CHAPTER
TWO

ROBOT ARM KINEMATICS

And see! she stirs, she starts,
she moves, she seems to feel
the thrill of life!
Henry Wadsworth Longfellow

2.1 INTRODUCTION

A mechanical manipulator can be modeled as an open-loop articulated chain with several rigid bodies (links) connected in series by either revolute or prismatic joints driven by actuators. One end of the chain is attached to a supporting base while the other end is free and attached with a tool (the end-effector) to manipulate objects or perform assembly tasks. The relative motion of the joints results in the motion of the links that positions the hand in a desired orientation. In most robotic applications, one is interested in the spatial description of the end-effector of the manipulator with respect to a fixed reference coordinate system.

Robot arm kinematics deals with the analytical study of the geometry of motion of a robot arm with respect to a fixed reference coordinate system as a function of time without regard to the forces/moments that cause the motion. Thus, it deals with the analytical description of the spatial displacement of the robot as a function of time, in particular the relations between the joint-variable space and the position and orientation of the end-effector of a robot arm. This chapter addresses two fundamental questions of both theoretical and practical interest in robot arm kinematics:

1. For a given manipulator, given the joint angle vector $\mathbf{q}(t) = (q_1(t), q_2(t), \ldots, q_n(t))^T$ and the geometric link parameters, where n is the number of degrees of freedom, what is the position and orientation of the end-effector of the manipulator with respect to a reference coordinate system?
2. Given a desired position and orientation of the end-effector of the manipulator and the geometric link parameters with respect to a reference coordinate system, can the manipulator reach the desired prescribed manipulator hand position and orientation? And if it can, how many different manipulator configurations will satisfy the same condition?

The first question is usually referred to as the *direct (or forward) kinematics* problem, while the second question is the *inverse kinematics (or arm solution)* problem.

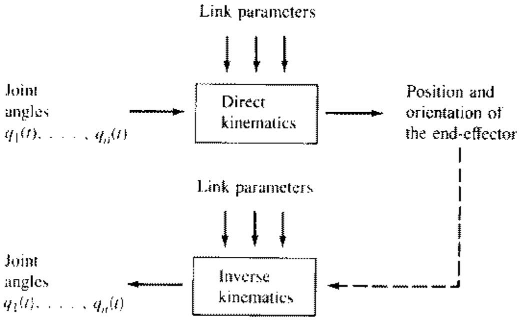


Figure 2.1 The direct and inverse kinematics problems.

Since the independent variables in a robot arm are the joint variables and a task is usually stated in terms of the reference coordinate frame, the inverse kinematics problem is used more frequently. A simple block diagram indicating the relationship between these two problems is shown in Fig. 2.1.

Since the links of a robot arm may rotate and/or translate with respect to a reference coordinate frame, the total spatial displacement of the end-effector is due to the angular rotations and linear translations of the links. Denavit and Hartenberg [1955] proposed a systematic and generalized approach of utilizing matrix algebra to describe and represent the spatial geometry of the links of a robot arm with respect to a fixed reference frame. This method uses a 4×4 homogeneous transformation matrix to describe the spatial relationship between two adjacent rigid mechanical links and reduces the direct kinematics problem to finding an equivalent 4×4 homogeneous transformation matrix that relates the spatial displacement of the "hand coordinate frame" to the reference coordinate frame. These homogeneous transformation matrices are also useful in deriving the dynamic equations of motion of a robot arm.

In general, the inverse kinematics problem can be solved by several techniques. Most commonly used methods are the matrix algebraic, iterative, or geometric approaches. A geometric approach based on the link coordinate systems and the manipulator configuration will be presented in obtaining a closed form joint solution for simple manipulators with rotary joints. Then a more general approach using 4×4 homogeneous matrices will be explored in obtaining a joint solution for simple manipulators.

2.2 THE DIRECT KINEMATICS PROBLEM

Vector and matrix algebra† are utilized to develop a systematic and generalized approach to describe and represent the location of the links of a robot arm with

† Vectors are represented in lowercase bold letters; matrices are in uppercase bold.

respect to a fixed reference frame. Since the links of a robot arm may rotate and/or translate with respect to a reference coordinate frame, a body-attached coordinate frame will be established along the joint axis for each link. The direct kinematics problem is reduced to finding a transformation matrix that relates the body-attached coordinate frame to the reference coordinate frame. A 3×3 rotation matrix is used to describe the rotational operations of the body-attached frame with respect to the reference frame. The homogeneous coordinates are then used to represent position vectors in a three-dimensional space, and the rotation matrices will be expanded to 4×4 homogeneous transformation matrices to include the translational operations of the body-attached coordinate frames. This matrix representation of a rigid mechanical link to describe the spatial geometry of a robot arm was first used by Denavit and Hartenberg [1955]. The advantage of using the Denavit-Hartenberg representation of linkages is its algorithmic universality in deriving the kinematic equation of a robot arm.

2.2.1 Rotation Matrices

A 3×3 rotation matrix can be defined as a transformation matrix which operates on a position vector in a three-dimensional euclidean space and maps its coordinates expressed in a rotated coordinate system *OUVW* (body-attached frame) to a reference coordinate system *OXYZ*. In Fig. 2.2, we are given two right-hand rectangular coordinate systems, namely, the *OXYZ* coordinate system with *OX*, *OY*, and *OZ* as its coordinate axes and the *OUVW* coordinate system with *OU*, *OV*, and *OW* as its coordinate axes. Both coordinate systems have their origins coincident at point *O*. The *OXYZ* coordinate system is fixed in the three-dimensional space and is considered to be the reference frame. The *OUVW* coordinate frame is rotating with respect to the reference frame *OXYZ*. Physically, one can consider the *OUVW* coordinate system to be a body-attached coordinate frame. That is, it is

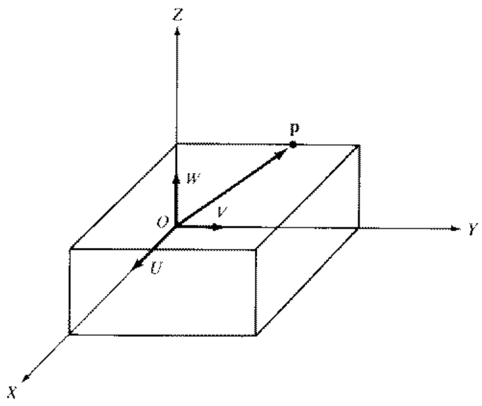


Figure 2.2 Reference and body-attached coordinate systems.

permanently and conveniently attached to the rigid body (e.g., an aircraft or a link of a robot arm) and moves together with it. Let ($\mathbf{i}_x$, $\mathbf{j}_y$, $\mathbf{k}_z$) and ($\mathbf{i}_u$, $\mathbf{j}_v$, $\mathbf{k}_w$) be the unit vectors along the coordinate axes of the *OXYZ* and *OUVW* systems, respectively. A point $\mathbf{p}$ in the space can be represented by its coordinates with respect to both coordinate systems. For ease of discussion, we shall assume that $\mathbf{p}$ is at rest and fixed with respect to the *OUVW* coordinate frame. Then the point $\mathbf{p}$ can be represented by its coordinates with respect to the *OUVW* and *OXYZ* coordinate systems, respectively, as

$$\mathbf{p}_{uvw} = (p_u, p_v, p_w)^T \qquad \text{and} \qquad \mathbf{p}_{xyz} = (p_x, p_y, p_z)^T \tag{2.2-1}$$

where $\mathbf{p}_{xyz}$ and $\mathbf{p}_{uvw}$ represent the same point $\mathbf{p}$ in the space with reference to different coordinate systems, and the superscript T on vectors and matrices denotes the transpose operation.

We would like to find a 3×3 transformation matrix $\mathbf{R}$ that will transform the coordinates of $\mathbf{p}_{uvw}$ to the coordinates expressed with respect to the *OXYZ* coordinate system, after the *OUVW* coordinate system has been rotated. That is,

$$\mathbf{p}_{xyz} = \mathbf{R}\mathbf{p}_{uvw} \tag{2.2-2}$$

Note that physically the point $\mathbf{p}_{uvw}$ has been rotated together with the *OUVW* coordinate system.

Recalling the definition of the components of a vector, we have

$$\mathbf{p}_{uvw} = p_u\mathbf{i}_u + p_v\mathbf{j}_v + p_w\mathbf{k}_w \tag{2.2-3}$$

where p_x, p_y, and p_z represent the components of $\mathbf{p}$ along the *OX*, *OY*, and *OZ* axes, respectively, or the projections of $\mathbf{p}$ onto the respective axes. Thus, using the definition of a scalar product and Eq. (2.2-3),

$$\begin{aligned} p_x &= \mathbf{i}_x \cdot \mathbf{p} = \mathbf{i}_x \cdot \mathbf{i}_u p_u + \mathbf{i}_x \cdot \mathbf{j}_v p_v + \mathbf{i}_x \cdot \mathbf{k}_w p_w \\ p_y &= \mathbf{j}_y \cdot \mathbf{p} = \mathbf{j}_y \cdot \mathbf{i}_u p_u + \mathbf{j}_y \cdot \mathbf{j}_v p_v + \mathbf{j}_y \cdot \mathbf{k}_w p_w \\ p_z &= \mathbf{k}_z \cdot \mathbf{p} = \mathbf{k}_z \cdot \mathbf{i}_u p_u + \mathbf{k}_z \cdot \mathbf{j}_v p_v + \mathbf{k}_z \cdot \mathbf{k}_w p_w \end{aligned} \tag{2.2-4}$$

or expressed in matrix form,

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} \mathbf{i}_x \cdot \mathbf{i}_u & \mathbf{i}_x \cdot \mathbf{j}_v & \mathbf{i}_x \cdot \mathbf{k}_w \\ \mathbf{j}_y \cdot \mathbf{i}_u & \mathbf{j}_y \cdot \mathbf{j}_v & \mathbf{j}_y \cdot \mathbf{k}_w \\ \mathbf{k}_z \cdot \mathbf{i}_u & \mathbf{k}_z \cdot \mathbf{j}_v & \mathbf{k}_z \cdot \mathbf{k}_w \end{bmatrix} \begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} \tag{2.2-5}$$

Using this notation, the matrix **R** in Eq. (2.2-2) is given by

$$\mathbf{R} = \begin{bmatrix} \mathbf{i}_x \cdot \mathbf{i}_u & \mathbf{i}_x \cdot \mathbf{j}_v & \mathbf{i}_x \cdot \mathbf{k}_w \\ \mathbf{j}_y \cdot \mathbf{i}_u & \mathbf{j}_y \cdot \mathbf{j}_v & \mathbf{j}_y \cdot \mathbf{k}_w \\ \mathbf{k}_z \cdot \mathbf{i}_u & \mathbf{k}_z \cdot \mathbf{j}_v & \mathbf{k}_z \cdot \mathbf{k}_w \end{bmatrix} \tag{2.2-6}$$

Similarly, one can obtain the coordinates of $\mathbf{p}_{uvw}$ from the coordinates of $\mathbf{p}_{xyz}$:

$$\mathbf{p}_{uvw} = \mathbf{Q}\mathbf{p}_{xyz} \tag{2.2-7}$$

or

$$\begin{bmatrix} p_u \\ p_v \\ p_w \end{bmatrix} = \begin{bmatrix} \mathbf{i}_u \cdot \mathbf{i}_x & \mathbf{i}_u \cdot \mathbf{j}_y & \mathbf{i}_u \cdot \mathbf{k}_z \\ \mathbf{j}_v \cdot \mathbf{i}_x & \mathbf{j}_v \cdot \mathbf{j}_y & \mathbf{j}_v \cdot \mathbf{k}_z \\ \mathbf{k}_w \cdot \mathbf{i}_x & \mathbf{k}_w \cdot \mathbf{j}_y & \mathbf{k}_w \cdot \mathbf{k}_z \end{bmatrix} \begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} \tag{2.2-8}$$

Since dot products are commutative, one can see from Eqs. (2.2-6) to (2.2-8) that

$$\mathbf{Q} = \mathbf{R}^{-1} = \mathbf{R}^T \tag{2.2-9}$$

and

$$\mathbf{Q}\mathbf{R} = \mathbf{R}^T\mathbf{R} = \mathbf{R}^{-1}\mathbf{R} = \mathbf{I}_3 \tag{2.2-10}$$

where $\mathbf{I}_3$ is a 3×3 identity matrix. The transformation given in Eq. (2.2-2) or (2.2-7) is called an *orthogonal* transformation and since the vectors in the dot products are all unit vectors, it is also called an *orthonormal* transformation.

The primary interest in developing the above transformation matrix is to find the rotation matrices that represent rotations of the *OUVW* coordinate system about each of the three principal axes of the reference coordinate system *OXYZ*. If the *OUVW* coordinate system is rotated an α angle about the *OX* axis to arrive at a new location in the space, then the point $\mathbf{p}_{uvw}$ having coordinates $(p_u, p_v, p_w)^T$ with respect to the *OUVW* system will have different coordinates $(p_x, p_y, p_z)^T$ with respect to the reference system *OXYZ*. The necessary transformation matrix $\mathbf{R}_{x,\alpha}$ is called the rotation matrix about the *OX* axis with α angle. $\mathbf{R}_{x,\alpha}$ can be derived from the above transformation matrix concept, that is

$$\mathbf{p}_{xyz} = \mathbf{R}_{x,\alpha}\, \mathbf{p}_{uvw} \tag{2.2-11}$$

with $\mathbf{i}_x \equiv \mathbf{i}_u$, and

$$\mathbf{R}_{x,\alpha} = \begin{bmatrix} \mathbf{i}_x \cdot \mathbf{i}_u & \mathbf{i}_x \cdot \mathbf{j}_v & \mathbf{i}_x \cdot \mathbf{k}_w \\ \mathbf{j}_y \cdot \mathbf{i}_u & \mathbf{j}_y \cdot \mathbf{j}_v & \mathbf{j}_y \cdot \mathbf{k}_w \\ \mathbf{k}_z \cdot \mathbf{i}_u & \mathbf{k}_z \cdot \mathbf{j}_v & \mathbf{k}_z \cdot \mathbf{k}_w \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix} \tag{2.2-12}$$

Similarly, the 3×3 rotation matrices for rotation about the *OY* axis with ϕ angle and about the *OZ* axis with θ angle are, respectively (see Fig. 2.3),

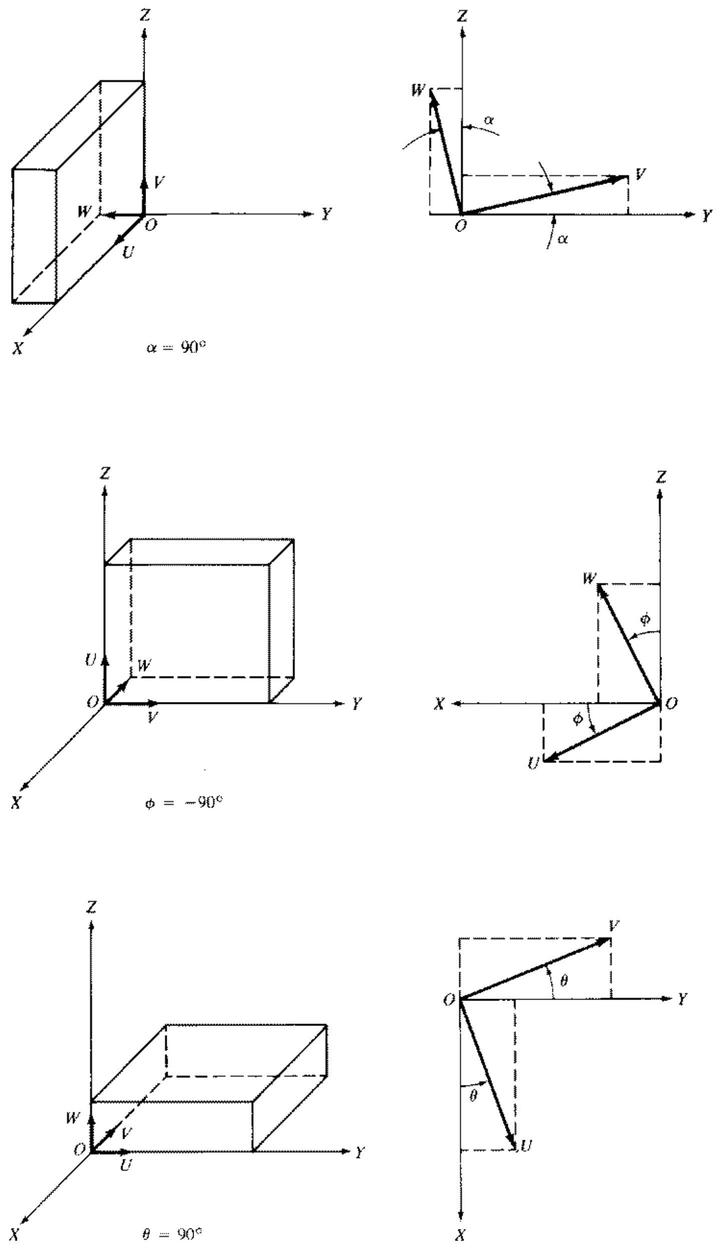


Figure 2.3 Rotating coordinate systems.

$$\mathbf{R}_{y,\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi \\ 0 & 1 & 0 \\ -\sin\phi & 0 & \cos\phi \end{bmatrix} \qquad \mathbf{R}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 \\ \sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \tag{2.2-13}$$

The matrices $\mathbf{R}_{x,\alpha}$, $\mathbf{R}_{y,\phi}$, and $\mathbf{R}_{z,\theta}$ are called the *basic rotation matrices*. Other finite rotation matrices can be obtained from these matrices.

Example: Given two points $\mathbf{a}_{uvw} = (4, 3, 2)^T$ and $\mathbf{b}_{uvw} = (6, 2, 4)^T$ with respect to the rotated *OUVW* coordinate system, determine the corresponding points $\mathbf{a}_{xyz}$, $\mathbf{b}_{xyz}$ with respect to the reference coordinate system if it has been rotated 60° about the *OZ* axis.

SOLUTION: $\mathbf{a}_{xyz} = \mathbf{R}_{z,60°}\,\mathbf{a}_{uvw}$ and $\mathbf{b}_{xyz} = \mathbf{R}_{z,60°}\,\mathbf{b}_{uvw}$

$$\mathbf{a}_{xyz} = \begin{bmatrix} 0.500 & -0.866 & 0 \\ 0.866 & 0.500 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix}$$

$$= \begin{bmatrix} 4(0.5)+3(-0.866)+2(0) \\ 4(0.866)+3(0.5)+2(0) \\ 4(0)+3(0)+2(1) \end{bmatrix} = \begin{bmatrix} -0.598 \\ 4.964 \\ 2.0 \end{bmatrix}$$

$$\mathbf{b}_{xyz} = \begin{bmatrix} 0.500 & -0.866 & 0 \\ 0.866 & 0.500 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 1.268 \\ 6.196 \\ 4.0 \end{bmatrix}$$

Thus, $\mathbf{a}_{xyz}$ and $\mathbf{b}_{xyz}$ are equal to $(-0.598, 4.964, 2.0)^T$ and $(1.268, 6.196, 4.0)^T$, respectively, when expressed in terms of the reference coordinate system. □

Example: If $\mathbf{a}_{xyz} = (4, 3, 2)^T$ and $\mathbf{b}_{xyz} = (6, 2, 4)^T$ are the coordinates with respect to the reference coordinate system, determine the corresponding points $\mathbf{a}_{uvw}$, $\mathbf{b}_{uvw}$ with respect to the rotated *OUVW* coordinate system if it has been rotated 60° about the *OZ* axis.

