\sigma_{23}$, $\sigma_{31}' = \sigma_{31}$

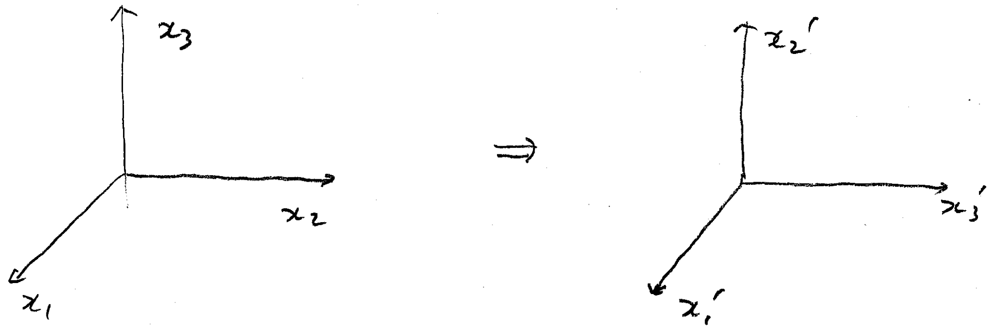
$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & 0 & 0 & 0 \\ & C_{22} & C_{23} & 0 & 0 & 0 \\ & & C_{33} & 0 & 0 & 0 \\ & \text{sym.} & & C_{44} & 0 & 0 \\ & & & & C_{55} & 0 \\ & & & & & C_{66} \end{bmatrix}$$

9 constants for orthotropic material

No coupling between
- normal stress - shear strains
- normal strain - shear stresses

Directional Independence (Interchange Axes)

(Cubic material)

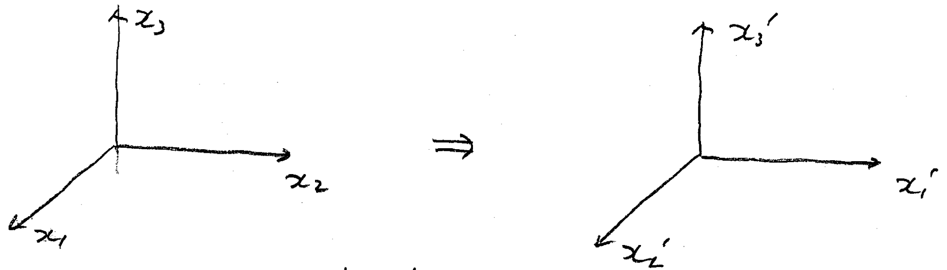


$$\underset{\sim}{R} = \begin{array}{c} \begin{array}{ccc} x_1 & x_2 & x_3 \end{array} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \end{array} \begin{array}{c} \\ x_1' \\ x_2' \\ x_3' \end{array}$$

from $\underset{\sim}{C}' = \underset{\sim}{C}$ $\quad \sigma_{11}' = \sigma_{11}, \quad \sigma_{22}' = \sigma_{33}, \quad \sigma_{33}' = \sigma_{22}$

$\sigma_{12}' = \sigma_{13}, \quad \sigma_{23}' = \sigma_{32}, \quad \sigma_{13}' = \sigma_{12}$

$\Rightarrow \quad C_{22} = C_{33}, \quad C_{44} = C_{66}, \quad C_{12} = C_{13},$



$$\underset{\sim}{R} = \begin{array}{c} \begin{array}{ccc} x_1' & x_2' & x_3' \end{array} \\ \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \end{array} \begin{array}{c} \\ x_1 \\ x_2 \\ x_3 \end{array}$$

from $\underset{\sim}{C}' = \underset{\sim}{C}$, $\quad \sigma_{11}' = \sigma_{33}, \quad \sigma_{22}' = \sigma_{22}, \quad \sigma_{33}' = \sigma_{11}$

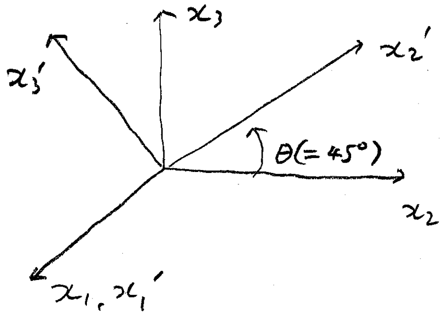
$\sigma_{12}' = \sigma_{12}, \quad \sigma_{23}' = \sigma_{31}, \quad \sigma_{31}' = \sigma_{32}$