SOLUTION: $\mathbf{a}_{uvw} = (\mathbf{R}_{z,\,60})^T\mathbf{a}_{xyz}$ and $\mathbf{b}_{uvw} = (\mathbf{R}_{z,\,60})^T\mathbf{b}_{xyz}$

$$\mathbf{a}_{uvw} = \begin{bmatrix} 0.500 & 0.866 & 0 \\ -0.866 & 0.500 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 4(0.5)+3(0.866)+2(0) \\ 4(-0.866)+3(0.5)+2(0) \\ 4(0)+3(0)+2(1) \end{bmatrix}$$

$$= \begin{bmatrix} 4.598 \\ -1.964 \\ 2.0 \end{bmatrix}$$

$$\mathbf{b}_{uvw} = \begin{bmatrix} 0.500 & 0.866 & 0 \\ -0.866 & 0.500 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \end{bmatrix} = \begin{bmatrix} 4.732 \\ -4.196 \\ 4.0 \end{bmatrix}$$ □

2.2.2 Composite Rotation Matrix

Basic rotation matrices can be multiplied together to represent a sequence of finite rotations about the principal axes of the *OXYZ* coordinate system. Since matrix multiplications do not commute, the order or sequence of performing rotations is important. For example, to develop a rotation matrix representing a rotation of α angle about the *OX* axis followed by a rotation of θ angle about the *OZ* axis followed by a rotation of ϕ angle about the *OY* axis, the resultant rotation matrix representing these rotations is

$$\mathbf{R} = \mathbf{R}_{y,\phi}\,\mathbf{R}_{z,\theta}\,\mathbf{R}_{x,\alpha} = \begin{bmatrix} C\phi & 0 & S\phi \\ 0 & 1 & 0 \\ -S\phi & 0 & C\phi \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

$$= \begin{bmatrix} C\phi C\theta & S\phi S\alpha - C\phi S\theta C\alpha & C\phi S\theta S\alpha + S\phi C\alpha \\ S\theta & C\theta C\alpha & -C\theta S\alpha \\ -S\phi C\theta & S\phi S\theta C\alpha + C\phi S\alpha & C\phi C\alpha - S\phi S\theta S\alpha \end{bmatrix} \qquad (2.2\text{-}14)$$

where $C\phi \equiv \cos\phi$; $S\phi \equiv \sin\phi$; $C\theta \equiv \cos\theta$; $S\theta \equiv \sin\theta$; $C\alpha \equiv \cos\alpha$; $S\alpha \equiv \sin\alpha$. That is different from the rotation matrix which represents a rotation of ϕ angle about the *OY* axis followed by a rotation of θ angle about the *OZ* axis followed by a rotation of α angle about the *OX* axis. The resultant rotation matrix is:

$$\mathbf{R} = \mathbf{R}_{x,\alpha}\,\mathbf{R}_{z,\theta}\,\mathbf{R}_{y,\phi} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\phi & 0 & S\phi \\ 0 & 1 & 0 \\ -S\phi & 0 & C\phi \end{bmatrix}$$

$$= \begin{bmatrix} C\theta C\phi & -S\theta & C\theta S\phi \\ C\alpha S\theta C\phi + S\alpha S\phi & C\alpha C\theta & C\alpha S\theta S\phi - S\alpha C\phi \\ S\alpha S\theta C\phi - C\alpha S\phi & S\alpha C\theta & S\alpha S\theta S\phi + C\alpha C\phi \end{bmatrix} \qquad (2.2\text{-}15)$$

In addition to rotating about the principal axes of the reference frame *OXYZ*, the rotating coordinate system *OUVW* can also rotate about its own principal axes. In this case, the resultant or composite rotation matrix may be obtained from the following simple rules:

1. Initially both coordinate systems are coincident, hence the rotation matrix is a 3×3 identity matrix, $\mathbf{I}_3$.
2. If the rotating coordinate system *OUVW* is rotating about one of the principal axes of the *OXYZ* frame, then *premultiply* the previous (resultant) rotation matrix with an appropriate basic rotation matrix.
3. If the rotating coordinate system *OUVW* is rotating about its own principal axes, then *postmultiply* the previous (resultant) rotation matrix with an appropriate basic rotation matrix.

Example: Find the resultant rotation matrix that represents a rotation of ϕ angle about the *OY* axis followed by a rotation of θ angle about the *OW* axis followed by a rotation of α angle about the *OU* axis.

SOLUTION:

$$\mathbf{R} = \mathbf{R}_{y,\phi}\,\mathbf{I}_3\,\mathbf{R}_{w,\theta}\,\mathbf{R}_{u,\alpha} = \mathbf{R}_{y,\phi}\,\mathbf{R}_{w,\theta}\,\mathbf{R}_{u,\alpha}$$

$$= \begin{bmatrix} C\phi & 0 & S\phi \\ 0 & 1 & 0 \\ -S\phi & 0 & C\phi \end{bmatrix} \begin{bmatrix} C\theta & -S\theta & 0 \\ S\theta & C\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

$$= \begin{bmatrix} C\phi C\theta & S\phi S\alpha - C\phi S\theta C\alpha & C\phi S\theta S\alpha + S\phi C\alpha \\ S\theta & C\theta C\alpha & -C\theta S\alpha \\ -S\phi C\theta & S\phi S\theta C\alpha + C\phi S\alpha & C\phi C\alpha - S\phi S\theta S\alpha \end{bmatrix}$$

Note that this example is chosen so that the resultant matrix is the same as Eq. (2.2-14), but the sequence of rotations is different from the one that generates Eq. (2.2-14). □

2.2.3 Rotation Matrix About an Arbitrary Axis

Sometimes the rotating coordinate system *OUVW* may rotate ϕ angle about an arbitrary axis $\mathbf{r}$ which is a unit vector having components of r_x, r_y, and r_z and passing through the origin O. The advantage is that for certain angular motions the *OUVW* frame can make one rotation about the axis $\mathbf{r}$ instead of several rotations about the principal axes of the *OUVW* and/or *OXYZ* coordinate frames. To derive this rotation matrix $\mathbf{R}_{r,\phi}$, we can first make some rotations about the principal axes of the *OXYZ* frame to align the axis $\mathbf{r}$ with the *OZ* axis. Then make the rotation about the $\mathbf{r}$ axis with ϕ angle and rotate about the principal axes of the *OXYZ* frame again to return the $\mathbf{r}$ axis back to its original location. With reference to Fig. 2.4, aligning the *OZ* axis with the $\mathbf{r}$ axis can be done by rotating about the *OX* axis with α angle (the axis $\mathbf{r}$ is in the *XZ* plane), followed by a rotation of $-\beta$ angle about the *OY* axis (the axis $\mathbf{r}$ now aligns with the *OZ* axis). After the rotation of ϕ angle about the *OZ* or $\mathbf{r}$ axis, reverse the above sequence of rotations with their respective opposite angles. The resultant rotation matrix is

$$\mathbf{R}_{r,\phi} = \mathbf{R}_{x,-\alpha}\mathbf{R}_{y,\beta}\mathbf{R}_{z,\phi}\mathbf{R}_{y,-\beta}\mathbf{R}_{x,\alpha}$$

$$= \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & S\alpha \\ 0 & -S\alpha & C\alpha \end{bmatrix} \begin{bmatrix} C\beta & 0 & S\beta \\ 0 & 1 & 0 \\ -S\beta & 0 & C\beta \end{bmatrix} \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} C\beta & 0 & -S\beta \\ 0 & 1 & 0 \\ S\beta & 0 & C\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\alpha & -S\alpha \\ 0 & S\alpha & C\alpha \end{bmatrix}$$

From Fig. 2.4, we easily find that

$$\sin\alpha = \frac{r_y}{\sqrt{r_y^2+r_z^2}} \qquad \cos\alpha = \frac{r_z}{\sqrt{r_y^2+r_z^2}}$$

$$\sin\beta = r_x \qquad \cos\beta = \sqrt{r_y^2+r_z^2}$$

Substituting into the above equation,

$$\mathbf{R}_{r,\phi} = \begin{bmatrix} r_x^2V\phi + C\phi & r_xr_yV\phi - r_zS\phi & r_xr_zV\phi + r_yS\phi \\ r_xr_yV\phi + r_zS\phi & r_y^2V\phi + C\phi & r_yr_zV\phi - r_xS\phi \\ r_xr_zV\phi - r_yS\phi & r_yr_zV\phi + r_xS\phi & r_z^2V\phi + C\phi \end{bmatrix} \tag{2.2-16}$$

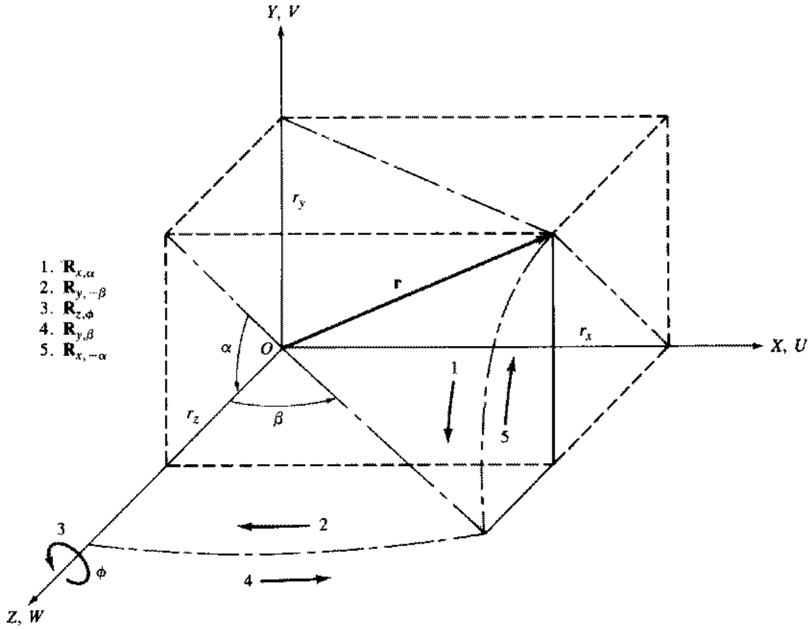


Figure 2.4 Rotation about an arbitrary axis.

where $V\phi = \text{vers}\,\phi = 1 - \cos\phi$. This is a very useful rotation matrix.

Example: Find the rotation matrix $\mathbf{R}_{r,\phi}$ that represents the rotation of ϕ angle about the vector $\mathbf{r} = (1, 1, 1)^T$.

SOLUTION: Since the vector $\mathbf{r}$ is not a unit vector, we need to normalize it and find its components along the principal axes of the $OXYZ$ frame. Therefore,

$$r_x = \frac{1}{\sqrt{r_x^2 + r_y^2 + r_z^2}} = \frac{1}{\sqrt{3}} \quad r_y = \frac{1}{\sqrt{3}} \quad r_z = \frac{1}{\sqrt{3}}$$

Substituting into Eq. (2.2-16), we obtain the $\mathbf{R}_{r,\phi}$ matrix:

$$\mathbf{R}_{r,\phi} = \begin{bmatrix} \frac{1}{3}V\phi + C\phi & \frac{1}{3}V\phi - \frac{1}{\sqrt{3}}S\phi & \frac{1}{3}V\phi + \frac{1}{\sqrt{3}}S\phi \\ \frac{1}{3}V\phi + \frac{1}{\sqrt{3}}S\phi & \frac{1}{3}V\phi + C\phi & \frac{1}{3}V\phi - \frac{1}{\sqrt{3}}S\phi \\ \frac{1}{3}V\phi - \frac{1}{\sqrt{3}}S\phi & \frac{1}{3}V\phi + \frac{1}{\sqrt{3}}S\phi & \frac{1}{3}V\phi + C\phi \end{bmatrix}$$ □

2.2.4 Rotation Matrix with Euler Angles Representation

The matrix representation for rotation of a rigid body simplifies many operations, but it needs nine elements to completely describe the orientation of a rotating rigid body. It does not lead directly to a complete set of generalized coordinates. Such a set of generalized coordinates can describe the orientation of a rotating rigid body with respect to a reference coordinate frame. They can be provided by three angles called Euler angles ϕ, θ, and ψ. Although Euler angles describe the orientation of a rigid body with respect to a fixed reference frame, there are many different types of Euler angle representations. The three most widely used Euler angles representations are tabulated in Table 2.1.

The first Euler angle representation in Table 2.1 is usually associated with gyroscopic motion. This representation is usually called the eulerian angles, and corresponds to the following sequence of rotations (see Fig. 2.5):

Table 2.1 Three types of Euler angle representations

	Eulerian angles system I	Euler angles system II	Roll, pitch, and yaw system III
Sequence	ϕ about OZ axis	ϕ about OZ axis	ψ about OX axis
of	θ about OU axis	θ about OV axis	θ about OY axis
rotations	ψ about OW axis	ψ about OW axis	ϕ about OZ axis

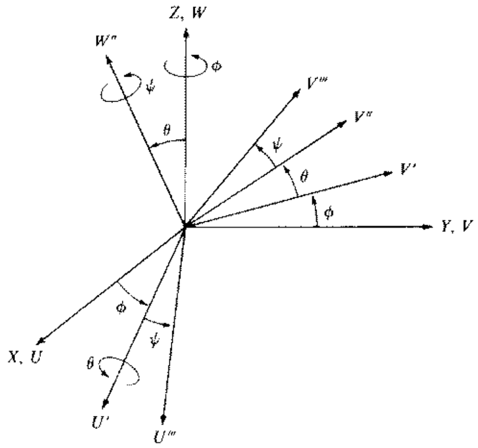


Figure 2.5 Eulerian angles system I.

1. A rotation of ϕ angle about the OZ axis ($\mathbf{R}_{z,\phi}$)
2. A rotation of θ angle about the rotated OU axis ($\mathbf{R}_{u,\theta}$)
3. Finally a rotation of ψ angle about the rotated OW axis ($\mathbf{R}_{w,\psi}$)

The resultant eulerian rotation matrix is

$$\mathbf{R}_{\phi,\theta,\psi} = \mathbf{R}_{z,\phi}\,\mathbf{R}_{u,\theta}\,\mathbf{R}_{w,\psi}$$

$$= \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \begin{bmatrix} C\psi & -S\psi & 0 \\ S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\phi C\psi - S\phi C\theta S\psi & -C\phi S\psi - S\phi C\theta C\psi & S\phi S\theta \\ S\phi C\psi + C\phi C\theta S\psi & -S\phi S\psi + C\phi C\theta C\psi & -C\phi S\theta \\ S\theta S\psi & S\theta C\psi & C\theta \end{bmatrix} \quad (2.2\text{-}17)$$

The above eulerian angle rotation matrix $\mathbf{R}_{\phi,\theta,\psi}$ can also be specified in terms of the rotations about the principal axes of the reference coordinate system: a rotation of ψ angle about the OZ axis followed by a rotation of θ angle about the OX axis and finally a rotation of ϕ angle about the OZ axis.

With reference to Fig. 2.6, another set of Euler angles ϕ, θ, and ψ representation corresponds to the following sequence of rotations:

1. A rotation of ϕ angle about the OZ axis ($\mathbf{R}_{z,\phi}$)
2. A rotation of θ angle about the rotated OV axis ($\mathbf{R}_{v,\theta}$)
3. Finally a rotation of ψ angle about the rotated OW axis ($\mathbf{R}_{w,\psi}$)

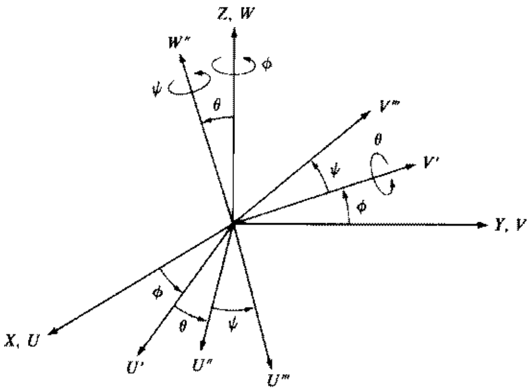


Figure 2.6 Eulerian angles system II.

The resultant rotation matrix is

$$\mathbf{R}_{\phi,\theta,\psi} = \mathbf{R}_{z,\phi}\,\mathbf{R}_{v,\theta}\,\mathbf{R}_{w,\psi}$$

$$= \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} C\psi & -S\psi & 0 \\ S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} C\phi C\theta C\psi - S\phi S\psi & -C\phi C\theta S\psi - S\phi C\psi & C\phi S\theta \\ S\phi C\theta C\psi + C\phi S\psi & -S\phi C\theta S\psi + C\phi C\psi & S\phi S\theta \\ -S\theta C\psi & S\theta S\psi & C\theta \end{bmatrix} \qquad (2.2\text{-}18)$$

The above Euler angle rotation matrix $\mathbf{R}_{\phi,\theta,\psi}$ can also be specified in terms of the rotations about the principal axes of the reference coordinate system: a rotation of ψ angle about the OZ axis followed by a rotation of θ angle about the OY axis and finally a rotation of ϕ angle about the OZ axis.

Another set of Euler angles representation for rotation is called *roll*, *pitch*, and *yaw* (RPY). This is mainly used in aeronautical engineering in the analysis of space vehicles. They correspond to the following rotations in sequence:

1. A rotation of ψ about the OX axis ($\mathbf{R}_{x,\psi}$)—yaw
2. A rotation of θ about the OY axis ($\mathbf{R}_{y,\theta}$)—pitch
3. A rotation of ϕ about the OZ axis ($\mathbf{R}_{z,\phi}$)—roll

The resultant rotation matrix is

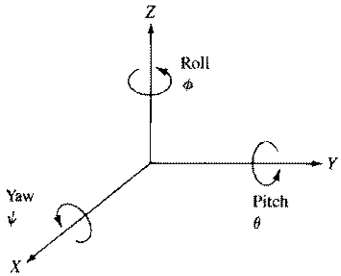


Figure 2.7 Roll, pitch and yaw.

$$\mathbf{R}_{\phi,\theta,\psi} = \mathbf{R}_{z,\phi}\,\mathbf{R}_{y,\theta}\,\mathbf{R}_{x,\psi} = \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\theta & 0 & S\theta \\ 0 & 1 & 0 \\ -S\theta & 0 & C\theta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\psi & -S\psi \\ 0 & S\psi & C\psi \end{bmatrix}$$

$$= \begin{bmatrix} C\phi C\theta & C\phi S\theta S\psi - S\phi C\psi & C\phi S\theta C\psi + S\phi S\psi \\ S\phi C\theta & S\phi S\theta S\psi + C\phi C\psi & S\phi S\theta C\psi - C\phi S\psi \\ -S\theta & C\theta S\psi & C\theta C\psi \end{bmatrix} \tag{2.2-19}$$

The above rotation matrix $\mathbf{R}_{\phi,\theta,\psi}$ for roll, pitch, and yaw can be specified in terms of the rotations about the principal axes of the reference coordinate system and the rotating coordinate system: a rotation of ϕ angle about the OZ axis followed by a rotation of θ angle about the rotated OV axis and finally a rotation of ϕ angle about the rotated OU axis (see Fig. 2.7).

2.2.5 Geometric Interpretation of Rotation Matrices

It is worthwhile to interpret the basic rotation matrices geometrically. Let us choose a point $\mathbf{p}$ fixed in the $OUVW$ coordinate system to be $(1, 0, 0)^T$, that is, $\mathbf{p}_{uvw} \equiv \mathbf{i}_u$. Then the first column of the rotation matrix represents the coordinates of this point with respect to the $OXYZ$ coordinate system. Similarly, choosing $\mathbf{p}$ to be $(0, 1, 0)^T$ and $(0, 0, 1)^T$, one can identify that the second- and third-column elements of a rotation matrix represent the OV and OW axes, respectively, of the $OUVW$ coordinate system with respect to the $OXYZ$ coordinate system. Thus, given a reference frame $OXYZ$ and a rotation matrix, the column vectors of the rotation matrix represent the principal axes of the $OUVW$ coordinate system with respect to the reference frame and one can draw the location of all the principal axes of the $OUVW$ coordinate frame with respect to the reference frame. In other words, a rotation matrix geometrically represents the principal axes of the rotated coordinate system with respect to the reference coordinate system.

Since the inverse of a rotation matrix is equivalent to its transpose, the row vectors of the rotation matrix represent the principal axes of the reference system *OXYZ* with respect to the rotated coordinate system *OUVW*. This geometric interpretation of the rotation matrices is an important concept that provides insight into many robot arm kinematics problems. Several useful properties of rotation matrices are listed as follows:

1. Each column vector of the rotation matrix is a representation of the rotated axis unit vector expressed in terms of the axis unit vectors of the reference frame, and each row vector is a representation of the axis unit vector of the reference frame expressed in terms of the rotated axis unit vectors of the *OUVW* frame.
2. Since each row and column is a unit vector representation, the magnitude of each row and column should be equal to 1. This is a direct property of orthonormal coordinate systems. Furthermore, the determinant of a rotation matrix is $+1$ for a right-hand coordinate system and -1 for a left-hand coordinate system.
3. Since each row is a vector representation of orthonormal vectors, the inner product (dot product) of each row with each other row equals zero. Similarly, the inner product of each column with each other column equals zero.
4. The inverse of a rotation matrix is the transpose of the rotation matrix.

$$\mathbf{R}^{-1} = \mathbf{R}^T \qquad \text{and} \qquad \mathbf{R}\,\mathbf{R}^T = \mathbf{I}_3$$

where $\mathbf{I}_3$ is a 3×3 identity matrix.

Properties 3 and 4 are especially useful in checking the results of rotation matrix multiplications, and in determining an erroneous row or column vector.

Example: If the *OU*, *OV*, and *OW* coordinate axes were rotated with α angle about the *OX* axis, what would the representation of the coordinate axes of the reference frame be in terms of the rotated coordinate system *OUVW*?

Solution: The new coordinate axis unit vectors become $\mathbf{i}_u = (1, 0, 0)^T$, $\mathbf{j}_v = (0, 1, 0)^T$, and $\mathbf{k}_w = (0, 0, 1)^T$ since they are expressed in terms of themselves. The original unit vectors are then

$$\mathbf{i}_x = 1\mathbf{i}_u + 0\mathbf{j}_v + 0\mathbf{k}_w = (1, 0, 0)^T$$

$$\mathbf{j}_y = 0\mathbf{i}_u + \cos\alpha\,\mathbf{j}_v - \sin\alpha\,\mathbf{k}_w = (0, \cos\alpha, -\sin\alpha)^T$$

$$\mathbf{k}_z = 0\mathbf{i}_u + \sin\alpha\,\mathbf{j}_v + \cos\alpha\,\mathbf{k}_w = (0, \sin\alpha, \cos\alpha)^T$$

Applying property 1 and considering these as rows of the rotation matrix, the $\mathbf{R}_{x,\alpha}$ matrix can be reconstructed as

$$\mathbf{R}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & \sin\alpha \\ 0 & -\sin\alpha & \cos\alpha \end{bmatrix}$$

which is the same as the transpose of Eq. (2.2-12). □

2.2.6 Homogeneous Coordinates and Transformation Matrix

Since a 3×3 rotation matrix does not give us any provision for translation and scaling, a fourth coordinate or component is introduced to a position vector $\mathbf{p} = (p_x, p_y, p_z)^T$ in a three-dimensional space which makes it $\hat{\mathbf{p}} = (wp_x, wp_y, wp_z, w)^T$. We say that the position vector $\hat{\mathbf{p}}$ is expressed in *homogeneous coordinates*. In this section, we use a "hat" (i.e., $\hat{\mathbf{p}}$) to indicate the representation of a cartesian vector in homogeneous coordinates. Later, if no confusion exists, these "hats" will be lifted. The concept of a homogeneous-coordinate representation of points in a three-dimensional euclidean space is useful in developing matrix transformations that include rotation, translation, scaling, and perspective transformation. In general, the representation of an N-component position vector by an $(N+1)$-component vector is called *homogeneous coordinate representation*. In a homogeneous coordinate representation, the transformation of an N-dimensional vector is performed in the $(N+1)$-dimensional space, and the physical N-dimensional vector is obtained by dividing the homogeneous coordinates by the $(N+1)$th coordinate, w. Thus, in a three-dimensional space, a position vector $\mathbf{p} = (p_x, p_y, p_z)^T$ is represented by an augmented vector $(wp_x, wp_y, wp_z, w)^T$ in the homogeneous coordinate representation. The physical coordinates are related to the homogeneous coordinates as follows:

$$p_x = \frac{wp_x}{w} \qquad p_y = \frac{wp_y}{w} \qquad p_z = \frac{wp_z}{w}$$

There is no unique homogeneous coordinates representation for a position vector in the three-dimensional space. For example, $\hat{\mathbf{p}}_1 = (w_1 p_x, w_1 p_y, w_1 p_z, w_1)^T$ and $\hat{\mathbf{p}}_2 = (w_2 p_x, w_2 p_y, w_2 p_z, w_2)^T$ are all homogeneous coordinates representing the same position vector $\mathbf{p} = (p_x, p_y, p_z)^T$. Thus, one can view the the fourth component of the homogeneous coordinates, w, as a scale factor. If this coordinate is unity ($w = 1$), then the transformed homogeneous coordinates of a position vector are the same as the physical coordinates of the vector. In robotics applications, this scale factor will always be equal to 1, although it is commonly used in computer graphics as a universal scale factor taking on any positive values.

The homogeneous transformation matrix is a 4×4 matrix which maps a position vector expressed in homogeneous coordinates from one coordinate system to another coordinate system. A homogeneous transformation matrix can be considered to consist of four submatrices:

$$\mathbf{T} = \begin{bmatrix} \mathbf{R}_{3\times3} & | & \mathbf{p}_{3\times1} \\ - & | & - \\ \mathbf{f}_{1\times3} & | & 1\times1 \end{bmatrix} = \begin{bmatrix} \text{rotation matrix} & | & \text{position vector} \\ - & | & - \\ \text{perspective transformation} & | & \text{scaling} \end{bmatrix} \tag{2.2-20}$$

The upper left 3×3 submatrix represents the rotation matrix; the upper right 3×1 submatrix represents the position vector of the origin of the rotated coordinate system with respect to the reference system; the lower left 1×3 submatrix represents perspective transformation; and the fourth diagonal element is the global scaling factor. The homogeneous transformation matrix can be used to explain the geometric relationship between the body-attached frame *OUVW* and the reference coordinate system *OXYZ*.

If a position vector $\mathbf{p}$ in a three-dimensional space is expressed in homogeneous coordinates [i.e., $\hat{\mathbf{p}} = (p_x, p_y, p_z, 1)^T$], then using the transformation matrix concept, a 3×3 rotation matrix can be extended to a 4×4 homogeneous transformation matrix $\mathbf{T}_{\text{rot}}$ for pure rotation operations. Thus, Eqs. (2.2-12) and (2.2-13), expressed as homogeneous rotation matrices, become

$$\mathbf{T}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T}_{y,\phi} = \begin{bmatrix} \cos\phi & 0 & \sin\phi & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\phi & 0 & \cos\phi & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\mathbf{T}_{z,\theta} = \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2-21}$$

These 4×4 rotation matrices are called the *basic homogeneous rotation matrices*.

The upper right 3×1 submatrix of the homogeneous transformation matrix has the effect of translating the *OUVW* coordinate system which has axes parallel to the reference coordinate system *OXYZ* but whose origin is at (dx, dy, dz) of the reference coordinate system:

$$\mathbf{T}_{\text{tran}} = \begin{bmatrix} 1 & 0 & 0 & dx \\ 0 & 1 & 0 & dy \\ 0 & 0 & 1 & dz \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2-22}$$

This 4×4 transformation matrix is called the *basic homogeneous translation matrix*.

The lower left 1×3 submatrix of the homogeneous transformation matrix represents perspective transformation, which is useful for computer vision and the calibration of camera models, as discussed in Chap. 7. In the present discussion, the elements of this submatrix are set to zero to indicate null perspective transformation.

The principal diagonal elements of a homogeneous transformation matrix produce local and global scaling. The first three diagonal elements produce local stretching or scaling, as in

$$\begin{bmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} ax \\ by \\ cz \\ 1 \end{bmatrix} \tag{2.2-23}$$

Thus, the coordinate values are stretched by the scalars a, b, and c, respectively. Note that the basic rotation matrices, $\mathbf{T}_{\text{rot}}$, do not produce any local scaling effect.

The fourth diagonal element produces global scaling as in

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & s \end{bmatrix} \begin{bmatrix} x \\ y \\ z \\ 1 \end{bmatrix} = \begin{bmatrix} x \\ y \\ z \\ s \end{bmatrix} \tag{2.2-24}$$

where $s > 0$. The physical cartesian coordinates of the vector are

$$p_x = \frac{x}{s} \qquad p_y = \frac{y}{s} \qquad p_z = \frac{z}{s} \qquad w = \frac{s}{s} = 1 \tag{2.2-25}$$

Therefore, the fourth diagonal element in the homogeneous transformation matrix has the effect of globally reducing the coordinates if $s > 1$ and of enlarging the coordinates if $0 < s < 1$.

In summary, a 4×4 homogeneous transformation matrix maps a vector expressed in homogeneous coordinates with respect to the *OUVW* coordinate system to the reference coordinate system *OXYZ*. That is, with $w = 1$,

$$\hat{\mathbf{p}}_{xyz} = \mathbf{T}\hat{\mathbf{p}}_{uvw} \tag{2.2-26a}$$

and

$$\mathbf{T} = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2-26b}$$

2.2.7 Geometric Interpretation of Homogeneous Transformation Matrices

In general, a homogeneous transformation matrix for a three-dimensional space can be represented as in Eq. (2.2-26*b*). Let us choose a point $\mathbf{p}$ fixed in the *OUVW* coordinate system and expressed in homogeneous coordinates as $(0, 0, 0, 1)^T$; that is, $\mathbf{p}_{uvw}$ is the origin of the *OUVW* coordinate system. Then the upper right 3×1 submatrix indicates the position of the origin of the *OUVW* frame with respect to the *OXYZ* reference coordinate frame. Next, let us choose the point $\mathbf{p}$ to be $(1, 0, 0\ 1)^T$; that is $\mathbf{p}_{uvw} \equiv \mathbf{i}_u$. Furthermore, we assume that the origins of both coordinate systems coincide at a point O. This has the effect of making the elements in the upper right 3×1 submatrix a null vector. Then the first column (or $\mathbf{n}$ vector) of the homogeneous transformation matrix represents the coordinates of the *OU* axis of *OUVW* with respect to the *OXYZ* coordinate system. Similarly, choosing $\mathbf{p}$ to be $(0, 1, 0, 1)^T$ and $(0, 0, 1, 1)^T$, one can identify that the second-column (or $\mathbf{s}$ vector) and third-column (or $\mathbf{a}$ vector) elements of the homogeneous transformation matrix represent the *OV* and *OW* axes, respectively, of the *OUVW* coordinate system with respect to the reference coordinate system. Thus, given a reference frame *OXYZ* and a homogeneous transformation matrix $\mathbf{T}$, the column vectors of the rotation submatrix represent the principal axes of the *OUVW* coordinate system with respect to the reference coordinate frame, and one can draw the orientation of all the principal axes of the *OUVW* coordinate frame with respect to the reference coordinate frame. The fourth-column vector of the homogeneous transformation matrix represents the position of the origin of the *OUVW* coordinate system with respect to the reference system. In other words, a homogeneous transformation matrix geometrically represents the *location* of a rotated coordinate system (position and orientation) with respect to a reference coordinate system.

Since the inverse of a rotation submatrix is equivalent to its transpose, the row vectors of a rotation submatrix represent the principal axes of the reference coordinate system with respect to the rotated coordinate system *OUVW*. However, the inverse of a homogeneous transformation matrix is *not* equivalent to its transpose. The position of the origin of the reference coordinate system with respect to the *OUVW* coordinate system can only be found after the inverse of the homogeneous transformation matrix is determined. In general, the inverse of a homogeneous transformation matrix can be found to be

$$\mathbf{T}^{-1} = \begin{bmatrix} n_x & n_y & n_z & -\mathbf{n}^T\mathbf{p} \\ s_x & s_y & s_z & -\mathbf{s}^T\mathbf{p} \\ a_x & a_y & a_z & -\mathbf{a}^T\mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} & \mathbf{R}^T_{3\times 3} & & -\mathbf{n}^T\mathbf{p} \\ & & & -\mathbf{s}^T\mathbf{p} \\ & & & -\mathbf{a}^T\mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2-27}$$

Thus, from Eq. (2.2-27), the column vectors of the inverse of a homogeneous transformation matrix represent the principal axes of the reference system with respect to the rotated coordinate system *OUVW*, and the upper right 3×1 submatrix represents the position of the origin of the reference frame with respect to the *OUVW* system. This geometric interpretation of the homogeneous transformation matrices is an important concept used frequently throughout this book.

2.2.8 Composite Homogeneous Transformation Matrix

The homogeneous rotation and translation matrices can be multiplied together to obtain a composite homogeneous transformation matrix (we shall call it the **T** matrix). However, since matrix multiplication is not commutative, careful attention must be paid to the order in which these matrices are multiplied. The following rules are useful for finding a composite homogeneous transformation matrix:

1. Initially both coordinate systems are coincident, hence the homogeneous transformation matrix is a 4×4 identity matrix, $\mathbf{I}_4$.
2. If the rotating coordinate system *OUVW* is rotating/translating about the principal axes of the *OXYZ* frame, then *premultiply* the previous (resultant) homogeneous transformation matrix with an appropriate basic homogeneous rotation/translation matrix.
3. If the rotating coordinate system *OUVW* is rotating/translating about its own principal axes, then *postmultiply* the previous (resultant) homogeneous transformation matrix with an appropriate basic homogeneous rotation/translation matrix.

Example: Two points $\mathbf{a}_{uvw} = (4, 3, 2)^T$ and $\mathbf{b}_{uvw} = (6, 2, 4)^T$ are to be translated a distance $+5$ units along the *OX* axis and -3 units along the *OZ* axis. Using the appropriate homogeneous transformation matrix, determine the new points $\mathbf{a}_{xyz}$ and $\mathbf{b}_{xyz}$.

SOLUTION:

$$\hat{\mathbf{a}}_{xyz} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4(1) + 1(5) \\ 3(1) + 1(0) \\ 2(1) + 1(-3) \\ 1(1) \end{bmatrix} = \begin{bmatrix} 9 \\ 3 \\ -1 \\ 1 \end{bmatrix}$$

$$\hat{\mathbf{b}}_{xyz} = \begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -3 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 6 \\ 2 \\ 4 \\ 1 \end{bmatrix} = \begin{bmatrix} 11 \\ 2 \\ 1 \\ 1 \end{bmatrix}$$

The translated points are $\mathbf{a}_{xyz} = (9, 3, -1)^T$ and $\mathbf{b}_{xyz} = (11, 2, 1)^T$. □

Example: A **T** matrix is to be determined that represents a rotation of α angle about the *OX* axis, followed by a translation of b units along the rotated *OV* axis.

SOLUTION: This problem can be tricky but illustrates some of the fundamental components of the **T** matrix. Two approaches will be utilized, an unorthodox approach which is illustrative, and the orthodox approach, which is simpler. After the rotation $\mathbf{T}_{x,\alpha}$, the rotated OV axis is (in terms of the unit vectors $\mathbf{i}_x$, $\mathbf{j}_y$, $\mathbf{k}_z$ of the reference system) $\mathbf{j}_v = \cos\alpha\mathbf{j}_y + \sin\alpha\mathbf{k}_z$; i.e., column 2 of Eq. (2.2-21). Thus, a translation along the rotated OV axis of b units is $b\,\mathbf{j}_v = b\cos\alpha\mathbf{j}_y + b\sin\alpha\mathbf{k}_z$. So the **T** matrix is

$$\mathbf{T} = \mathbf{T}_{v,b}\,\mathbf{T}_{x,\alpha} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b\cos\alpha \\ 0 & 0 & 1 & b\sin\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & b\cos\alpha \\ 0 & \sin\alpha & \cos\alpha & b\sin\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

In the orthodox approach, following the rules as stated earlier, one should realize that since the $\mathbf{T}_{x,\alpha}$ matrix will rotate the OY axis to the OV axis, then translation along the OV axis will accomplish the same goal, that is,

$$\mathbf{T} = \mathbf{T}_{x,\alpha}\,\mathbf{T}_{v,b} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & b \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & b\cos\alpha \\ 0 & \sin\alpha & \cos\alpha & b\sin\alpha \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ □

Example: Find a homogeneous transformation matrix **T** that represents a rotation of α angle about the OX axis, followed by a translation of a units along the OX axis, followed by a translation of d units along the OZ axis, followed by a rotation of θ angle about the OZ axis.

SOLUTION:

$$\mathbf{T} = \mathbf{T}_{z,\theta}\,\mathbf{T}_{z,d}\,\mathbf{T}_{x,a}\,\mathbf{T}_{x,\alpha}$$

$$= \begin{bmatrix} \cos\theta & -\sin\theta & 0 & 0 \\ \sin\theta & \cos\theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & a \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha & 0 \\ 0 & \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\theta & -\cos\alpha\sin\theta & \sin\alpha\sin\theta & a\cos\theta \\ \sin\theta & \cos\alpha\cos\theta & -\sin\alpha\cos\theta & a\sin\theta \\ 0 & \sin\alpha & \cos\alpha & d \\ 0 & 0 & 0 & 1 \end{bmatrix}$$ □

We have identified two coordinate systems, the fixed reference coordinate frame *OXYZ* and the moving (translating and rotating) coordinate frame *OUVW*. To describe the spatial displacement relationship between these two coordinate systems, a 4×4 homogeneous transformation matrix is used. Homogeneous transformation matrices have the combined effect of rotation, translation, perspective, and global scaling when operating on position vectors expressed in homogeneous coordinates.

If these two coordinate systems are assigned to each link of a robot arm, say link $i-1$ and link i, respectively, then the link $i-1$ coordinate system is the reference coordinate system and the link i coordinate system is the moving coordinate system, when joint i is activated. Using the **T** matrix, we can specify a point $\mathbf{p}_i$ at rest in link i and expressed in the link i (or *OUVW*) coordinate system in terms of the link $i-1$ (or *OXYZ*) coordinate system as

$$\mathbf{p}_{i-1} = \mathbf{T}\,\mathbf{p}_i \tag{2.2-28}$$

where

$\mathbf{T}$ = 4×4 homogeneous transformation matrix relating the two coordinate systems

$\mathbf{p}_i$ = 4×1 augmented position vector $(x_i, y_i, z_i, 1)^T$ representing a point in the link i coordinate system expressed in homogeneous coordinates

$\mathbf{p}_{i-1}$ = is the 4×1 augmented position vector $(x_{i-1}, y_{i-1}, z_{i-1}, 1)^T$ representing the same point $\mathbf{p}_i$ in terms of the link $i-1$ coordinate system

2.2.9 Links, Joints, and Their Parameters

A mechanical manipulator consists of a sequence of rigid bodies, called links, connected by either revolute or prismatic joints (see Fig. 2.8). Each joint-link pair constitutes 1 degree of freedom. Hence, for an N degree of freedom manipulator, there are N joint-link pairs with link 0 (not considered part of the robot) attached to a supporting base where an inertial coordinate frame is usually established for this dynamic system, and the last link is attached with a tool. The joints and links

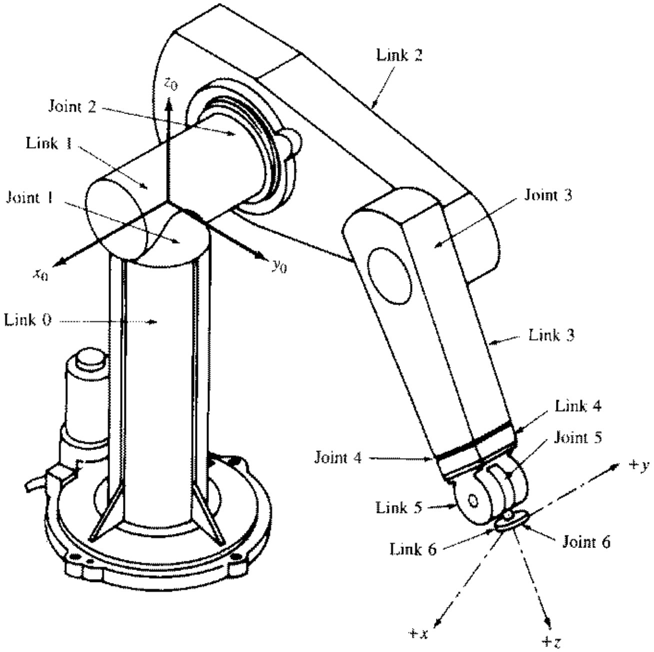


Figure 2.8 A PUMA robot arm illustrating joints and links.

are numbered outwardly from the base; thus, joint 1 is the point of connection between link 1 and the supporting base. Each link is connected to, at most, two others so that no closed loops are formed.

In general, two links are connected by a *lower-pair* joint which has two surfaces sliding over one another while remaining in contact. Only six different lower-pair joints are possible: revolute (rotary), prismatic (sliding), cylindrical, spherical, screw, and planar (see Fig. 2.9). Of these, only rotary and prismatic joints are common in manipulators.

A joint axis (for joint i) is established at the connection of two links (see Fig. 2.10). This joint axis will have two normals connected to it, one for each of the links. The relative position of two such connected links (link $i - 1$ and link i) is given by d_i which is the distance measured along the joint axis between the normals. The joint angle θ_i between the normals is measured in a plane normal to the joint axis. Hence, d_i and θ_i may be called the *distance* and the *angle* between the adjacent links, respectively. They determine the relative position of neighboring links.

A link i ($i = 1, \ldots, 6$) is connected to, at most, two other links (e.g., link $i - 1$ and link $i + 1$); thus, two joint axes are established at both ends of the connection. The significance of links, from a kinematic perspective, is that they maintain a fixed configuration between their joints which can be characterized by two

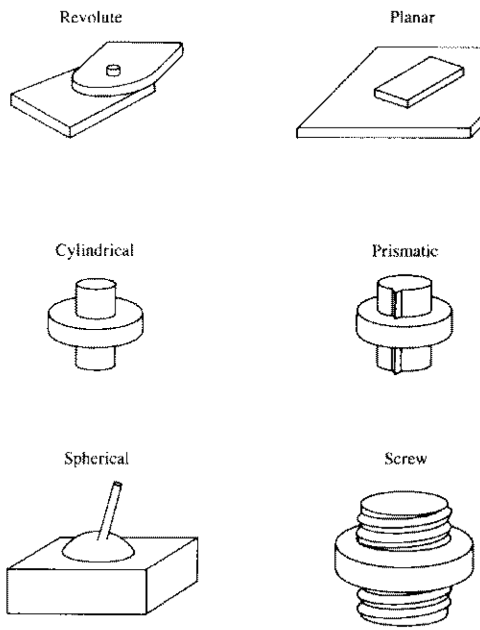


Figure 2.9 The lower pair.

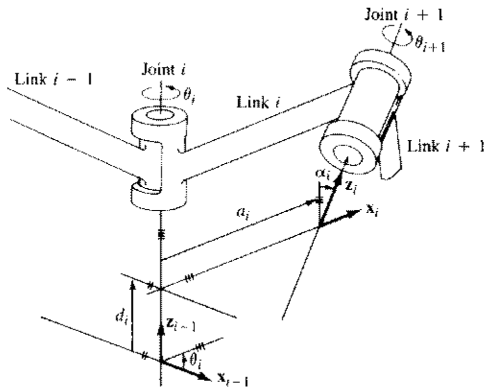


Figure 2.10 Link coordinate system and its parameters.

parameters: a_i and α_i. The parameter a_i is the shortest distance measured along the common normal between the joint axes (i.e., the $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$ axes for joint i and joint $i+1$, respectively), and α_i is the angle between the joint axes measured in a plane perpendicular to a_i. Thus, a_i and α_i may be called the *length* and the *twist angle* of the link i, respectively. They determine the structure of link i.