$\Rightarrow \quad C_{11} = C_{33}, \quad C_{55} = C_{66}, \quad C_{12} = C_{23}$

$$C_{ij} = \begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{11} & C_{12} & 0 & 0 & 0 \\ & & C_{11} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{sym.} & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix}$$

3 independent constants for a cubic material

Rotational Independence (Isotropic material)



$$\underset{\sim}{R} = \begin{array}{c} \begin{matrix} x_1 & x_2 & x_3 \end{matrix} \\ \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & +\sin\theta \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \end{array} \begin{matrix} x_1' \\ x_2' \\ x_3' \end{matrix}$$

from $\underset{\sim}{C}' = \underset{\sim}{C}$, $\underset{\sim}{\sigma}' = \underset{\sim}{R}^T \underset{\sim}{\sigma}' \underset{\sim}{R}$, $\underset{\sim}{\varepsilon} = \underset{\sim}{R}^T \underset{\sim}{\varepsilon}' \underset{\sim}{R}$

$\underset{\sim}{\sigma} = \underset{\sim}{C}\,\underset{\sim}{\varepsilon}$, $\underset{\sim}{\sigma}' = \underset{\sim}{C}'\underset{\sim}{\varepsilon}'$

$\Rightarrow \; C_{11} - C_{12} = C_{44}$

$$C_{ij} = \begin{bmatrix} C_{12}+C_{44} & C_{12} & C_{12} & 0 & 0 & 0 \\ & C_{12}+C_{44} & C_{12} & 0 & 0 & 0 \\ & & C_{12}+C_{44} & 0 & 0 & 0 \\ & & & C_{44} & 0 & 0 \\ & \text{sym.} & & & C_{44} & 0 \\ & & & & & C_{44} \end{bmatrix}$$

set $C_{12} \to \lambda$, $C_{44} \to 2\mu$

$$C_{ij} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & 2\mu & & \\ & & & & 2\mu & \\ & & & & & 2\mu \end{bmatrix}$$

$$\lambda = \frac{\nu E}{(1+\nu)(1-2\nu)} \qquad \mu = G = \frac{E}{2(1+\nu)}$$

λ, μ = Lame' constants

$$\sigma_{ij} = \lambda \varepsilon_{kk} \delta_{ij} + 2\mu \varepsilon_{ij}$$

If we define the strains as $\underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix}$

$$\underset{\sim}{\sigma} = \underset{\sim}{E}\, \underset{\sim}{\varepsilon}$$

$$\underset{\sim}{E} = \begin{bmatrix} \lambda+2\mu & \lambda & \lambda & & & \\ \lambda & \lambda+2\mu & \lambda & & \underset{\sim}{0} & \\ \lambda & \lambda & \lambda+2\mu & & & \\ & & & \mu & & \\ & \underset{\sim}{0} & & & \mu & \\ & & & & & \mu \end{bmatrix}$$

Strain - Displacement Relationship

$$\underset{\sim}{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} \qquad \underset{\sim}{\varepsilon} = \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \varepsilon_{12} \\ \varepsilon_{23} \\ \varepsilon_{31} \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{21} \end{bmatrix}$$

$$\varepsilon_{11} = \frac{\partial u}{\partial x}, \qquad \varepsilon_{22} = \frac{\partial v}{\partial y}, \qquad \varepsilon_{33} = \frac{\partial w}{\partial z}$$

$$\varepsilon_{12} = \frac{1}{2}\left(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}\right) \qquad \gamma_{12} = \frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}$$

$$\varepsilon_{23} = \frac{1}{2}\left(\frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}\right) \qquad \gamma_{23} = \frac{\partial v}{\partial z} + \frac{\partial w}{\partial y}$$

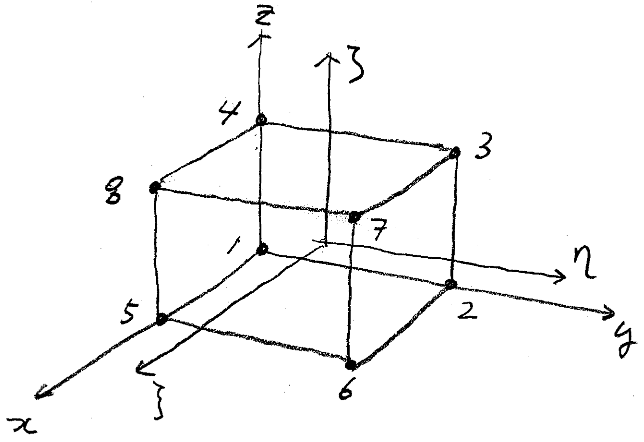
$$\varepsilon_{31} = \frac{1}{2}\left(\frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}\right) \qquad \gamma_{31} = \frac{\partial w}{\partial x} + \frac{\partial u}{\partial z}$$