In summary, four parameters, a_i, α_i, d_i, and θ_i, are associated with each link of a manipulator. If a sign convention for each of these parameters has been established, then these parameters constitute a sufficient set to completely determine the kinematic configuration of each link of a robot arm. Note that these four parameters come in pairs: the link parameters (a_i, α_i) which determine the structure of the link and the joint parameters (d_i, θ_i) which determine the relative position of neighboring links.

2.2.10 The Denavit-Hartenberg Representation

To describe the translational and rotational relationships between adjacent links, Denavit and Hartenberg [1955] proposed a matrix method of systematically establishing a coordinate system (body-attached frame) to each link of an articulated chain. The Denavit-Hartenberg (D-H) representation results in a 4×4 homogeneous transformation matrix representing each link's coordinate system at the joint with respect to the previous link's coordinate system. Thus, through sequential transformations, the end-effector expressed in the "hand coordinates" can be transformed and expressed in the "base coordinates" which make up the inertial frame of this dynamic system.

An orthonormal cartesian coordinate system $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$† can be established for each link at its joint axis, where $i = 1, 2, \ldots, n$ (n = number of degrees of freedom) plus the base coordinate frame. Since a rotary joint has only 1 degree of freedom, each $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ coordinate frame of a robot arm corresponds to joint $i + 1$ and is fixed in link i. When the joint actuator activates joint i, link i will move with respect to link $i - 1$. Since the ith coordinate system is fixed in link i, it moves together with the link i. Thus, the nth coordinate frame moves with the hand (link n). The base coordinates are defined as the 0th coordinate frame ($\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{z}_0$) which is also the inertial coordinate frame of the robot arm. Thus, for a six-axis PUMA-like robot arm, we have seven coordinate frames, namely, $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0), (\mathbf{x}_1, \mathbf{y}_1, \mathbf{z}_1), \ldots, (\mathbf{x}_6, \mathbf{y}_6, \mathbf{z}_6)$.

Every coordinate frame is determined and established on the basis of three rules:

1. The $\mathbf{z}_{i-1}$ axis lies along the axis of motion of the ith joint.
2. The $\mathbf{x}_i$ axis is normal to the $\mathbf{z}_{i-1}$ axis, and pointing away from it.
3. The $\mathbf{y}_i$ axis completes the right-handed coordinate system as required.

By these rules, one is free to choose the location of coordinate frame 0 anywhere in the supporting base, as long as the $\mathbf{z}_0$ axis lies along the axis of motion of the first joint. The last coordinate frame (nth frame) can be placed anywhere in the hand, as long as the $\mathbf{x}_n$ axis is normal to the $\mathbf{z}_{n-1}$ axis.

The D-H representation of a rigid link depends on four geometric parameters associated with each link. These four parameters completely describe any revolute

† $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ actually represent the unit vectors along the principal axes of the coordinate frame i, respectively, but are used here to denote the coordinate frame i.

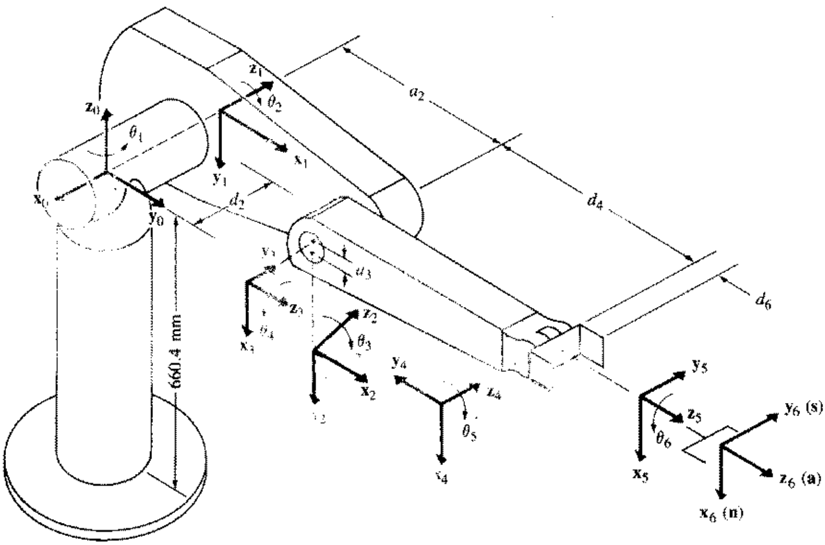
or prismatic joint. Referring to Fig. 2.10, these four parameters are defined as follows:

θ_i is the joint angle from the $\mathbf{x}_{i-1}$ axis to the $\mathbf{x}_i$ axis about the $\mathbf{z}_{i-1}$ axis (using the right-hand rule).

d_i is the distance from the origin of the $(i-1)$th coordinate frame to the intersection of the $\mathbf{z}_{i-1}$ axis with the $\mathbf{x}_i$ axis along the $\mathbf{z}_{i-1}$ axis.

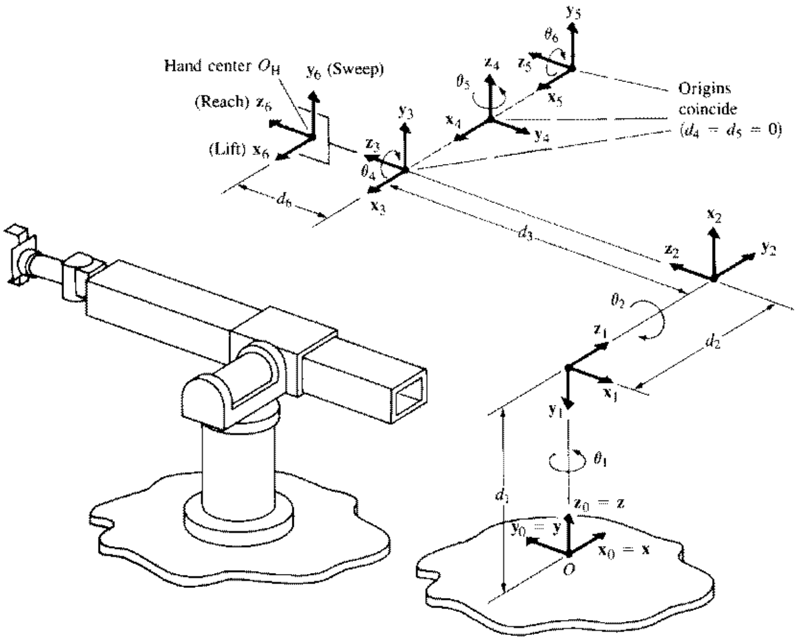
a_i is the offset distance from the intersection of the $\mathbf{z}_{i-1}$ axis with the $\mathbf{x}_i$ axis to the origin of the ith frame along the $\mathbf{x}_i$ axis (or the shortest distance between the $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$ axes).

α_i is the offset angle from the $\mathbf{z}_{i-1}$ axis to the $\mathbf{z}_i$ axis about the $\mathbf{x}_i$ axis (using the right-hand rule).



PUMA robot arm link coordinate parameters					
Joint i	θ_i	α_i	a_i	d_i	Joint range
1	90	−90	0	0	−160 to +160
2	0	0	431.8 mm	149.09 mm	−225 to 45
3	90	90	−20.32 mm	0	−45 to 225
4	0	−90	0	433.07 mm	−110 to 170
5	0	90	0	0	−100 to 100
6	0	0	0	56.25 mm	−266 to 266

Figure 2.11 Establishing link coordinate systems for a PUMA robot.



Stanford robot arm link coordinate parameters				
Joint i	θ_i	α_i	a_i	d_i
1	$\theta_1 = -90$	−90	0	d_1
2	$\theta_2 = -90$	90	0	d_2
3	−90	0	0	d_3
4	$\theta_4 = 0$	−90	0	0
5	$\theta_5 = 0$	90	0	0
6	$\theta_6 = 0$	0	0	d_6

Figure 2.12 Establishing link coordinate systems for a Stanford robot.

For a rotary joint, d_i, a_i, and α_i are the joint parameters and remain constant for a robot, while θ_i is the joint variable that changes when link i moves (or rotates) with respect to link $i - 1$. For a prismatic joint, θ_i, a_i, and α_i are the joint parameters and remain constant for a robot, while d_i is the joint variable. For the remainder of this book, *joint variable* refers to θ_i (or d_i), that is, the varying quantity, and *joint parameters* refer to the remaining three geometric constant values (d_i, a_i, α_i) for a rotary joint, or $(\theta_i, a_i, \alpha_i)$ for a prismatic joint.

With the above three basic rules for establishing an orthonormal coordinate system for each link and the geometric interpretation of the joint and link parameters, a procedure for establishing *consistent* orthonormal coordinate systems for a robot is outlined in Algorithm 2.1. Examples of applying this algorithm to a six-axis PUMA-like robot arm and a Stanford arm are given in Figs. 2.11 and 2.12, respectively.

Algorithm 2.1: Link Coordinate System Assignment. Given an n degree of freedom robot arm, this algorithm assigns an orthonormal coordinate system to each link of the robot arm according to arm configurations similar to those of human arm geometry. The labeling of the coordinate systems begins from the supporting base to the end-effector of the robot arm. The relations between adjacent links can be represented by a 4×4 homogeneous transformation matrix. The significance of this assignment is that it will aid the development of a consistent procedure for deriving the joint solution as discussed in the later sections. (Note that the assignment of coordinate systems is not unique.)

D1. *Establish the base coordinate system.* Establish a right-handed orthonormal coordinate system $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ at the supporting base with the $\mathbf{z}_0$ axis lying along the axis of motion of joint 1 and pointing toward the shoulder of the robot arm. The $\mathbf{x}_0$ and $\mathbf{y}_0$ axes can be conveniently established and are normal to the $\mathbf{z}_0$ axis.

D2. *Initialize and loop.* For each i, $i = 1, \ldots, n-1$, perform steps D3 to D6.

D3. *Establish joint axis.* Align the $\mathbf{z}_i$ with the axis of motion (rotary or sliding) of joint $i+1$. For robots having left-right arm configurations, the $\mathbf{z}_1$ and $\mathbf{z}_2$ axes are pointing away from the shoulder and the "trunk" of the robot arm.

D4. *Establish the origin of the* ith *coordinate system.* Locate the origin of the ith coordinate system at the intersection of the $\mathbf{z}_i$ and $\mathbf{z}_{i-1}$ axes or at the intersection of common normal between the $\mathbf{z}_i$ and $\mathbf{z}_{i-1}$ axes and the $\mathbf{z}_i$ axis.

D5. *Establish* $\mathbf{x}_i$ *axis.* Establish $\mathbf{x}_i = \pm (\mathbf{z}_{i-1} \times \mathbf{z}_i)/\|\mathbf{z}_{i-1} \times \mathbf{z}_i\|$ or along the common normal between the $\mathbf{z}_{i-1}$ and $\mathbf{z}_i$ axes when they are parallel.

D6. *Establish* $\mathbf{y}_i$ *axis.* Assign $\mathbf{y}_i = +(\mathbf{z}_i \times \mathbf{x}_i)/\|\mathbf{z}_i \times \mathbf{x}_i\|$ to complete the right-handed coordinate system. (Extend the $\mathbf{z}_i$ and the $\mathbf{x}_i$ axes if necessary for steps D9 to D12).

D7. *Establish the hand coordinate system.* Usually the nth joint is a rotary joint. Establish $\mathbf{z}_n$ along the direction of $\mathbf{z}_{n-1}$ axis and pointing away from the robot. Establish $\mathbf{x}_n$ such that it is normal to both $\mathbf{z}_{n-1}$ and $\mathbf{z}_n$ axes. Assign $\mathbf{y}_n$ to complete the right-handed coordinate system. (See Sec. 2.2.11 for more detail.)

D8. *Find joint and link parameters.* For each i, $i = 1, \ldots, n$, perform steps D9 to D12.

D9. *Find* d_i. d_i is the distance from the origin of the $(i-1)$th coordinate system to the intersection of the $\mathbf{z}_{i-1}$ axis and the $\mathbf{x}_i$ axis along the $\mathbf{z}_{i-1}$ axis. It is the joint variable if joint i is prismatic.

D10. *Find* a_i. a_i is the distance from the intersection of the $\mathbf{z}_{i-1}$ axis and the $\mathbf{x}_i$ axis to the origin of the ith coordinate system along the $\mathbf{x}_i$ axis.

D11. *Find* θ_i. θ_i is the angle of rotation from the $\mathbf{x}_{i-1}$ axis to the $\mathbf{x}_i$ axis about the $\mathbf{z}_{i-1}$ axis. It is the joint variable if joint i is rotary.

D12. *Find* α_i. α_i is the angle of rotation from the $\mathbf{z}_{i-1}$ axis to the $\mathbf{z}_i$ axis about the $\mathbf{x}_i$ axis.

Once the D-H coordinate system has been established for each link, a homogeneous transformation matrix can easily be developed relating the ith coordinate frame to the $(i-1)$th coordinate frame. Looking at Fig. 2.10, it is obvious that a point $\mathbf{r}_i$ expressed in the ith coordinate system may be expressed in the $(i-1)$th coordinate system as $\mathbf{r}_{i-1}$ by performing the following successive transformations:

1. Rotate about the $\mathbf{z}_{i-1}$ axis an angle of θ_i to align the $\mathbf{x}_{i-1}$ axis with the $\mathbf{x}_i$ axis ($\mathbf{x}_{i-1}$ axis is parallel to $\mathbf{x}_i$ and pointing in the same direction).
2. Translate along the $\mathbf{z}_{i-1}$ axis a distance of d_i to bring the $\mathbf{x}_{i-1}$ and $\mathbf{x}_i$ axes into coincidence.
3. Translate along the $\mathbf{x}_i$ axis a distance of a_i to bring the two origins as well as the $\mathbf{x}$ axis into coincidence.
4. Rotate about the $\mathbf{x}_i$ axis an angle of α_i to bring the two coordinate systems into coincidence.

Each of these four operations can be expressed by a basic homogeneous rotation-translation matrix and the product of these four basic homogeneous transformation matrices yields a composite homogeneous transformation matrix, ${}^{i-1}\mathbf{A}_i$, known as the D-H transformation matrix for adjacent coordinate frames, i and $i-1$. Thus,

$$
\begin{aligned}
{}^{i-1}\mathbf{A}_i &= \mathbf{T}_{z,d}\,\mathbf{T}_{z,\theta}\,\mathbf{T}_{x,a}\,\mathbf{T}_{x,\alpha} \\
&= \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} \cos\theta_i & -\sin\theta_i & 0 & 0 \\ \sin\theta_i & \cos\theta_i & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & a_i \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\alpha_i & -\sin\alpha_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \\
&= \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\end{aligned}
\tag{2.2-29}
$$

Using Eq. (2.2-27), the inverse of this transformation can be found to be

$$
[{}^{i-1}\mathbf{A}_i]^{-1} = {}^{i}\mathbf{A}_{i-1} = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & -a_i \\ -\cos\alpha_i\sin\theta_i & \cos\alpha_i\cos\theta_i & \sin\alpha_i & -d_i\sin\alpha_i \\ \sin\alpha_i\sin\theta_i & -\sin\alpha_i\cos\theta_i & \cos\alpha_i & -d_i\cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
\tag{2.2-30}
$$

where α_i, a_i, d_i are constants while θ_i is the joint variable for a revolute joint.

For a prismatic joint, the joint variable is d_i, while α_i, a_i, and θ_i are constants. In this case, ${}^{i-1}\mathbf{A}_i$ becomes

$$ {}^{i-1}\mathbf{A}_i = \mathbf{T}_{z,\theta}\,\mathbf{T}_{z,d}\,\mathbf{T}_{x,\alpha} = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\,\sin\theta_i & \sin\alpha_i\,\sin\theta_i & 0 \\ \sin\theta_i & \cos\alpha_i\,\cos\theta_i & -\sin\alpha_i\,\cos\theta_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2-31} $$

and its inverse is

$$ [{}^{i-1}\mathbf{A}_i]^{-1} = {}^{i}\mathbf{A}_{i-1} = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 & 0 \\ -\cos\alpha_i\sin\theta_i & \cos\alpha_i\,\cos\theta_i & \sin\alpha_i & -d_i\,\sin\alpha_i \\ \cos\alpha_i\sin\theta_i & -\sin\alpha_i\cos\theta_i & \cos\alpha_i & -d_i\cos\alpha_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2-32} $$

Using the ${}^{i-1}\mathbf{A}_i$ matrix, one can relate a point $\mathbf{p}_i$ at rest in link i, and expressed in homogeneous coordinates with respect to coordinate system i, to the coordinate system $i-1$ established at link $i-1$ by

$$ \mathbf{p}_{i-1} = {}^{i-1}\mathbf{A}_i\,\mathbf{p}_i \tag{2.2-33} $$

where $\mathbf{p}_{i-1} = (x_{i-1}, y_{i-1}, z_{i-1}, 1)^T$ and $\mathbf{p}_i = (x_i, y_i, z_i, 1)^T$.

The six ${}^{i-1}\mathbf{A}_i$ transformation matrices for the six-axis PUMA robot arm have been found on the basis of the coordinate systems established in Fig. 2.11. These ${}^{i-1}\mathbf{A}_i$ matrices are listed in Fig. 2.13.

2.2.11 Kinematic Equations for Manipulators

The homogeneous matrix ${}^0\mathbf{T}_i$ which specifies the location of the ith coordinate frame with respect to the base coordinate system is the chain product of successive coordinate transformation matrices of ${}^{i-1}\mathbf{A}_i$, and is expressed as

$$ {}^0\mathbf{T}_i = {}^0\mathbf{A}_1\,{}^1\mathbf{A}_2 \cdots {}^{i-1}\mathbf{A}_i = \prod_{j=1}^{i} {}^{j-1}\mathbf{A}_j \qquad \text{for } i = 1, 2, \ldots, n $$

$$ = \begin{bmatrix} \mathbf{x}_i & \mathbf{y}_i & \mathbf{z}_i & \mathbf{p}_i \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{R}_i & {}^0\mathbf{p}_i \\ 0 & 1 \end{bmatrix} \tag{2.2-34} $$

where

$[\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i]$ = orientation matrix of the ith coordinate system established at link i with respect to the base coordinate system. It is the upper left 3×3 partitioned matrix of ${}^0\mathbf{T}_i$.

$\mathbf{p}_i$ = position vector which points from the origin of the base coordinate system to the origin of the ith coordinate system. It is the upper right 3×1 partitioned matrix of ${}^0\mathbf{T}_i$.

$$
{}^{i-1}\mathbf{A}_i = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{0}\mathbf{A}_1 = \begin{bmatrix} C_1 & 0 & -S_1 & 0 \\ S_1 & 0 & C_1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{1}\mathbf{A}_2 = \begin{bmatrix} C_2 & -S_2 & 0 & a_2 C_2 \\ S_2 & C_2 & 0 & a_2 S_2 \\ 0 & 0 & 1 & d_2 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{2}\mathbf{A}_3 = \begin{bmatrix} C_3 & 0 & S_3 & a_3 C_3 \\ S_3 & 0 & -C_3 & a_3 S_3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{3}\mathbf{A}_4 = \begin{bmatrix} C_4 & 0 & -S_4 & 0 \\ S_4 & 0 & C_4 & 0 \\ 0 & -1 & 0 & d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^{4}\mathbf{A}_5 = \begin{bmatrix} C_5 & 0 & S_5 & 0 \\ S_5 & 0 & -C_5 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^{5}\mathbf{A}_6 = \begin{bmatrix} C_6 & -S_6 & 0 & 0 \\ S_6 & C_6 & 0 & 0 \\ 0 & 0 & 1 & d_6 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\mathbf{T}_1 \equiv {}^{0}\mathbf{A}_1 \, {}^{1}\mathbf{A}_2 \, {}^{2}\mathbf{A}_3 = \begin{bmatrix} C_1 C_{23} & -S_1 & C_1 S_{23} & a_2 C_1 C_2 + a_3 C_1 C_{23} - d_2 S_1 \\ S_1 C_{23} & C_1 & S_1 S_{23} & a_2 S_1 C_2 + a_3 S_1 C_{23} + d_2 C_1 \\ -S_{23} & 0 & C_{23} & -a_2 S_2 - a_3 S_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
\mathbf{T}_2 \equiv {}^{3}\mathbf{A}_4 \, {}^{4}\mathbf{A}_5 \, {}^{5}\mathbf{A}_6 = \begin{bmatrix} C_4 C_5 C_6 - S_4 S_6 & -C_4 C_5 S_6 - S_4 C_6 & C_4 S_5 & d_6 C_4 S_5 \\ S_4 C_5 C_6 + C_4 S_6 & -S_4 C_5 S_6 + C_4 C_6 & S_4 S_5 & d_6 S_4 S_5 \\ -S_5 C_6 & S_5 S_6 & C_5 & d_6 C_5 + d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where $C_i \equiv \cos\theta_i$; $S_i \equiv \sin\theta_i$; $C_{ij} \equiv \cos(\theta_i + \theta_j)$; $S_{ij} \equiv \sin(\theta_i + \theta_j)$.