$$\begin{bmatrix} \varepsilon_{11} \\ \varepsilon_{22} \\ \varepsilon_{33} \\ \gamma_{12} \\ \gamma_{23} \\ \gamma_{31} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & 0 & 0 \\ 0 & \frac{\partial}{\partial y} & 0 \\ 0 & 0 & \frac{\partial}{\partial z} \\ \frac{\partial}{\partial y} & \frac{\partial}{\partial x} & 0 \\ 0 & \frac{\partial}{\partial z} & \frac{\partial}{\partial y} \\ \frac{\partial}{\partial z} & 0 & \frac{\partial}{\partial x} \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{d}\, \underset{\sim}{u}$$

General Solids

8 - node Solid Element (Isoparametric Element)



$$\begin{cases} u = f_1 u_1 + f_2 u_2 + f_3 u_3 + f_4 u_4 + \cdots + f_8 u_8 = \sum_i^8 f_i u_i \\ v = \sum f_i v_i \\ w = \sum f_i w_i \end{cases}$$

$$\begin{cases} x = \sum f_i x_i \\ y = \sum f_i y_i \\ z = \sum f_i z_i \end{cases}$$

node 1 $(-1\ -1\ -1)$ $\quad f_1 = \frac{1}{8}(1-\xi)(1-\eta)(1-\zeta)$

node 2 $(-1\ 1\ -1)$ $\quad f_2 = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)$

node 3 $(-1\ 1\ 1)$ $\quad f_3 = \frac{1}{8}(1-\xi)(1+\eta)(1-\zeta)$

$\vdots$

node 8 $(1\ -1\ 1)$ $\quad f_8 = \frac{1}{8}(1+\xi)(1-\eta)(1+\zeta)$

$$\Rightarrow f_i = \frac{1}{8}(1+\xi\xi_i)(1+\eta\eta_i)(1+\zeta\zeta_i)$$

$$\underset{\sim}{u} = \begin{bmatrix} u \\ v \\ w \end{bmatrix} = \left[\begin{array}{ccc|ccc|c|ccc} f_1 & 0 & 0 & f_2 & 0 & 0 & & f_8 & 0 & 0 \\ 0 & f_1 & 0 & 0 & f_2 & 0 & \cdots & 0 & f_8 & 0 \\ 0 & 0 & f_1 & 0 & 0 & f_2 & & 0 & 0 & f_8 \end{array}\right] \begin{bmatrix} u_1 \\ v_1 \\ w_1 \\ \vdots \\ u_8 \\ v_8 \\ w_8 \end{bmatrix}$$

$$\underset{\sim}{u} = \underset{\sim}{f}\ \underset{\sim}{q}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{B}\,\underset{\sim}{g} \qquad \underset{(6\times 24)}{\underset{\sim}{B}} = \underset{(6\times 3)}{\underset{\sim}{d}}\;\underset{(3\times 24)}{\underset{\sim}{f}}$$

$$\underset{\sim}{B} = \left[\begin{array}{ccc|c|ccc}
\frac{\partial f_1}{\partial x} & 0 & 0 & & \frac{\partial f_8}{\partial x} & 0 & 0 \\
0 & \frac{\partial f_1}{\partial y} & 0 & & 0 & \frac{\partial f_8}{\partial y} & 0 \\
0 & 0 & \frac{\partial f_1}{\partial z} & - - - - & 0 & 0 & \frac{\partial f_8}{\partial z} \\
\frac{\partial f_1}{\partial y} & \frac{\partial f_1}{\partial x} & 0 & & \frac{\partial f_8}{\partial y} & \frac{\partial f_8}{\partial x} & 0 \\
0 & \frac{\partial f_1}{\partial z} & \frac{\partial f_1}{\partial y} & & 0 & \frac{\partial f_8}{\partial z} & \frac{\partial f_8}{\partial y} \\
\frac{\partial f_1}{\partial z} & 0 & \frac{\partial f_1}{\partial x} & & \frac{\partial f_8}{\partial z} & 0 & \frac{\partial f_8}{\partial x}
\end{array}\right]$$