Figure 2.13 PUMA link coordinate transformation matrices.

Specifically, for $i = 6$, we obtain the $\mathbf{T}$ matrix, $\mathbf{T} = {}^{0}\mathbf{A}_6$, which specifies the position and orientation of the endpoint of the manipulator with respect to the base coordinate system. This $\mathbf{T}$ matrix is used so frequently in robot arm kinematics that it is called the "arm matrix." Consider the $\mathbf{T}$ matrix to be of the form

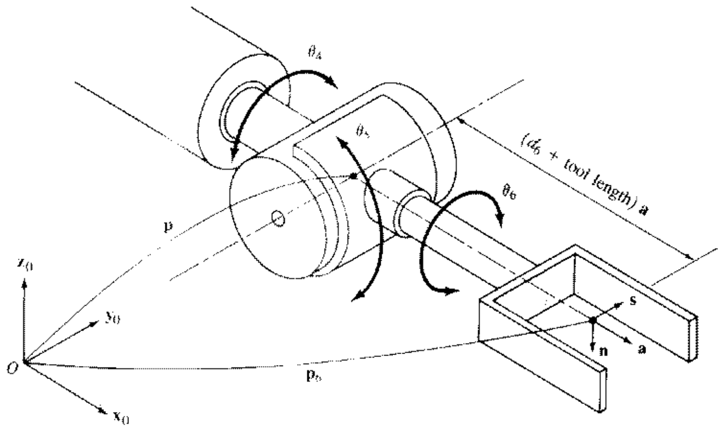


Figure 2.14 Hand coordinate system and $\{\mathbf{n}, \mathbf{s}, \mathbf{a}\}$.

$$\mathbf{T} = \begin{bmatrix} \mathbf{x}_6 & \mathbf{y}_6 & \mathbf{z}_6 & \mathbf{p}_6 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} {}^0\mathbf{R}_6 & {}^0\mathbf{p}_6 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2.2\text{-}35)$$

where (see Fig. 2.14)

$\mathbf{n}$ = normal vector of the hand. Assuming a parallel-jaw hand, it is orthogonal to the fingers of the robot arm.

$\mathbf{s}$ = sliding vector of the hand. It is pointing in the direction of the finger motion as the gripper opens and closes.

$\mathbf{a}$ = approach vector of the hand. It is pointing in the direction normal to the palm of the hand (i.e., normal to the tool mounting plate of the arm).

$\mathbf{p}$ = position vector of the hand. It points from the origin of the base coordinate system to the origin of the hand coordinate system, which is usually located at the center point of the fully closed fingers.

If the manipulator is related to a reference coordinate frame by a transformation $\mathbf{B}$ and has a tool attached to its last joint's mounting plate described by $\mathbf{H}$, then the endpoint of the tool can be related to the reference coordinate frame by multiplying the matrices $\mathbf{B}$, ${}^0\mathbf{T}_6$, and $\mathbf{H}$ together as

$${}^{\text{ref}}\mathbf{T}_{\text{tool}} = \mathbf{B}\,{}^0\mathbf{T}_6\,\mathbf{H} \qquad (2.2\text{-}36)$$

Note that $\mathbf{H} \equiv {}^6\mathbf{A}_{\text{tool}}$ and $\mathbf{B} \equiv {}^{\text{ref}}\mathbf{A}_0$.

The direct kinematics solution of a six-link manipulator is, therefore, simply a matter of calculating $\mathbf{T} = {}^0\mathbf{A}_6$ by chain multiplying the six ${}^{i-1}\mathbf{A}_i$ matrices and evaluating each element in the $\mathbf{T}$ matrix. Note that the direct kinematics solution yields a unique $\mathbf{T}$ matrix for a given $\mathbf{q} = (q_1, q_2, \ldots, q_6)^T$ and a given set of coordinate systems, where $q_i = \theta_i$ for a rotary joint and $q_i = d_i$ for a prismatic joint. The only constraints are the physical bounds of θ_i for each joint of the robot arm. The table in Fig. 2.11 lists the joint constraints of a PUMA 560 series robot based on the coordinate system assigned in Fig. 2.11.

Having obtained all the coordinate transformation matrices ${}^{i-1}\mathbf{A}_i$ for a robot arm, the next task is to find an efficient method to compute $\mathbf{T}$ on a general purpose digital computer. The most efficient method is by multiplying all six ${}^{i-1}\mathbf{A}_i$ matrices together manually and evaluating the elements of $\mathbf{T}$ matrix out explicitly on a computer program. The disadvantages of this method are (1) it is laborious to multiply all six ${}^{i-1}\mathbf{A}_i$ matrices together, and (2) the arm matrix is applicable only to a particular robot for a specific set of coordinate systems (it is not flexible enough). On the other extreme, one can input all six ${}^{i-1}\mathbf{A}_i$ matrices and let the computer do the multiplication. This method is very flexible but at the expense of computation time as the fourth row of ${}^{i-1}\mathbf{A}_i$ consists mostly of zero elements.

A method that has both fast computation and flexibility is to "hand" multiply the first three ${}^{i-1}\mathbf{A}_i$ matrices together to form $\mathbf{T}_1 = {}^0\mathbf{A}_1\,{}^1\mathbf{A}_2\,{}^2\mathbf{A}_3$ and also the last three ${}^{i-1}\mathbf{A}_i$ matrices together to form $\mathbf{T}_2 = {}^3\mathbf{A}_4\,{}^4\mathbf{A}_5\,{}^5\mathbf{A}_6$, which is a fairly straightforward task. Then, we express the elements of $\mathbf{T}_1$ and $\mathbf{T}_2$ out in a computer program explicitly and let the computer multiply them together to form the resultant arm matrix $\mathbf{T} = \mathbf{T}_1\mathbf{T}_2$.

For a PUMA 560 series robot, $\mathbf{T}_1$ is found from Fig. 2.13 to be

$$\mathbf{T}_1 = {}^0\mathbf{A}_3 = {}^0\mathbf{A}_1\,{}^1\mathbf{A}_2\,{}^2\mathbf{A}_3$$

$$= \begin{bmatrix} C_1C_{23} & -S_1 & C_1S_{23} & a_2C_1C_2 + a_3C_1C_{23} - d_2S_1 \\ S_1C_{23} & C_1 & S_1S_{23} & a_2S_1C_2 + a_3S_1C_{23} + d_2C_1 \\ -S_{23} & 0 & C_{23} & -a_2S_2 - a_3S_{23} \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2-37}$$

and the $\mathbf{T}_2$ matrix is found to be

$$\mathbf{T}_2 = {}^3\mathbf{A}_6 = {}^3\mathbf{A}_4\,{}^4\mathbf{A}_5\,{}^5\mathbf{A}_6$$

$$= \begin{bmatrix} C_4C_5C_6 - S_4S_6 & -C_4C_5S_6 - S_4C_6 & C_4S_5 & d_6C_4S_5 \\ S_4C_5C_6 + C_4S_6 & -S_4C_5S_6 + C_4C_6 & S_4S_5 & d_6S_4S_5 \\ -S_5C_6 & S_5S_6 & C_5 & d_6C_5 + d_4 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2-38}$$

where $C_{ij} \equiv \cos(\theta_i + \theta_j)$ and $S_{ij} \equiv \sin(\theta_i + \theta_j)$.

The arm matrix T for the PUMA robot arm shown in Fig. 2.11 is found to be

$$\mathbf{T} = \mathbf{T}_1\,\mathbf{T}_2 = {}^0\mathbf{A}_1\,{}^1\mathbf{A}_2\,{}^2\mathbf{A}_3\,{}^3\mathbf{A}_4\,{}^4\mathbf{A}_5\,{}^5\mathbf{A}_6 = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2-39}$$

where

$$n_x = C_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] - S_1(S_4C_5C_6 + C_4S_6)$$

$$n_y = S_1[C_{23}(C_4C_5C_6 - S_4S_6) - S_{23}S_5C_6] + C_1(S_4C_5C_6 + C_4S_6) \tag{2.2-40}$$

$$n_z = -S_{23}[C_4C_5C_6 - S_4S_6] - C_{23}S_5C_6$$

$$s_x = C_1[-C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6] - S_1(-S_4C_5S_6 + C_4C_6)$$

$$s_y = S_1[-C_{23}(C_4C_5S_6 + S_4C_6) + S_{23}S_5S_6] + C_1(-S_4C_5S_6 + C_4C_6)$$

$$s_z = S_{23}(C_4C_5S_6 + S_4C_6) + C_{23}S_5S_6 \tag{2.2-41}$$

$$a_x = C_1(C_{23}C_4S_5 + S_{23}C_5) - S_1S_4S_5$$

$$a_y = S_1(C_{23}C_4S_5 + S_{23}C_5) + C_1S_4S_5 \tag{2.2-42}$$

$$a_z = -S_{23}C_4S_5 + C_{23}C_5$$

$$p_x = C_1[d_6(C_{23}C_4S_5 + S_{23}C_5) + S_{23}d_4 + a_3C_{23} + a_2C_2] - S_1(d_6S_4S_5 + d_2)$$

$$p_y = S_1[d_6(C_{23}C_4S_5 + S_{23}C_5) + S_{23}d_4 + a_3C_{23} + a_2C_2] + C_1(d_6S_4S_5 + d_2)$$

$$p_z = d_6(C_{23}C_5 - S_{23}C_4S_5) + C_{23}d_4 - a_3S_{23} - a_2S_2 \tag{2.2-43}$$

As a check, if $\theta_1 = 90°, \theta_2 = 0°, \theta_3 = 90°, \theta_4 = 0°, \theta_5 = 0°, \theta_6 = 0°$, then the $\mathbf{T}$ matrix is

$$\mathbf{T} = \begin{bmatrix} 0 & -1 & 0 & -149.09 \\ 0 & 0 & 1 & 921.12 \\ -1 & 0 & 0 & 20.32 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

which agrees with the coordinate systems established in Fig. 2.11.

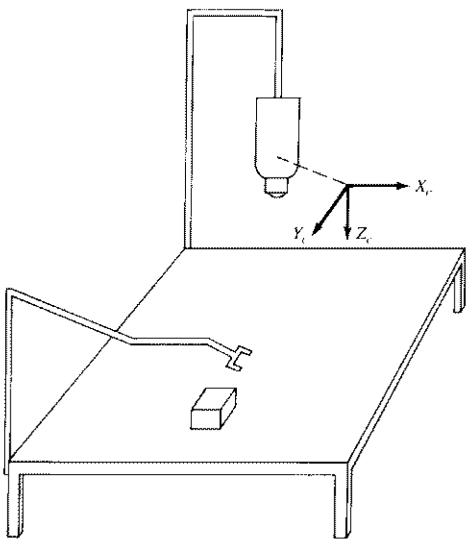
From Eqs. (2.2-40) through (2.2-43), the arm matrix $\mathbf{T}$ requires 12 transcendental function calls, 40 multiplications and 20 additions if we only compute the upper right 3×3 submatrix of $\mathbf{T}$ and the normal vector $\mathbf{n}$ is found from the cross-product of $(\mathbf{n} = \mathbf{s} \times \mathbf{a})$. Furthermore, if we combine d_6 with the tool length of the terminal device, then $d_6 = 0$ and the new tool length will be increased by d_6 unit. This reduces the computation to 12 transcendental function calls, 35 multiplications, and 16 additions.

Example: A robot work station has been set up with a TV camera (see the figure). The camera can see the origin of the base coordinate system where a six-joint robot is attached. It can also see the center of an object (assumed to be a cube) to be manipulated by the robot. If a local coordinate system has been established at the center of the cube, this object as seen by the camera can be represented by a homogeneous transformation matrix $\mathbf{T}_1$. If the origin of the base coordinate system as seen by the camera can also be expressed by a homogeneous transformation matrix $\mathbf{T}_2$ and

$$\mathbf{T}_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & -1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad \mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & -1 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(*a*) What is the position of the center of the cube with respect to the base coordinate system?

(*b*) Assume that the cube is within the arm's reach. What is the orientation matrix $[\mathbf{n}, \mathbf{s}, \mathbf{a}]$ if you want the gripper (or finger) of the hand to be aligned with the $\mathbf{y}$ axis of the object and at the same time pick up the object from the top?



SOLUTION:

$$^{\text{camera}}\mathbf{T}_{\text{cube}} \equiv \mathbf{T}_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & -1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

and

$$^{\text{camera}}\mathbf{T}_{\text{base}} \equiv \mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & -1 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

To find $^{\text{base}}\mathbf{T}_{\text{cube}}$, we use the "chain product" rule:

$$^{\text{base}}\mathbf{T}_{\text{cube}} = {}^{\text{base}}\mathbf{T}_{\text{camera}} \, {}^{\text{camera}}\mathbf{T}_{\text{cube}} = (\mathbf{T}_2)^{-1}\,\mathbf{T}_1$$

Using Eq. (2.2-27) to invert the $\mathbf{T}_2$ matrix, we obtain the resultant transformation matrix:

$$^{\text{base}}\mathbf{T}_{\text{cube}} = \begin{bmatrix} 1 & 0 & 0 & 10 \\ 0 & -1 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & -1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & 1 & 0 & 11 \\ -1 & 0 & 0 & 10 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

Therefore, the cube is at location $(11, 10, 1)^T$ from the base coordinate system. Its $\mathbf{x}, \mathbf{y}$, and $\mathbf{z}$ axes are parallel to the $-\mathbf{y}, \mathbf{x}$, and $\mathbf{z}$ axes of the base coordinate system, respectively.

To find $[\mathbf{n}, \mathbf{s}, \mathbf{a}]$, we make use of

$$^{0}\mathbf{T}_6 = \begin{bmatrix} \mathbf{n} & \mathbf{s} & \mathbf{a} & \mathbf{p} \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

where $\mathbf{p} = (11, 10, 1)^T$ from the above solution. From the above figure, we want the approach vector $\mathbf{a}$ to align with the negative direction of the OZ axis of the base coordinate system [i.e., $\mathbf{a} = (0, 0, -1)^T$]; the $\mathbf{s}$ vector can be aligned in either direction of the $\mathbf{y}$ axis of $^{\text{base}}\mathbf{T}_{\text{cube}}$ [i.e., $\mathbf{s} = (\pm 1, 0, 0)^T$];

and the **n** vector can be obtained from the cross product of **s** and **a**:

$$\mathbf{n} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ s_x & s_y & s_z \\ a_x & a_y & a_z \end{vmatrix} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ \pm 1 & 0 & 0 \\ 0 & 0 & -1 \end{vmatrix} = \begin{bmatrix} 0 \\ \pm 1 \\ 0 \end{bmatrix}$$

Therefore, the orientation matrix $[\mathbf{n}, \mathbf{s}, \mathbf{a}]$ is found to be

$$[\mathbf{n}, \mathbf{s}, \mathbf{a}] = \begin{bmatrix} 0 & 1 & 0 \\ +1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix} \quad \text{or} \quad \begin{bmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$ □

2.2.12 Other Specifications of the Location of the End-Effector

In previous sections, we analyzed the translations and rotations of rigid bodies (or links) and introduced the homogeneous transformation matrix for describing the position and orientation of a link coordinate frame. Of particular interest is the arm matrix ${}^0\mathbf{T}_6$ which describes the position and orientation of the hand with respect to the base coordinate frame. The upper left 3×3 submatrix of ${}^0\mathbf{T}_6$ describes the orientation of the hand. This rotation submatrix is equivalent to ${}^0\mathbf{R}_6$. There are other specifications which can be used to describe the location of the end-effector.

Euler Angle Representation for Orientation. As indicated in Sec. 2.2.4, this matrix representation for rotation of a rigid body simplifies many operations, but it does not lead directly to a complete set of generalized coordinates. Such a set of generalized coordinates can be provided by three Euler angles (ϕ, θ, and ψ).

Using the rotation matrix with eulerian angle representation as in Eq. (2.2-17), the arm matrix ${}^0\mathbf{T}_6$ can be expressed as:

$${}^0\mathbf{T}_6 = \begin{bmatrix} C\phi C\psi - S\phi C\theta S\psi & -C\phi S\psi - S\phi C\theta C\psi & S\phi S\theta & p_x \\ S\phi C\psi + C\phi C\theta S\psi & -S\phi S\psi + C\phi C\theta C\psi & -C\phi S\theta & p_y \\ S\theta S\psi & S\theta C\psi & C\theta & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2-44}$$

Another advantage of using Euler angle representation for the orientation is that the storage for the position and orientation of an object is reduced to a six-element vector $XYZ\phi\theta\psi$. From this vector, one can construct the arm matrix ${}^0\mathbf{T}_6$ by Eq. (2.2-44).

Roll, Pitch, and Yaw Representation for Orientation. Another set of Euler angle representation for rotation is roll, pitch, and yaw (RPY). Again, using Eq.

(2.2-19), the rotation matrix representing roll, pitch, and yaw can be used to obtain the arm matrix ${}^0\mathbf{T}_6$ as:

$$ {}^0\mathbf{T}_6 = \begin{bmatrix} C\phi C\theta & C\phi S\theta S\psi - S\phi C\psi & C\phi S\theta C\psi + S\phi S\psi & p_x \\ S\phi C\theta & S\phi S\theta S\psi + C\phi C\psi & S\phi S\theta C\psi - C\phi S\psi & p_y \\ -S\theta & C\theta S\psi & C\theta C\psi & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.2\text{-}45) $$

As discussed in Chap. 1, there are different types of robot arms according to their joint motion (*XYZ*, cylindrical, spherical, and articulated arm). Thus, one can specify the position of the hand $(p_x, p_y, p_z)^T$ in other coordinates such as cylindrical or spherical. The resultant arm transformation matrix can be obtained by

$$ {}^0\mathbf{T}_6 = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & & 0 \\ & {}^0\mathbf{R}_6 & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad (2.2\text{-}46) $$

where ${}^0\mathbf{R}_6$ = rotation matrix expressed in either Euler angles or $[\mathbf{n}, \mathbf{s}, \mathbf{a}]$ or roll, pitch, and yaw.

Cylindrical Coordinates for Positioning Subassembly. In a cylindrical coordinate representation, the position of the end-effector can be specified by the following translations/rotations (see Fig. 2.15):

1. A translation of r unit along the OX axis ($\mathbf{T}_{x,r}$)
2. A rotation of α angle about the OZ axis ($\mathbf{T}_{z,\alpha}$)
3. A translation of d unit along the OZ axis ($\mathbf{T}_{z,d}$)

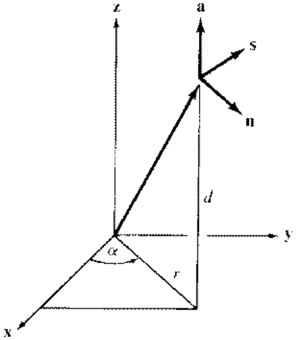


Figure 2.15 Cylindrical coordinate system representation.

The homogeneous transformation matrix that represents the above operations can be expressed as:

$$\mathbf{T}_{\text{cylindrical}} = \mathbf{T}_{z,d}\,\mathbf{T}_{z,\alpha}\,\mathbf{T}_{x,r} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\alpha & -S\alpha & 0 & 0 \\ S\alpha & C\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & r \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\alpha & -S\alpha & 0 & rC\alpha \\ S\alpha & C\alpha & 0 & rS\alpha \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2.2\text{-}47)$$

Since we are only interested in the position vectors (i.e., the fourth column of $\mathbf{T}_{\text{cylindrical}}$), the arm matrix ${}^0\mathbf{T}_6$ can be obtained utilizing Eq. (2.2-46).

$${}^0\mathbf{T}_6 = \begin{bmatrix} 1 & 0 & 0 & rC\alpha \\ 0 & 1 & 0 & rS\alpha \\ 0 & 0 & 1 & d \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & & 0 \\ & {}^0\mathbf{R}_6 & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad (2.2\text{-}48)$$

and $p_x \equiv rC\alpha,\ p_y \equiv rS\alpha,\ p_z \equiv d$.

Spherical Coordinates for Positioning Subassembly. We can also utilize the spherical coordinate system for specifying the position of the end-effector. This involves the following translations/rotations (see Fig. 2.16):

1. A translation of r unit along the OZ axis ($\mathbf{T}_{z,r}$)
2. A rotation of β angle about the OY axis ($\mathbf{T}_{y,\beta}$)
3. A rotation of α angle about the OZ axis ($\mathbf{T}_{z,\alpha}$)

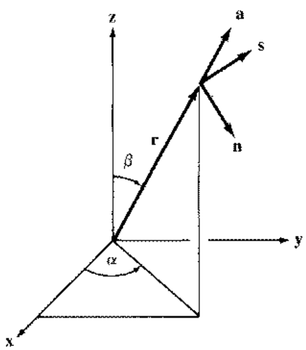


Figure 2.16 Spherical coordinate system representation.