$$\underset{\sim}{J} = \begin{bmatrix}
\frac{\partial x}{\partial \xi} & \frac{\partial y}{\partial \xi} & \frac{\partial z}{\partial \xi} \\
\frac{\partial x}{\partial \eta} & \frac{\partial y}{\partial \eta} & \frac{\partial z}{\partial \eta} \\
\frac{\partial x}{\partial \zeta} & \frac{\partial y}{\partial \zeta} & \frac{\partial z}{\partial \zeta}
\end{bmatrix}$$

$$\frac{\partial x}{\partial \xi} = \sum_i^8 \frac{\partial f_i}{\partial \xi} x_i \qquad \frac{\partial x}{\partial \eta} = \sum \frac{\partial f_i}{\partial \eta} x_i \qquad \frac{\partial x}{\partial \zeta} = \sum \frac{\partial f_i}{\partial \zeta} x_i$$

$$\frac{\partial y}{\partial \xi} = \sum \frac{\partial f_i}{\partial \xi} y_i \qquad \frac{\partial y}{\partial \eta} = \sum \frac{\partial f_i}{\partial \eta} y_i \qquad \frac{\partial y}{\partial \zeta} = \sum \frac{\partial f_i}{\partial \zeta} y_i$$

$$\frac{\partial z}{\partial \xi} = \sum \frac{\partial f_i}{\partial \xi} z_i \qquad \frac{\partial z}{\partial \eta} = \sum \frac{\partial f_i}{\partial \eta} z_i \qquad \frac{\partial z}{\partial \zeta} = \sum \frac{\partial f_i}{\partial \zeta} z_i$$

$$\underset{\sim}{J}^{*} = \underset{\sim}{J}^{-1} = \begin{bmatrix} \frac{\partial \xi}{\partial x} & \frac{\partial \eta}{\partial x} & \frac{\partial \zeta}{\partial x} \\ \frac{\partial \xi}{\partial y} & \frac{\partial \eta}{\partial y} & \frac{\partial \zeta}{\partial y} \\ \frac{\partial \xi}{\partial z} & \frac{\partial \eta}{\partial z} & \frac{\partial \zeta}{\partial z} \end{bmatrix}$$

$$\frac{\partial f_i}{\partial x} = \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial x} + \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial x} + \frac{\partial f_i}{\partial \zeta}\frac{\partial \zeta}{\partial x}$$

$$\frac{\partial f_i}{\partial y} = \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial y} + \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial y} + \frac{\partial f_i}{\partial \zeta}\frac{\partial \zeta}{\partial y}$$

$$\frac{\partial f_i}{\partial z} = \frac{\partial f_i}{\partial \xi}\frac{\partial \xi}{\partial z} + \frac{\partial f_i}{\partial \eta}\frac{\partial \eta}{\partial z} + \frac{\partial f_i}{\partial \zeta}\frac{\partial \zeta}{\partial z}$$

$$\underset{\sim}{K} = \int \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, dV$$

$$= \iiint_{-1}^{1} \underbrace{\underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{B}\, |\underset{\sim}{J}|}_{\underset{\sim}{f}}\, d\xi\, d\eta\, d\zeta$$

Using Gaussian Quadrature

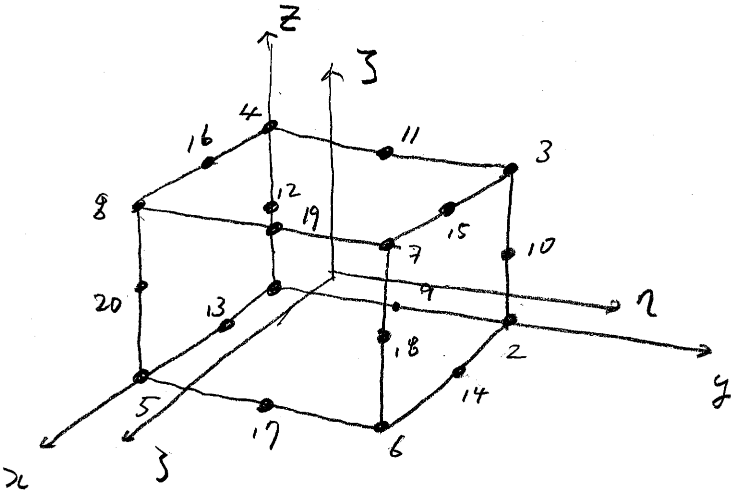
$$\underset{\sim}{K} = \sum_{i}^{l} \sum_{j}^{m} \sum_{k}^{n} R_i R_j R_k\, \underset{\sim}{f}(\xi_i\, \eta_j\, \zeta_k)$$