The transformation matrix for the above operations is

$$\mathbf{T}_{sph} = \mathbf{T}_{z,\alpha}\mathbf{R}_{y,\beta}\mathbf{T}_{z,r} = \begin{bmatrix} C\alpha & -S\alpha & 0 & 0 \\ S\alpha & C\alpha & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} C\beta & 0 & S\beta & 0 \\ 0 & 1 & 0 & 0 \\ -S\beta & 0 & C\beta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$\times \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & r \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\alpha C\beta & -S\alpha & C\alpha S\beta & rC\alpha S\beta \\ S\alpha C\beta & C\alpha & S\alpha S\beta & rS\alpha S\beta \\ -S\beta & 0 & C\beta & rC\beta \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2-49}$$

Again, our interest is the position vector with respect to the base coordinate system, therefore, the arm matrix ${}^0\mathbf{T}_6$ whose position vector is expressed in spherical coordinates and the orientation matrix is expressed in $[\mathbf{n}, \mathbf{s}, \mathbf{a}]$ or Euler angles or roll, pitch, and yaw can be obtained by:

$${}^0\mathbf{T}_6 = \begin{bmatrix} 1 & 0 & 0 & rC\alpha S\beta \\ 0 & 1 & 0 & rS\alpha S\beta \\ 0 & 0 & 1 & rC\beta \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} & & & 0 \\ & {}^0\mathbf{R}_6 & & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.2-50}$$

where $p_x \equiv rC\alpha S\beta$, $p_y \equiv rS\alpha S\beta$, $p_z \equiv rC\beta$.

In summary, there are several methods (or coordinate systems) that one can choose to describe the position and orientation of the end-effector. For positioning, the position vector can be expressed in cartesian $(p_x, p_y, p_z)^T$, cylindrical $(rC\alpha, rS\alpha, d)^T$, or spherical $(rC\alpha S\beta, rS\alpha S\beta, rC\beta)^T$ terms. For describing the orientation of the end-effector with respect to the base coordinate system, we have cartesian $[\mathbf{n}, \mathbf{s}, \mathbf{a}]$, Euler angles (ϕ, θ, ψ), and (roll, pitch, and yaw). The result of the above discussion is tabulated in Table 2.2.

Table 2.2 Various positioning/orientation representations

Positioning	Orientation
Cartesian $(p_x, p_y, p_z)^T$	Cartesian $[\mathbf{n}, \mathbf{s}, \mathbf{a}]$
Cylindrical $(rC\alpha, rS\alpha, d)^T$	Euler angles (ϕ, θ, ψ)
Spherical $(rC\alpha S\beta, rS\alpha S\beta, rC\beta)^T$	Roll, pitch, and yaw
$\mathbf{T}_{\text{position}} = \begin{bmatrix} 1 & 0 & 0 & p_x \\ 0 & 1 & 0 & p_y \\ 0 & 0 & 1 & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\mathbf{T}_{\text{rot}} = \begin{bmatrix} & & & 0 \\ [\mathbf{n}, \mathbf{s}, \mathbf{a}] & \text{or} & \mathbf{R}_{\phi,\theta,\psi} & 0 \\ & & & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

$${}^0\mathbf{T}_6 = \mathbf{T}_{\text{position}}\mathbf{T}_{\text{rot}}$$

2.2.13 Classification of Manipulators

A manipulator consists of a group of rigid bodies or links, with the first link connected to a supporting base and the last link containing the terminal device (or tool). In addition, each link is connected to, at most, two others so that closed loops are not formed. We made the assumption that the connection between links (the joints) have only 1 degree of freedom. With this restriction, two types of joints are of interest: revolute (or rotary) and prismatic. A revolute joint only permits rotation about an axis, while the prismatic joint allows sliding along an axis with no rotation (sliding with rotation is called a *screw joint*). These links are connected and powered in such a way that they are forced to move relative to one another in order to position the end-effector (a hand or tool) in a particular *position* and *orientation.*

Hence, a manipulator, considered to be a combination of links and joints, with the first link connected to ground and the last link containing the "hand," may be classified by the *type of joints* and their *order* (from the base to the hand). With this convention, the PUMA robot arm may be classified as 6R and the Stanford arm as 2R-P-3R, where R is a revolute joint and P is a prismatic joint.

2.3 THE INVERSE KINEMATICS PROBLEM

This section addresses the second problem of robot arm kinematics: the inverse kinematics or arm solution for a six-joint manipulator. Computer-based robots are usually servoed in the joint-variable space, whereas objects to be manipulated are usually expressed in the world coordinate system. In order to control the position and orientation of the end-effector of a robot to reach its object, the inverse kinematics solution is more important. In other words, given the position and orientation of the end-effector of a six-axis robot arm as ${}^0\mathbf{T}_6$ and its joint and link parameters, we would like to find the corresponding joint angles $\mathbf{q} = (q_1, q_2, q_3, q_4, q_5, q_6)^T$ of the robot so that the end-effector can be positioned as desired.

In general, the inverse kinematics problem can be solved by various methods, such as inverse transform (Paul et al. [1981]), screw algebra (Kohli and Soni [1975]), dual matrices (Denavit [1956]), dual quaternian (Yang and Freudenstein [1964]), iterative (Uicker et al. [1964]), and geometric approaches (Lee and Ziegler [1984]). Pieper [1968] presented the kinematics solution for any 6 degree of freedom manipulator which has revolute or prismatic pairs for the first three joints and the joint axes of the last three joints intersect at a point. The solution can be expressed as a fourth-degree polynomial in one unknown, and closed form solution for the remaining unknowns. Paul et al. [1981] presented an inverse transform technique using the 4×4 homogeneous transformation matrices in solving the kinematics solution for the same class of simple manipulators as discussed by Pieper. Although the resulting solution is correct, it suffers from the fact that the solution does not give a clear indication on how to select an appropriate solution from the several possible solutions for a particular arm configuration. The user often needs to rely on his or her intuition to pick the right answer. We shall discuss Pieper's approach in solving the inverse solution for Euler angles. Uicker et al. [1964] and Milenkovic and Huang [1983] presented iterative solutions for most industrial robots. The iterative solution often requires more computation and it does not guarantee convergence to the correct solution, especially in the singular and degenerate cases. Furthermore, as with the inverse transform technique, there is no indication on how to choose the correct solution for a particular arm configuration.

It is desirable to find a closed-form arm solution for manipulators. Fortunately, most of the commercial robots have either one of the following sufficient conditions which make the closed-form arm solution possible:

1. Three adjacent joint axes intersecting
2. Three adjacent joint axes parallel to one another

Both PUMA and Stanford robot arms satisfy the first condition while ASEA and MINIMOVER robot arms satisfy the second condition for finding the closed-form solution.

From Eq. (2.2-39), we have the arm transformation matrix given as

$$\mathbf{T}_6 = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} = {}^0\mathbf{A}_1 \, {}^1\mathbf{A}_2 \, {}^2\mathbf{A}_3 \, {}^3\mathbf{A}_4 \, {}^4\mathbf{A}_5 \, {}^5\mathbf{A}_6 \tag{2.3-1}$$

The above equation indicates that the arm matrix $\mathbf{T}$ is a function of sine and cosine of $\theta_1, \theta_2, \ldots, \theta_6$. For example, for a PUMA robot arm, equating the elements of the matrix equations as in Eqs. (2.2-40) to (2.2-43), we have twelve equations with six unknowns (joint angles) and these equations involve complex trigonometric functions. Since we have more equations than unknowns, one can immediately conclude that multiple solutions exist for a PUMA-like robot arm. We shall explore two methods for finding the inverse solution: inverse transform technique for finding Euler angles solution, which can also be used to find the joint solution of a PUMA-like robot arm, and a geometric approach which provides more insight into solving simple manipulators with rotary joints.

2.3.1 Inverse Transform Technique for Euler Angles Solution

In this section, we shall show the basic concept of the inverse transform technique by applying it to solve for the Euler angles. Since the 3×3 rotation matrix can be expressed in terms of the Euler angles (ϕ, θ, ψ) as in Eq. (2.2-17), and given

$$\begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} = \mathbf{R}_{z,\phi} \, \mathbf{R}_{u,\theta} \, \mathbf{R}_{w,\psi}$$

$$= \begin{bmatrix} C\phi C\psi - S\phi C\theta S\psi & -C\phi S\psi - S\phi C\theta C\psi & S\phi S\theta \\ S\phi C\psi + C\phi C\theta S\psi & -S\phi S\psi + C\phi C\theta C\psi & -C\phi S\theta \\ S\theta S\psi & S\theta C\psi & C\theta \end{bmatrix} \tag{2.3-2}$$

we would like to find the corresponding value of ϕ, θ, ψ. Equating the elements of the above matrix equation, we have:

$$n_x = C\phi C\psi - S\phi C\theta S\psi \tag{2.3-3a}$$

$$n_y = S\phi C\psi + C\phi C\theta S\psi \tag{2.3-3b}$$

$$n_z = S\theta S\psi \tag{2.3-3c}$$

$$s_x = -C\phi S\psi - S\phi C\theta C\psi \tag{2.3-3d}$$

$$s_y = -S\phi S\psi + C\phi C\theta C\psi \tag{2.3-3e}$$

$$s_z = S\theta C\psi \tag{2.3-3f}$$

$$a_x = S\phi S\theta \tag{2.3-3g}$$

$$a_y = -C\phi S\theta \tag{2.3-3h}$$

$$a_z = C\theta \tag{2.3-3i}$$

Using Eqs. (2.3-3*i*), (2.3-3*f*), and (2.3-3*h*), a solution to the above nine equations is:

$$\theta = \cos^{-1}(a_z) \tag{2.3-4}$$

$$\psi = \cos^{-1}\left[\frac{s_z}{S\theta}\right] \tag{2.3-5}$$

$$\phi = \cos^{-1}\left[\frac{-a_y}{S\theta}\right] \tag{2.3-6}$$

The above solution is inconsistent and ill-conditioned because:

1. The arc cosine function does not behave well as its accuracy in determining the angle is dependent on the angle. That is, $\cos(\theta) = \cos(-\theta)$.
2. When $\sin(\theta)$ approaches zero, that is, $\theta \approx 0°$ or $\theta \approx \pm 180°$, Eqs. (2.3-5) and (2.3-6) give inaccurate solutions or are undefined.

We must, therefore, find a more consistent approach to determining the Euler angle solution and a more consistent arc trigonometric function in evaluating the angle solution. In order to evaluate θ for $-\pi \leqslant \theta \leqslant \pi$, an arc tangent function, atan2 (y, x), which returns $\tan^{-1}(y/x)$ adjusted to the proper quadrant will be used. It is defined as:

$$\theta = \text{atan2}(y, x) = \begin{cases} 0° \leqslant \theta \leqslant 90° & \text{for } +x \text{ and } +y \\ 90° \leqslant \theta \leqslant 180° & \text{for } -x \text{ and } +y \\ -180° \leqslant \theta \leqslant -90° & \text{for } -x \text{ and } -y \\ -90° \leqslant \theta \leqslant 0° & \text{for } +x \text{ and } -y \end{cases} \tag{2.3-7}$$

Using the arc tangent function (atan2) with two arguments, we shall take a look at a general solution proposed by Paul et al. [1981].

From the matrix equation in Eq. (2.3-2), the elements of the matrix on the left hand side (LHS) of the matrix equation are given, while the elements of the three matrices on the right-hand side (RHS) are unknown and they are dependent on ϕ, θ, ψ. Paul et al. [1981] suggest *premultiplying* the above matrix equation by its unknown inverse transforms successively and from the elements of the resultant matrix equation determine the unknown angle. That is, we move one unknown (by its inverse transform) from the RHS of the matrix equation to the LHS and solve for the unknown, then move the next unknown to the LHS, and repeat the process until all the unknowns are solved.

Premultiplying the above matrix equation by $\mathbf{R}_{z,\phi}^{-1}$, we have one unknown (ϕ) on the LHS and two unknowns (θ, ψ) on the RHS of the matrix equation, thus we have

$$\begin{bmatrix} C\phi & S\phi & 0 \\ -S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \begin{bmatrix} C\psi & -S\psi & 0 \\ S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

or

$$\begin{bmatrix} C\phi n_x + S\phi n_y & C\phi s_x + S\phi s_y & C\phi a_x + S\phi a_y \\ -S\phi n_x + C\phi n_y & -S\phi s_x + C\phi s_y & -S\phi a_x + C\phi a_y \\ n_z & s_z & a_z \end{bmatrix} = \begin{bmatrix} C\psi & -S\psi & 0 \\ C\theta S\psi & C\theta C\psi & -S\theta \\ S\theta S\psi & S\theta C\psi & C\theta \end{bmatrix} \tag{2.3-8}$$

Equating the (1, 3) elements of both matrices in Eq. (2.3-8), we have:

$$C\phi a_x + S\phi a_y = 0 \tag{2.3-9}$$

which gives

$$\phi = \tan^{-1}\left(\frac{a_x}{-a_y}\right) = \text{atan2}(a_x, -a_y) \tag{2.3-10}$$

Equating the (1, 1) and (1, 2) elements of the both matrices, we have:

$$C\psi = C\phi n_x + S\phi n_y \tag{2.3-11a}$$

$$S\psi = -C\phi s_x - S\phi s_y \tag{2.3-11b}$$

which lead to the solution for ψ,

$$\psi = \tan^{-1}\left[\frac{S\psi}{C\psi}\right] = \tan^{-1}\left[\frac{-C\phi s_x - S\phi s_y}{C\phi n_x + S\phi n_y}\right]$$

$$= \operatorname{atan2}(-C\phi s_x - S\phi s_y,\ C\phi n_x + S\phi n_y) \tag{2.3-12}$$

Equating the (2, 3) and (3, 3) elements of the both matrices, we have:

$$S\theta = S\phi a_x - C\phi a_y$$
$$C\theta = a_z \tag{2.3-13}$$

which gives us the solution for θ,

$$\theta = \tan^{-1}\left[\frac{S\theta}{C\theta}\right] = \tan^{-1}\left[\frac{S\phi a_x - C\phi a_y}{a_z}\right] = \operatorname{atan2}(S\phi a_x - C\phi a_y,\ a_z) \tag{2.3-14}$$

Since the concept of inverse transform technique is to move one unknown to the LHS of the matrix equation at a time and solve for the unknown, we can try to solve the above matrix equation for ϕ, θ, ψ by *postmultiplying* the above matrix equation by its inverse transform $\mathbf{R}_{w,\psi}^{-1}$

$$\begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix}\begin{bmatrix} C\psi & S\psi & 0 \\ -S\psi & C\psi & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} C\phi & -S\phi & 0 \\ S\phi & C\phi & 0 \\ 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & 0 \\ 0 & C\theta & -S\theta \\ 0 & S\theta & C\theta \end{bmatrix}$$

Multiplying the matrices out, we have,

$$\begin{bmatrix} n_xC\psi - s_xS\psi & n_xS\psi + s_xC\psi & a_x \\ n_yC\psi - s_yS\psi & n_yS\psi + s_yC\psi & a_y \\ n_zC\psi - s_zS\psi & n_zS\psi + s_zC\psi & a_z \end{bmatrix} = \begin{bmatrix} C\phi & -S\phi C\theta & S\phi S\theta \\ S\phi & C\phi C\theta & -C\phi S\theta \\ 0 & S\theta & C\theta \end{bmatrix} \tag{2.3-15}$$

Again equating the (3, 1) elements of both matrices in the above matrix equation, we have

$$n_zC\psi - s_zS\psi = 0 \tag{2.3-16}$$

which gives

$$\psi = \tan^{-1}\left[\frac{n_z}{s_z}\right] = \operatorname{atan2}(n_z, s_z) \tag{2.3-17}$$

Equating the (3, 2) and (3, 3) elements of both matrices, we have:

$$S\theta = n_z S\psi + s_z C\psi \tag{2.3-18a}$$

$$C\theta = a_z \tag{2.3-18b}$$

which leads us to the solution for θ,

$$\theta = \tan^{-1}\left[\frac{n_z S\psi + s_z C\psi}{a_z}\right] = \operatorname{atan2}(n_z S\psi + s_z C\psi, a_z) \tag{2.3-19}$$

Equating the (1, 1) and (2, 1) elements of both matrices, we have

$$C\phi = n_x C\psi - s_x S\psi \tag{2.3-20a}$$

$$S\phi = n_y C\psi - s_y S\psi \tag{2.3-20b}$$

which gives

$$\phi = \tan^{-1}\left[\frac{n_y C\psi - s_y S\psi}{n_x C\psi - s_x S\psi}\right]$$

$$= \operatorname{atan2}(n_y C\psi - s_y S\psi, n_x C\psi - s_x S\psi) \tag{2.3-21}$$

Whether one should premultiply or postmultiply a given matrix equation is up to the user's discretion and it depends on the intuition of the user.

Let us apply this inverse transform technique to solve the Euler angles for a PUMA robot arm (OAT solution of a PUMA robot). PUMA robots use the symbols O, A, T to indicate the Euler angles and their definitions are given as follows (with reference to Fig. 2.17):

O (*orientation*) is the angle formed from the $\mathbf{y}_0$ axis to the projection of the tool **a** axis on the XY plane about the $\mathbf{z}_0$ axis.

A (*altitude*) is the angle formed from the XY plane to the tool **a** axis about the **s** axis of the tool.

T (*tool*) is the angle formed from the XY plane to the tool **s** axis about the **a** axis of the tool.

Initially the tool coordinate system (or the hand coordinate system) is aligned with the base coordinate system of the robot as shown in Fig. 2.18. That is, when $O = A = T = 0°$, the hand points in the negative $\mathbf{y}_0$ axis with the fingers in a horizontal plane, and the **s** axis is pointing to the positive $\mathbf{x}_0$ axis. The necessary

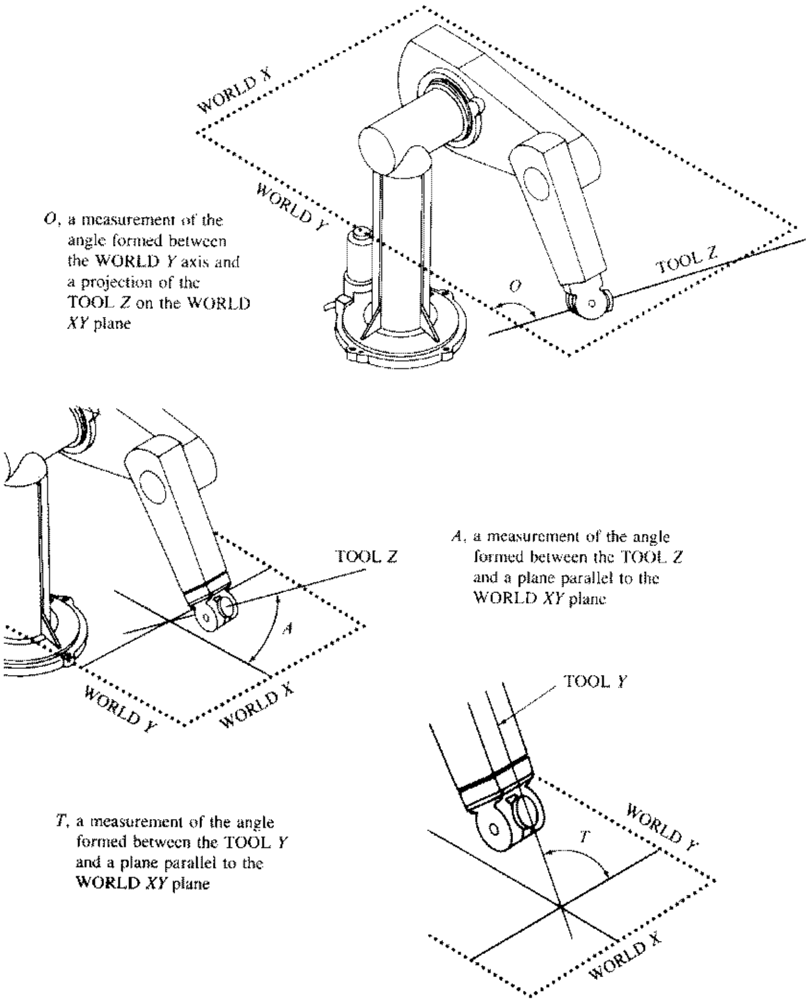


Figure 2.17 Definition of Euler angles O, A, and T. (Taken from PUMA robot manual 398H.)

transform that describes the orientation of the hand coordinate system $(\mathbf{n}, \mathbf{s}, \mathbf{a})$ with respect to the base coordinate system $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ is given by

$$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \tag{2.3-22}$$

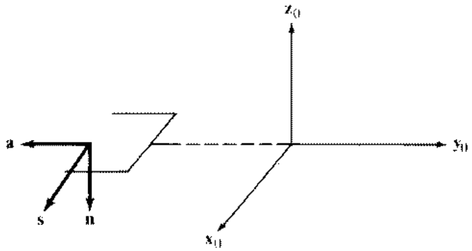


Figure 2.18 Initial alignment of tool coordinate system.