Equivalent nodal forces

$$\underset{\sim}{P}_b = \iiint \underset{\sim}{f}^T \underset{\sim}{b}\, |\underset{\sim}{J}|\, d\xi\, d\eta\, d\zeta$$

$$\underset{\sim}{P}_0 = \iiint \underset{\sim}{B}^T \underset{\sim}{E}\, \underset{\sim}{\varepsilon}_0\, |\underset{\sim}{J}|\, d\xi\, d\eta\, d\zeta$$

20 nodes Solid Element (Isoparametric Element)



see p160 and p161

$$\begin{cases} u = \sum f_i u_i \\ v = \sum f_i v_i \\ w = \sum f_i w_i \end{cases} \qquad \begin{cases} x = \sum f_i x_i \\ y = \sum f_i y_i \\ z = \sum f_i z_i \end{cases}$$

$$f_i = \frac{1}{8}(1+\xi_i\xi)(1+\eta_i\eta)(1+\zeta_i\zeta)(\xi_i\xi+\eta_i\eta+\zeta_i\zeta-2) \quad (i=1,2,\cdots,8)$$

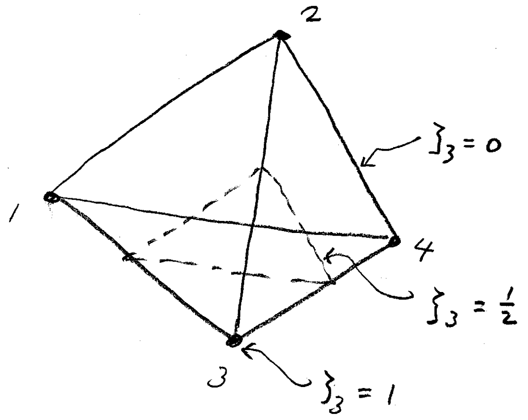
$$f_i = \frac{1}{4}(1-\xi^2)(1+\eta_i\eta)(1+\zeta_i\zeta) \quad i=9,11,17,19$$

$$f_i = \frac{1}{4}(1-\eta^2)(1+\xi_i\xi)(1+\zeta_i\zeta) \quad i=10,12,18,20$$

$$f_i = \frac{1}{4}(1-\zeta^2)(1+\xi_i\xi)(1+\eta_i\eta) \quad i=13,14,15,16$$

4-node Tetrahedron Element

volume coordinates



$$\zeta_1 = \frac{V_1}{V}$$

$$\zeta_2 = \frac{V_2}{V}$$

$$\zeta_3 = \frac{V_3}{V}$$

$$\zeta_4 = \frac{V_4}{V}$$

$$V_1 + V_2 + V_3 + V_4 = V$$

$$\zeta_1 + \zeta_2 + \zeta_3 + \zeta_4 = 1$$

$$\begin{cases} x = \Sigma f_i x_i = \Sigma \zeta_i x_i \\ y = \Sigma \zeta_i y_i \\ z = \Sigma f_i z_i \end{cases}$$

$f_i = \zeta_i$

$$\begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix} = \underbrace{\begin{bmatrix} 1 & 1 & 1 & 1 \\ x_1 & x_2 & x_3 & x_4 \\ y_1 & y_2 & y_3 & y_4 \\ z_1 & z_2 & z_3 & z_4 \end{bmatrix}}_{\underset{\sim}{H}} \begin{bmatrix} \zeta_1 \\ \zeta_2 \\ \zeta_3 \\ \zeta_4 \end{bmatrix}$$

$$\underset{\sim}{X} = \underset{\sim}{H}\, \underset{\sim}{\zeta}$$

$$\underset{\sim}{\xi} = \underset{\sim}{H}^{-1} \underset{\sim}{X}$$

$$\begin{bmatrix} \xi_1 \\ \xi_2 \\ \xi_3 \\ \xi_4 \end{bmatrix} = \frac{1}{|\underset{\sim}{H}|} \begin{bmatrix} H_{11}^* & H_{12}^* & H_{13}^* & H_{14}^* \\ H_{21}^* & H_{22}^* & H_{23}^* & H_{24}^* \\ H_{31}^* & H_{32}^* & H_{33}^* & H_{34}^* \\ H_{41}^* & H_{42}^* & H_{43}^* & H_{44}^* \end{bmatrix} \begin{bmatrix} 1 \\ x \\ y \\ z \end{bmatrix}$$