From the definition of the *OAT* angles and the initial alignment matrix [Eq. (2.3-22)], the relationship between the hand transform and the *OAT* angle is given by

$$\begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} = \mathbf{R}_{z,O} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \mathbf{R}_{s,A}\,\mathbf{R}_{a,T}$$

$$= \begin{bmatrix} CO & -SO & 0 \\ SO & CO & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix} \begin{bmatrix} CA & 0 & SA \\ 0 & 1 & 0 \\ -SA & 0 & CA \end{bmatrix} \begin{bmatrix} CT & -ST & 0 \\ ST & CT & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Postmultiplying the above matrix equation by the inverse transform of $\mathbf{R}_{a,T}$,

$$\begin{bmatrix} n_x & s_x & a_x \\ n_y & s_y & a_y \\ n_z & s_z & a_z \end{bmatrix} \begin{bmatrix} CT & ST & 0 \\ -ST & CT & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} CO & -SO & 0 \\ SO & CO & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ -1 & 0 & 0 \end{bmatrix}$$

$$\times \begin{bmatrix} CA & 0 & SA \\ 0 & 1 & 0 \\ -SA & 0 & CA \end{bmatrix}$$

and multiplying the matrices out, we have:

$$\begin{bmatrix} n_xCT - s_xST & n_xST + s_xCT & a_x \\ n_yCT - s_yST & n_yST + s_yCT & a_y \\ n_zCT - s_zST & n_zST + s_zCT & a_z \end{bmatrix} = \begin{bmatrix} -SOSA & CO & SOCA \\ COSA & SO & -COCA \\ -CA & 0 & -SA \end{bmatrix} \tag{2.3-23}$$

Equating the (3, 2) elements of the above matrix equation, we have:

$$n_zST + s_zCT = 0 \tag{2.3-24}$$

which gives the solution of T,

$$T = \tan^{-1}\left[\frac{s_z}{-n_z}\right] = \text{atan2}\,(s_z,\ -n_z) \qquad (2.3\text{-}25)$$

Equating the (3, 1) and (3, 3) elements of the both matrices, we have:

$$SA = -a_z \qquad (2.3\text{-}26a)$$

and

$$CA = -n_zCT + s_zST \qquad (2.3\text{-}26b)$$

then the above equations give

$$A = \tan^{-1}\left[\frac{-a_z}{-n_zCT + s_zST}\right] = \text{atan2}\,(-a_z,\ -n_zCT + s_zST) \qquad (2.3\text{-}27)$$

Equating the (1, 2) and (2, 2) elements of the both matrices, we have:

$$CO = n_xST + s_xCT \qquad (2.3\text{-}28a)$$

$$SO = n_yST + s_yCT \qquad (2.3\text{-}28b)$$

which give the solution of O,

$$O = \tan^{-1}\left[\frac{n_yST + s_yCT}{n_xST + s_xCT}\right]$$

$$= \text{atan2}\,(n_yST + s_yCT,\ n_xST + s_xCT) \qquad (2.3\text{-}29)$$

The above premultiplying or postmultiplying of the unknown inverse transforms can also be applied to find the joint solution of a PUMA robot. Details about the PUMA robot arm joint solution can be found in Paul et al. [1981].

Although the inverse transform technique provides a general approach in determining the joint solution of a manipulator, it does not give a clear indication on how to select an appropriate solution from the several possible solutions for a particular arm configuration. This has to rely on the user's geometric intuition. Thus, a geometric approach is more useful in deriving a consistent joint-angle solution given the arm matrix as in Eq. (2.2-39), and it provides a means for the user to select a unique solution for a particular arm configuration. This approach is presented in Sec. 2.3.2.

2.3.2 A Geometric Approach

This section presents a geometric approach to solving the inverse kinematics problem of six-link manipulators with rotary joints. The discussion focuses on a PUMA-like manipulator. Based on the link coordinate systems and human arm geometry, various arm configurations of a PUMA-like robot (Fig. 2.11) can be identified with the assistance of three configuration indicators (ARM, ELBOW, and WRIST)—two associated with the solution of the first three joints and the other with the last three joints. For a six-axis PUMA-like robot arm, there are four possible solutions to the first three joints and for each of these four solutions there are two possible solutions to the last three joints. The first two configuration indicators allow one to determine one solution from the possible four solutions for the first three joints. Similarly, the third indicator selects a solution from the possible two solutions for the last three joints. The arm configuration indicators are prespecified by a user for finding the inverse solution. The solution is calculated in two stages. First, a position vector pointing from the shoulder to the wrist is derived. This is used to derive the solution of each joint i ($i = 1, 2, 3$) for the first three joints by looking at the projection of the position vector onto the $\mathbf{x}_{i-1}\mathbf{y}_{i-1}$ plane. The last three joints are solved using the calculated joint solution from the first three joints, the orientation submatrices of ${}^0\mathbf{T}_i$ and ${}^{i-1}\mathbf{A}_i$ ($i = 4, 5, 6$), and the projection of the link coordinate frames onto the $\mathbf{x}_{i-1}\mathbf{y}_{i-1}$ plane. From the geometry, one can easily find the arm solution consistently. As a verification of the joint solution, the arm configuration indicators can be determined from the corresponding decision equations which are functions of the joint angles. With appropriate modification and adjustment, this approach can be generalized to solve the inverse kinematics problem of most present day industrial robots with rotary joints.

If we are given ${}^{\text{ref}}\mathbf{T}_{\text{tool}}$, then we can find ${}^0\mathbf{T}_6$ by premultiplying and postmultiplying ${}^{\text{ref}}\mathbf{T}_{\text{tool}}$ by $\mathbf{B}^{-1}$ and $\mathbf{H}^{-1}$, respectively, and the joint-angle solution can be applied to ${}^0\mathbf{T}_6$ as desired.

$$ {}^0\mathbf{T}_6 \equiv \mathbf{T} = \mathbf{B}^{-1}\,{}^{\text{ref}}\mathbf{T}_{\text{tool}}\,\mathbf{H}^{-1} = \begin{bmatrix} n_x & s_x & a_x & p_x \\ n_y & s_y & a_y & p_y \\ n_z & s_z & a_z & p_z \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2.3-30} $$

Definition of Various Arm Configurations. For the PUMA robot arm shown in Fig. 2.11 (and other rotary robot arms), various arm configurations are defined according to human arm geometry and the link coordinate systems which are established using Algorithm 2.1 as (Fig. 2.19)

RIGHT (shoulder) ARM: Positive θ_2 moves the wrist in the *positive* $\mathbf{z}_0$ direction while joint 3 is not activated.

LEFT (shoulder) ARM: Positive θ_2 moves the wrist in the *negative* $\mathbf{z}_0$ direction while joint 3 is not activated.

ABOVE ARM (elbow above wrist): Position of the wrist of the

$\left\{\begin{matrix}\text{RIGHT}\\ \text{LEFT}\end{matrix}\right\}$ arm with respect to the shoulder coordinate system has

$\left\{\begin{matrix} \text{negative} \\ \text{positive} \end{matrix}\right\}$ coordinate value along the $\mathbf{y}_2$ axis.

BELOW ARM (elbow below wrist): Position of the wrist of the

$\left\{\begin{matrix} \text{RIGHT} \\ \text{LEFT} \end{matrix}\right\}$ arm with respect to the shoulder coordinate system has

$\left\{\begin{matrix} \text{positive} \\ \text{negative} \end{matrix}\right\}$ coordinate value along the $\mathbf{y}_2$ axis.

WRIST DOWN: The $\mathbf{s}$ unit vector of the hand coordinate system and the $\mathbf{y}_5$ unit vector of the $(\mathbf{x}_5, \mathbf{y}_5, \mathbf{z}_5)$ coordinate system have a positive dot product.

WRIST UP: The $\mathbf{s}$ unit vector of the hand coordinate system and the $\mathbf{y}_5$ unit vector of the $(\mathbf{x}_5, \mathbf{y}_5, \mathbf{z}_5)$ coordinate system have a negative dot product.

(Note that the definition of the arm configurations with respect to the link coordinate systems may have to be slightly modified if one uses different link coordinate systems.)

With respect to the above definition of various arm configurations, two arm configuration *indicators* (ARM and ELBOW) are defined for each arm configuration. These two indicators are combined to give one solution out of the possible four joint solutions for the first three joints. For each of the four arm configurations (Fig. 2.19) defined by these two indicators, the third indicator (WRIST) gives one of the two possible joint solutions for the last three joints. These three indicators can be defined as:

$$\text{ARM} = \begin{cases} +1 & \text{RIGHT arm} \\ -1 & \text{LEFT arm} \end{cases} \tag{2.3-31}$$

$$\text{ELBOW} = \begin{cases} +1 & \text{ABOVE arm} \\ -1 & \text{BELOW arm} \end{cases} \tag{2.3-32}$$

$$\text{WRIST} = \begin{cases} +1 & \text{WRIST DOWN} \\ -1 & \text{WRIST UP} \end{cases} \tag{2.3-33}$$

In addition to these indicators, the user can define a "FLIP" toggle as:

$$\text{FLIP} = \begin{cases} +1 & \text{Flip the wrist orientation} \\ -1 & \text{Do not flip the wrist orientation} \end{cases} \tag{2.3-34}$$

The signed values of these indicators and the toggle are prespecified by a user for finding the inverse kinematics solution. These indicators can also be set from the knowledge of the joint angles of the robot arm using the corresponding decision equations. We shall later give the decision equations that determine these indicator

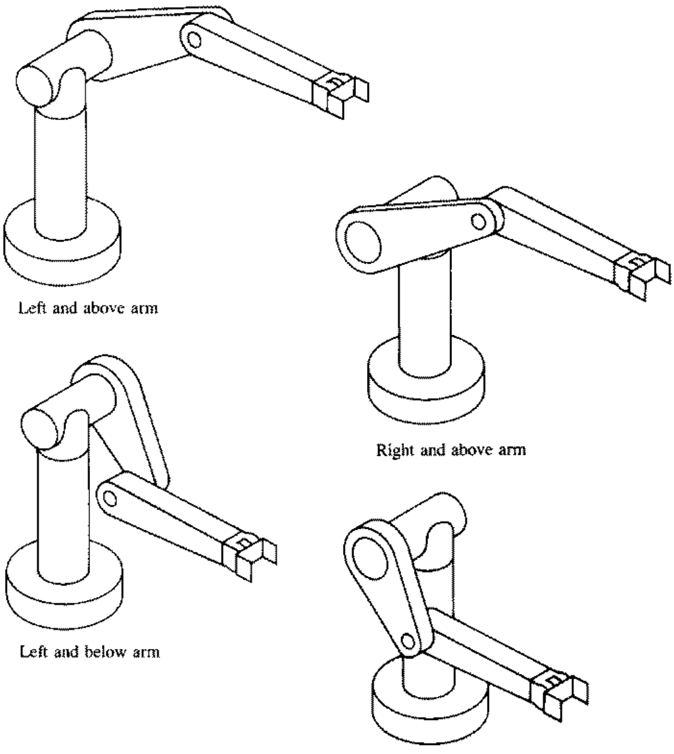


Figure 2.19 Definition of various arm configurations.

values. The decision equations can be used as a verification of the inverse kinematics solution.

Arm Solution for the First Three Joints. From the kinematics diagram of the PUMA robot arm in Fig. 2.11, we define a position vector $\mathbf{p}$ which points from the origin of the shoulder coordinate system $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ to the point where the last three joint axes intersect as (see Fig. 2.14):

$$\mathbf{p} = \mathbf{p}_6 - d_6 \mathbf{a} = (p_x, p_y, p_z)^T \tag{2.3-35}$$

which corresponds to the position vector of ${}^0\mathbf{T}_4$:

$$\begin{bmatrix} p_x \\ p_y \\ p_z \end{bmatrix} = \begin{bmatrix} C_1(a_2 C_2 + a_3 C_{23} + d_4 S_{23}) - d_2 S_1 \\ S_1(a_2 C_2 + a_3 C_{23} + d_4 S_{23}) + d_2 C_1 \\ d_4 C_{23} - a_3 S_{23} - a_2 S_2 \end{bmatrix} \tag{2.3-36}$$

Joint 1 solution. If we project the position vector **p** onto the $\mathbf{x}_0\mathbf{y}_0$ plane as in Fig. 2.20, we obtain the following equations for solving θ_1:

$$\theta_1^L = \phi - \alpha \qquad \theta_1^R = \pi + \phi + \alpha \tag{2.3-37}$$

$$r = \sqrt{p_x^2 + p_y^2 - d_2^2} \qquad R = \sqrt{p_x^2 + p_y^2} \tag{2.3-38}$$

$$\sin\phi = \frac{p_y}{R} \qquad \cos\phi = \frac{p_x}{R} \tag{2.3-39}$$

$$\sin\alpha = \frac{d_2}{R} \qquad \cos\alpha = \frac{r}{R} \tag{2.3-40}$$

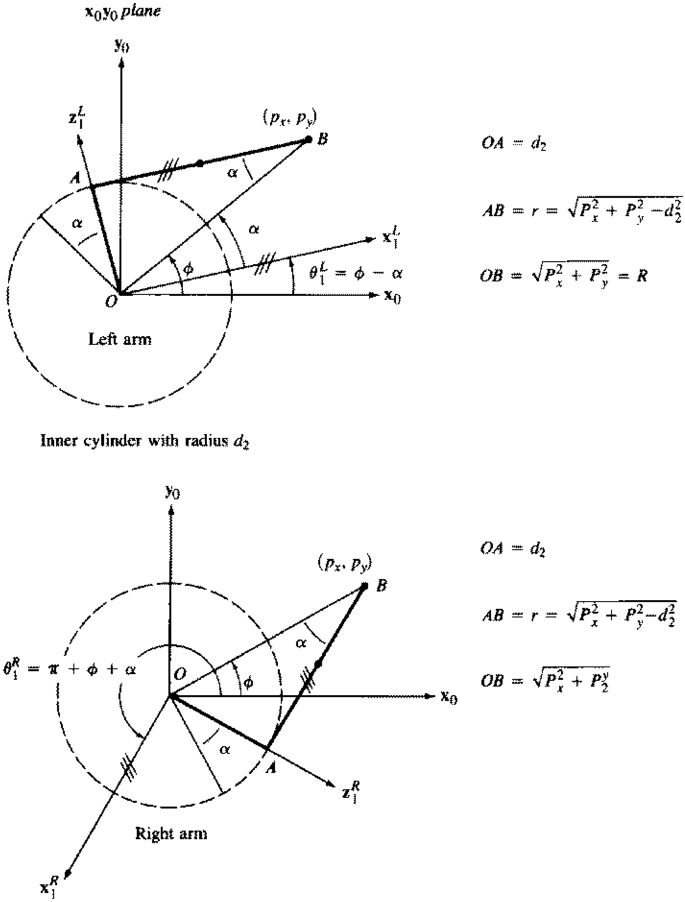


Figure 2.20 Solution for joint 1.

where the superscripts L and R on joint angles indicate the LEFT/RIGHT arm configurations. From Eqs. (2.3-37) to (2.3-40), we obtain the sine and cosine functions of θ_1 for LEFT/RIGHT arm configurations:

$$\sin\theta_1^L = \sin(\phi - \alpha) = \sin\phi\cos\alpha - \cos\phi\sin\alpha = \frac{p_y r - p_x d_2}{R^2} \qquad (2.3\text{-}41)$$

$$\cos\theta_1^L = \cos(\phi - \alpha) = \cos\phi\cos\alpha + \sin\phi\sin\alpha = \frac{p_x r + p_y d_2}{R^2} \qquad (2.3\text{-}42)$$

$$\sin\theta_1^R = \sin(\pi + \phi + \alpha) = \frac{-p_y r - p_x d_2}{R^2} \qquad (2.3\text{-}43)$$

$$\cos\theta_1^R = \cos(\pi + \phi + \alpha) = \frac{-p_x r + p_y d_2}{R^2} \qquad (2.3\text{-}44)$$

Combining Eqs. (2.3-41) to (2.3-44) and using the ARM indicator to indicate the LEFT/RIGHT arm configuration, we obtain the sine and cosine functions of θ_1, respectively:

$$\sin\theta_1 = \frac{-\text{ARM}\; p_y\sqrt{p_x^2 + p_y^2 - d_2^2} - p_x d_2}{p_x^2 + p_y^2} \qquad (2.3\text{-}45)$$

$$\cos\theta_1 = \frac{-\text{ARM}\; p_x\sqrt{p_x^2 + p_y^2 - d_2^2} + p_y d_2}{p_x^2 + p_y^2} \qquad (2.3\text{-}46)$$

where the positive square root is taken in these equations and ARM is defined as in Eq. (2.3-31). In order to evaluate θ_1 for $-\pi \leqslant \theta_1 \leqslant \pi$, an arc tangent function as defined in Eq. (2.3-7) will be used. From Eqs. (2.3-45) and (2.3-46), and using Eq. (2.3-7), θ_1 is found to be:

$$\theta_1 = \tan^{-1}\left[\frac{\sin\theta_1}{\cos\theta_1}\right]$$

$$= \tan^{-1}\left[\frac{-\text{ARM}\; p_y\sqrt{p_x^2 + p_y^2 - d_2^2} - p_x d_2}{-\text{ARM}\; p_x\sqrt{p_x^2 + p_y^2 - d_2^2} + p_y d_2}\right] \qquad -\pi \leqslant \theta_1 \leqslant \pi$$

$$(2.3\text{-}47)$$

Joint 2 solution. To find joint 2, we project the position vector $\mathbf{p}$ onto the $\mathbf{x}_1\mathbf{y}_1$ plane as shown in Fig. 2.21. From Fig. 2.21, we have four different arm configurations. Each arm configuration corresponds to different values of joint 2 as shown in Table 2.3, where $0° \leqslant \alpha \leqslant 360°$ and $0° \leqslant \beta \leqslant 90°$.

Table 2.3 Various arm configurations for joint 2

Arm configurations	θ_2	ARM	ELBOW	ARM · ELBOW
LEFT and ABOVE arm	$\alpha - \beta$	-1	$+1$	-1
LEFT and BELOW arm	$\alpha + \beta$	-1	-1	$+1$
RIGHT and ABOVE arm	$\alpha + \beta$	$+1$	$+1$	$+1$
RIGHT and BELOW arm	$\alpha - \beta$	$+1$	-1	-1

From the above table, θ_2 can be expressed in one equation for different arm and elbow configurations using the ARM and ELBOW indicators as:

$$\theta_2 = \alpha + (\text{ARM} \cdot \text{ELBOW})\beta = \alpha + K \cdot \beta \tag{2.3-48}$$

where the combined arm configuration indicator $K = \text{ARM} \cdot \text{ELBOW}$ will give an appropriate signed value and the "dot" represents a multiplication operation on the indicators. From the arm geometry in Fig. 2.21, we obtain:

$$R = \sqrt{p_x^2 + p_y^2 + p_z^2 - d_2^2} \qquad r = \sqrt{p_x^2 + p_y^2 - d_2^2} \tag{2.3-49}$$

$$\sin\alpha = -\frac{p_z}{R} = -\frac{p_z}{\sqrt{p_x^2 + p_y^2 + p_z^2 - d_2^2}} \tag{2.3-50}$$

$$\cos\alpha = -\frac{\text{ARM} \cdot r}{R} = -\frac{\text{ARM} \cdot \sqrt{p_x^2 + p_y^2 - d_2^2}}{\sqrt{p_x^2 + p_y^2 + p_z^2 - d_2^2}} \tag{2.3-51}$$

$$\cos\beta = \frac{a_2^2 + R^2 - (d_4^2 + a_3^2)}{2a_2R} \tag{2.3-52}$$

$$= \frac{p_x^2 + p_y^2 + p_z^2 + a_2^2 - d_2^2 - (d_4^2 + a_3^2)}{2a_2\sqrt{p_x^2 + p_y^2 + p_z^2 - d_2^2}}$$

$$\sin\beta = \sqrt{1 - \cos^2\beta} \tag{2.3-53}$$

From Eqs. (2.3-48) to (2.3-53), we can find the sine and cosine functions of θ_2:

$$\sin\theta_2 = \sin(\alpha + K \cdot \beta) = \sin\alpha\cos(K \cdot \beta) + \cos\alpha\sin(K \cdot \beta)$$

$$= \sin\alpha\cos\beta + (\text{ARM} \cdot \text{ELBOW})\cos\alpha\sin\beta \tag{2.3-54}$$

$$\cos\theta_2 = \cos(\alpha + K \cdot \beta)$$

$$= \cos\alpha\cos\beta - (\text{ARM} \cdot \text{ELBOW})\sin\alpha\sin\beta \tag{2.3-55}$$

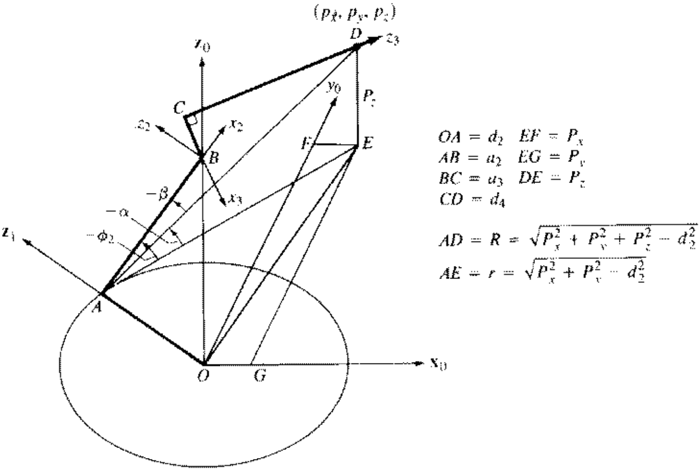


Figure 2.21 Solution for joint 2.