$$\begin{cases} u = \Sigma f_i u_i = \Sigma \xi_i u_i \\ v = \Sigma \xi_i v_i \\ w = \Sigma \xi_i w_i \end{cases}$$

$$\underset{\sim}{\varepsilon} = \underset{\sim}{B} \underset{\sim}{\delta}$$

$$\underset{(6\times 12)}{\underset{\sim}{B}} = \left[\begin{array}{ccc|c|ccc} f_{1,x} & 0 & 0 & & f_{4,x} & 0 & 0 \\ 0 & f_{1,y} & 0 & & 0 & f_{4,y} & 0 \\ 0 & 0 & f_{1,z} & \cdots & 0 & 0 & f_{4,z} \\ f_{1,y} & f_{1,x} & 0 & & f_{4,y} & f_{4,x} & 0 \\ 0 & f_{1,z} & f_{1,y} & & 0 & f_{4,z} & f_{4,y} \\ f_{1,z} & 0 & f_{1,x} & & f_{4,z} & 0 & f_{4,x} \end{array}\right]$$

$$\frac{\partial f_i}{\partial x} = \frac{\partial f_i}{\partial \xi_1}\frac{\partial \xi_1}{\partial x} + \frac{\partial f_i}{\partial \xi_2}\frac{\partial \xi_2}{\partial x} + \frac{\partial f_i}{\partial \xi_3}\frac{\partial \xi_3}{\partial x} + \frac{\partial f_i}{\partial \xi_4}\frac{\partial \xi_4}{\partial x}$$

$$= \frac{\partial f_i}{\partial \xi_1}\frac{1}{|\underset{\sim}{H}|}H_{12}^* + \frac{\partial f_i}{\partial \xi_2}\frac{1}{|\underset{\sim}{H}|}H_{22}^* + \frac{\partial f_i}{\partial \xi_3}\frac{1}{|\underset{\sim}{H}|}H_{32}^* + \frac{\partial f_i}{\partial \xi_4}\frac{1}{|\underset{\sim}{H}|}H_{42}^*$$

for 4node tetraheron element. $f_i \simeq \xi_i$

$$\frac{\partial f_i}{\partial x} = \frac{\partial \xi_i}{\partial \xi_i}\frac{1}{|\underset{\sim}{H}|}H_{i2}^* = \frac{1}{|H|}H_{i2}^*$$

$$\frac{\partial f_i}{\partial y} = \frac{1}{|\underset{\sim}{J}|} H_{i3}^{*} , \quad \frac{\partial f_i}{\partial z} = \frac{1}{|\underset{\sim}{J}|} H_{i4}^{*}$$

$$\underset{\sim}{K} = \int \underset{\sim}{B}^T \underset{\sim}{E} \underset{\sim}{B} \, dV = \underline{B}^T \underline{E} \, \underline{B} \, V \quad (\text{constant strain})$$

Generally

$$\underset{\sim}{K} = \int_V \zeta_1^a \zeta_2^b \zeta_3^c \zeta_4^d \, dV = \frac{a!\,b!\,c!\,d!}{(a+b+c+d+3)} (6V)$$

or

$$\underset{\sim}{K} = \frac{1}{6} \sum_{j=1}^{n} w_j \, f(\zeta_1, \zeta_2, \zeta_3, \zeta_4)_j \, |\underset{\sim}{J}(\zeta_1, \zeta_2, \zeta_3)_j|$$

p156 Table 4.1

$$\underset{\sim}{J} = \begin{bmatrix} x_{,\zeta_1} & y_{,\zeta_1} & z_{,\zeta_1} \\ x_{,\zeta_2} & y_{,\zeta_2} & z_{,\zeta_2} \\ x_{,\zeta_3} & y_{,\zeta_3} & z_{,\zeta_3} \end{bmatrix}$$

$$x_{,\zeta_1} = \frac{\partial x}{\partial \zeta_1} = \sum \frac{\partial f_i}{\partial \zeta_1} x_i \quad \cdots\cdots$$

$$x = \zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_3 + \zeta_4 x_4$$

$$= \zeta_1 x_1 + \zeta_2 x_2 + \zeta_3 x_3 + (1 - \zeta_1 - \zeta_2 - \zeta_3) x_4$$

$$x_{,\zeta_1} = x_1 - x_4 \Rightarrow \underset{\sim}{J} \Rightarrow \text{constants}$$

$$|\underset{\sim}{J}| = 6V$$

Tetrahedron 10 node Element

→ see p. 164