From Eqs. (2.3-54) and (2.3-55), we obtain the solution for θ_2:

$$\theta_2 = \tan^{-1}\left[\frac{\sin\theta_2}{\cos\theta_2}\right] \qquad -\pi \leqslant \theta_2 \leqslant \pi \tag{2.3-56}$$

Joint 3 solution. For joint 3, we project the position vector $\mathbf{p}$ onto the $\mathbf{x}_2\mathbf{y}_2$ plane as shown in Fig. 2.22. From Fig. 2.22, we again have four different arm configurations. Each arm configuration corresponds to different values of joint 3 as shown in Table 2.4, where $({}^2\mathbf{p}_4)_y$ is the y component of the position vector from the origin of $(\mathbf{x}_2, \mathbf{y}_2, \mathbf{z}_2)$ to the point where the last three joint axes intersect.

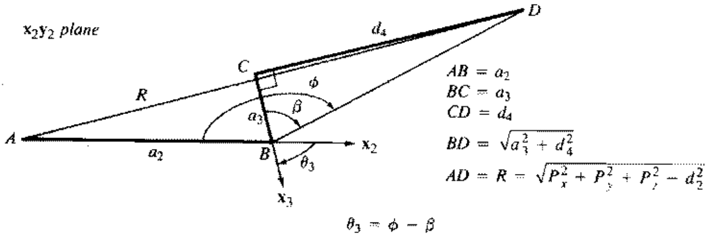
From the arm geometry in Fig. 2.22, we obtain the following equations for finding the solution for θ_3:

$$R = \sqrt{p_x^2 + p_y^2 + p_z^2 - d_2^2} \tag{2.3-57}$$

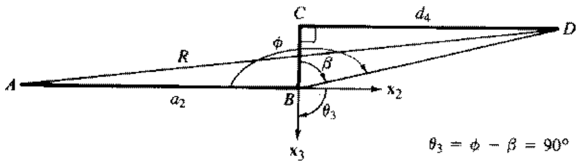
$$\cos\phi = \frac{a_2^2 + (d_4^2 + a_3^2) - R^2}{2a_2\sqrt{d_4^2 + a_3^2}} \tag{2.3-58}$$

$$\sin\phi = \text{ARM} \cdot \text{ELBOW}\sqrt{1 - \cos^2\phi}$$

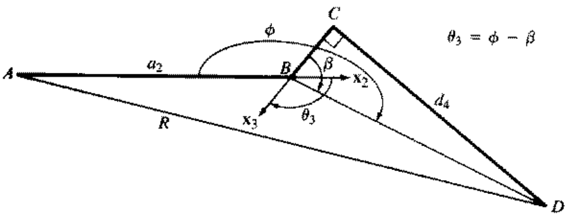
$$\sin\beta = \frac{d_4}{\sqrt{d_4^2 + a_3^2}} \qquad \cos\beta = \frac{|a_3|}{\sqrt{d_4^2 + a_3^2}} \tag{2.3-59}$$



Left and below arm



Left and below arm



Left and above arm

Figure 2.22 Solution for joint 3.

From Table 2.4, we can express θ_3 in one equation for different arm configurations:

$$\theta_3 = \phi - \beta \tag{2.3-60}$$

From Eq. (2.3-60), the sine and cosine functions of θ_3 are, respectively,

$$\sin\theta_3 = \sin(\phi - \beta) = \sin\phi\cos\beta - \cos\phi\sin\beta \tag{2.3-61}$$

$$\cos\theta_3 = \cos(\phi - \beta) = \cos\phi\cos\beta + \sin\phi\sin\beta \tag{2.3-62}$$

From Eqs. (2.3-61) and (2.3-62), and using Eqs. (2.3-57) to (2.3-59), we find the solution for θ_3:

Table 2.4 Various arm configurations for joint 3

Arm configurations	$({}^2\mathbf{p}_4)_y$	θ_3	ARM	ELBOW	ARM · ELBOW
LEFT and ABOVE arm	$\geqslant 0$	$\phi - \beta$	-1	$+1$	-1
LEFT and BELOW arm	$\leqslant 0$	$\phi - \beta$	-1	-1	$+1$
RIGHT and ABOVE arm	$\leqslant 0$	$\phi - \beta$	$+1$	$+1$	$+1$
RIGHT and BELOW arm	$\geqslant 0$	$\phi - \beta$	$+1$	-1	-1

$$\theta_3 = \tan^{-1}\left[\frac{\sin\theta_3}{\cos\theta_3}\right] \qquad -\pi \leqslant \theta_3 \leqslant \pi \tag{2.3-63}$$

Arm Solution for the Last Three Joints. Knowing the first three joint angles, we can evaluate the ${}^0\mathbf{T}_3$ matrix which is used extensively to find the solution of the last three joints. The solution of the last three joints of a PUMA robot arm can be found by setting these joints to meet the following criteria:

1. Set joint 4 such that a rotation about joint 5 will align the axis of motion of joint 6 with the given approach vector (**a** of **T**).
2. Set joint 5 to align the axis of motion of joint 6 with the approach vector.
3. Set joint 6 to align the given orientation vector (or sliding vector or $\mathbf{y}_6$) and normal vector.

Mathematically the above criteria respectively mean:

$$\mathbf{z}_4 = \frac{\pm(\mathbf{z}_3 \times \mathbf{a})}{\|\mathbf{z}_3 \times \mathbf{a}\|} \qquad \text{given } \mathbf{a} = (a_x, a_y, a_z)^T \tag{2.3-64}$$

$$\mathbf{a} = \mathbf{z}_5 \qquad \text{given } \mathbf{a} = (a_x, a_y, a_z)^T \tag{2.3-65}$$

$$\mathbf{s} = \mathbf{y}_6 \qquad \text{given } \mathbf{s} = (s_x, s_y, s_z)^T \text{ and } \mathbf{n} = (n_x, n_y, n_z)^T \tag{2.3-66}$$

In Eq. (2.3-64), the vector cross product may be taken to be positive or negative. As a result, there are two possible solutions for θ_4. If the vector cross product is zero (i.e., $\mathbf{z}_3$ is parallel to $\mathbf{a}$), it indicates the degenerate case. This happens when the axes of rotation for joint 4 and joint 6 are parallel. It indicates that at this particular arm configuration, a five-axis robot arm rather than a six-axis one would suffice.

Joint 4 solution. Both orientations of the wrist (UP and DOWN) are defined by looking at the orientation of the hand coordinate frame ($\mathbf{n}$, $\mathbf{s}$, $\mathbf{a}$) with respect to the ($\mathbf{x}_5$, $\mathbf{y}_5$, $\mathbf{z}_5$) coordinate frame. The sign of the vector cross product in Eq. (2.3-64) cannot be determined without referring to the orientation of either the $\mathbf{n}$ or $\mathbf{s}$ unit vector with respect to the $\mathbf{x}_5$ or $\mathbf{y}_5$ unit vector, respectively, which have a fixed relation with respect to the $\mathbf{z}_4$ unit vector from the assignment of the link coordinate frames. (From Fig. 2.11, we have the $\mathbf{z}_4$ unit vector pointing at the same direction as the $\mathbf{y}_5$ unit vector.)

We shall start with the assumption that the vector cross product in Eq. (2.3-64) has a positive sign. This can be indicated by an orientation indicator Ω which is defined as:

$$\Omega = \begin{cases} 0 & \text{if in the degenerate case} \\ \mathbf{s} \cdot \mathbf{y}_5 & \text{if } \mathbf{s} \cdot \mathbf{y}_5 \neq 0 \\ \mathbf{n} \cdot \mathbf{y}_5 & \text{if } \mathbf{s} \cdot \mathbf{y}_5 = 0 \end{cases} \tag{2.3-67}$$

From Fig. 2.11, $\mathbf{y}_5 = \mathbf{z}_4$, and using Eq. (2.3-64), the orientation indicator Ω can be rewritten as:

$$\Omega = \begin{cases} 0 & \text{if in the degenerate case} \\ \mathbf{s} \cdot \dfrac{(\mathbf{z}_3 \times \mathbf{a})}{\|\mathbf{z}_3 \times \mathbf{a}\|} & \text{if } \mathbf{s} \cdot (\mathbf{z}_3 \times \mathbf{a}) \neq 0 \\ \mathbf{n} \cdot \dfrac{(\mathbf{z}_3 \times \mathbf{a})}{\|\mathbf{z}_3 \times \mathbf{a}\|} & \text{if } \mathbf{s} \cdot (\mathbf{z}_3 \times \mathbf{a}) = 0 \end{cases} \tag{2.3-68}$$

If our assumption of the sign of the vector cross product in Eq. (2.3-64) is not correct, it will be corrected later using the combination of the WRIST indicator and the orientation indicator Ω. The Ω is used to indicate the initial orientation of the $\mathbf{z}_4$ unit vector (positive direction) from the link coordinate systems assignment, while the WRIST indicator specifies the user's preference of the orientation of the wrist subsystem according to the definition given in Eq. (2.3-33). If both the orientation Ω and the WRIST indicators have the same sign, then the assumption of the sign of the vector cross product in Eq. (2.3-64) is correct. Various wrist orientations resulting from the combination of the various values of the WRIST and orientation indicators are tabulated in Table 2.5.

Table 2.5 Various orientations for the wrist

Wrist orientation	$\Omega = \mathbf{s} \cdot \mathbf{y}_5$ or $\mathbf{n} \cdot \mathbf{y}_5$	WRIST	M = WRIST sign (Ω)
DOWN	$\geqslant 0$	+1	+1
DOWN	< 0	+1	−1
UP	$\geqslant 0$	−1	−1
UP	< 0	−1	+1

Again looking at the projection of the coordinate frame $(\mathbf{x}_4, \mathbf{y}_4, \mathbf{z}_4)$ on the $\mathbf{x}_3\mathbf{y}_3$ plane and from Table 2.5 and Fig. 2.23, it can be shown that the following are true (see Fig. 2.23):

$$\sin\theta_4 = -M(\mathbf{z}_4 \cdot \mathbf{x}_3) \qquad \cos\theta_4 = M(\mathbf{z}_4 \cdot \mathbf{y}_3) \tag{2.3-69}$$

where $\mathbf{x}_3$ and $\mathbf{y}_3$ are the x and y column vectors of ${}^0\mathbf{T}_3$, respectively, M = WRIST sign (Ω), and the sign function is defined as:

$$\text{sign}(x) = \begin{cases} +1 & \text{if } x \geqslant 0 \\ -1 & \text{if } x < 0 \end{cases} \tag{2.3-70}$$

Thus, the solution for θ_4 with the orientation and WRIST indicators is:

$$\theta_4 = \tan^{-1}\left[\frac{\sin\theta_4}{\cos\theta_4}\right]$$

$$= \tan^{-1}\left[\frac{M(C_1 a_y - S_1 a_x)}{M(C_1 C_{23} a_x + S_1 C_{23} a_y - S_{23} a_z)}\right] \qquad -\pi \leqslant \theta_4 \leqslant \pi \tag{2.3-71}$$

If the degenerate case occurs, any convenient value may be chosen for θ_4 as long as the orientation of the wrist (UP/DOWN) is satisfied. This can always be ensured by setting θ_4 equals to the current value of θ_4. In addition to this, the user can turn on the FLIP toggle to obtain the other solution of θ_4, that is, $\theta_4 = \theta_4 + 180°$

Joint 5 solution. To find θ_5, we use the criterion that aligns the axis of rotation of joint 6 with the approach vector (or $\mathbf{a} = \mathbf{z}_5$). Looking at the projection of the

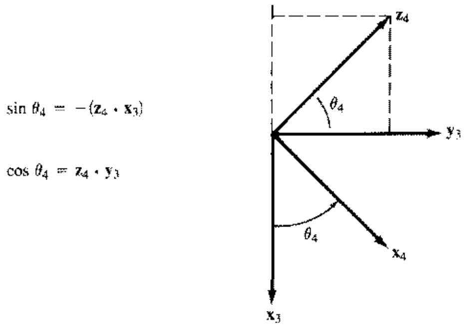


Figure 2.23 Solution for joint 4.

coordinate frame ($\mathbf{x}_5$, $\mathbf{y}_5$, $\mathbf{z}_5$) on the $\mathbf{x}_4\mathbf{y}_4$ plane, it can be shown that the following are true (see Fig. 2.24):

$$\sin\theta_5 = \mathbf{a}\cdot\mathbf{x}_4 \qquad \cos\theta_5 = -(\mathbf{a}\cdot\mathbf{y}_4) \tag{2.3-72}$$

where $\mathbf{x}_4$ and $\mathbf{y}_4$ are the x and y column vectors of ${}^0\mathbf{T}_4$, respectively, and $\mathbf{a}$ is the approach vector. Thus, the solution for θ_5 is:

$$\theta_5 = \tan^{-1}\left[\frac{\sin\theta_5}{\cos\theta_5}\right] \qquad -\pi \le \theta_5 \le \pi$$

$$= \tan^{-1}\left[\frac{(C_1C_{23}C_4 - S_1S_4)a_x + (S_1C_{23}C_4 + C_1S_4)a_y - C_4S_{23}a_z}{C_1S_{23}a_x + S_1S_{23}a_y + C_{23}a_z}\right] \tag{2.3-73}$$

If $\theta_5 \approx 0$, then the degenerate case occurs.

Joint 6 solution. Up to now, we have aligned the axis of joint 6 with the approach vector. Next, we need to align the orientation of the gripper to ease picking up the object. The criterion for doing this is to set $\mathbf{s} = \mathbf{y}_6$. Looking at the projection of the hand coordinate frame ($\mathbf{n}$, $\mathbf{s}$, $\mathbf{a}$) on the $\mathbf{x}_5\mathbf{y}_5$ plane, it can be shown that the following are true (see Fig. 2.25):

$$\sin\theta_6 = \mathbf{n}\cdot\mathbf{y}_5 \qquad \cos\theta_6 = \mathbf{s}\cdot\mathbf{y}_5 \tag{2.3-74}$$

where $\mathbf{y}_5$ is the y column vector of ${}^0\mathbf{T}_5$ and $\mathbf{n}$ and $\mathbf{s}$ are the normal and sliding vectors of ${}^0\mathbf{T}_6$, respectively. Thus, the solution for θ_6 is:

$$\theta_6 = \tan^{-1}\left[\frac{\sin\theta_6}{\cos\theta_6}\right] \qquad -\pi \le \theta_6 \le \pi$$

$$= \tan^{-1}\left[\frac{(-S_1C_4 - C_1C_{23}S_4)n_x + (C_1C_4 - S_1C_{23}S_4)n_y + (S_4S_{23})n_z}{(-S_1C_4 - C_1C_{23}S_4)s_x + (C_1C_4 - S_1C_{23}S_4)s_y + (S_4S_{23})s_z}\right] \tag{2.3-75}$$

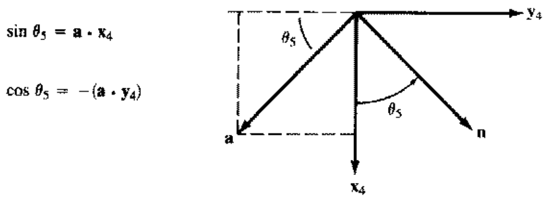


Figure 2.24 Solution for joint 5.

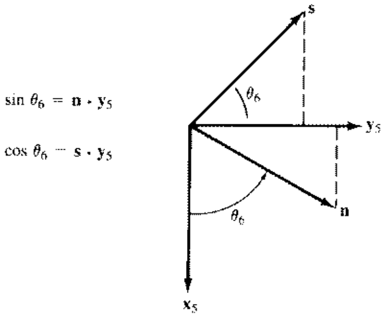


Figure 2.25 Solution for joint 6.

The above derivation of the inverse kinematics solution of a PUMA robot arm is based on the geometric interpretation of the position of the endpoint of link 3 and the hand (or tool) orientation requirement. There is one pitfall in the above derivation for θ_4, θ_5, and θ_6. The criterion for setting the axis of motion of joint 5 equal to the cross product of $\mathbf{z}_3$ and $\mathbf{a}$ may not be valid when $\sin\theta_5 \approx 0$, which means that $\theta_5 \approx 0$. In this case, the manipulator becomes *degenerate* with both the axes of motion of joints 4 and 6 aligned. In this state, only the sum of θ_4 and θ_6 is significant. If the degenerate case occurs, then we are free to choose any value for θ_4, and usually its current value is used and then we would like to have $\theta_4 + \theta_6$ equal to the total angle required to align the sliding vector $\mathbf{s}$ and the normal vector $\mathbf{n}$. If the FLIP toggle is on (i.e., FLIP $= 1$), then $\theta_4 = \theta_4 + \pi$, $\theta_5 = -\theta_5$, and $\theta_6 = \theta_6 + \pi$.

In summary, there are eight solutions to the inverse kinematics problem of a six-joint PUMA-like robot arm. The first three-joint solution (θ_1, θ_2, θ_3) positions the arm while the last three-joint solution, (θ_4, θ_5, θ_6), provides appropriate orientation for the hand. There are four solutions for the first three-joint solutions—two for the right shoulder arm configuration and two for the left shoulder arm configuration. For each arm configuration, Eqs. (2.3-47), (2.3-56), (2.3-63), (2.3-71), (2.3-73), and (2.3-75) give one set of solutions (θ_1, θ_2, θ_3, θ_4, θ_5, θ_6) and (θ_1, θ_2, θ_3, $\theta_4 + \pi$, $-\theta_5$, $\theta_6 + \pi$) (with the FLIP toggle on) gives another set of solutions.

Decision Equations for the Arm Configuration Indicators. The solution for the PUMA-like robot arm derived in the previous section is not unique and depends on the arm configuration indicators specified by the user. These arm configuration indicators (ARM, ELBOW, and WRIST) can also be determined from the joint angles. In this section, we derive the respective decision equation for each arm configuration indicator. The signed value of the decision equation (positive, zero, or negative) provides an indication of the arm configuration as defined in Eqs. (2.3-31) to (2.3-33).

For the ARM indicator, following the definition of the RIGHT/LEFT arm, a decision equation for the ARM indicator can be found to be:

$$g(\theta, \mathbf{p}) = \mathbf{z}_0 \cdot \frac{\mathbf{z}_1 \times \mathbf{p}'}{\|\mathbf{z}_1 \times \mathbf{p}'\|} = \mathbf{z}_0 \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ -\sin\theta_1 & \cos\theta_1 & 0 \\ p_x & p_y & 0 \end{vmatrix} \frac{1}{\|\mathbf{z}_1 \times \mathbf{p}'\|}$$

$$= \frac{-p_y \sin\theta_1 - p_x \cos\theta_1}{\|\mathbf{z}_1 \times \mathbf{p}'\|} \tag{2.3-76}$$

where $\mathbf{p}' = (p_x, p_y, 0)^T$ is the projection of the position vector $\mathbf{p}$ [Eq. (2.3-36)] onto the $\mathbf{x}_0 \mathbf{y}_0$ plane, $\mathbf{z}_1 = (-\sin\theta_1, \cos\theta_1, 0)^T$ from the third column vector of ${}^0\mathbf{T}_1$, and $\mathbf{z}_0 = (0, 0, 1)^T$. We have the following possibilities:

1. If $g(\theta, \mathbf{p}) > 0$, then the arm is in the RIGHT arm configuration.
2. If $g(\theta, \mathbf{p}) < 0$, then the arm is in the LEFT arm configuration.
3. If $g(\theta, \mathbf{p}) = 0$, then the criterion for finding the LEFT/RIGHT arm configuration cannot be uniquely determined. The arm is within the inner cylinder of radius d_2 in the workspace (see Fig. 2.19). In this case, it is default to the RIGHT arm (ARM $= +1$).

Since the denominator of the above decision equation is always positive, the determination of the LEFT/RIGHT arm configuration is reduced to checking the sign of the numerator of $g(\theta, \mathbf{p})$:

$$\text{ARM} = \text{sign}\,[g(\theta, \mathbf{p})] = \text{sign}\,(-p_x \cos\theta_1 - p_y \sin\theta_1) \tag{2.3-77}$$

where the sign function is defined in Eq. (2.3-70). Substituting the x and y components of $\mathbf{p}$ from Eq. (2.3-36), Eq. (2.3-77) becomes:

$$\text{ARM} = \text{sign}\,[g(\theta, \mathbf{p})] = \text{sign}\,[g(\theta)] = \text{sign}\,(-d_4 S_{23} - a_3 C_{23} - a_2 C_2) \tag{2.3-78}$$

Hence, from the decision equation in Eq. (2.3-78), one can relate its signed value to the ARM indicator for the RIGHT/LEFT arm configuration as:

$$\text{ARM} = \text{sign}\,(-d_4 S_{23} - a_3 C_{23} - a_2 C_2) = \begin{cases} +1 & \Rightarrow \text{RIGHT arm} \\ -1 & \Rightarrow \text{LEFT arm} \end{cases} \tag{2.3-79}$$

For the ELBOW arm indicator, we follow the definition of ABOVE/BELOW arm to formulate the corresponding decision equation. Using $({}^2\mathbf{p}_4)_y$ and the ARM indicator in Table 2.4, the decision equation for the ELBOW indicator is based on the sign of the y component of the position vector of ${}^2\mathbf{A}_3\,{}^3\mathbf{A}_4$ and the ARM indicator:

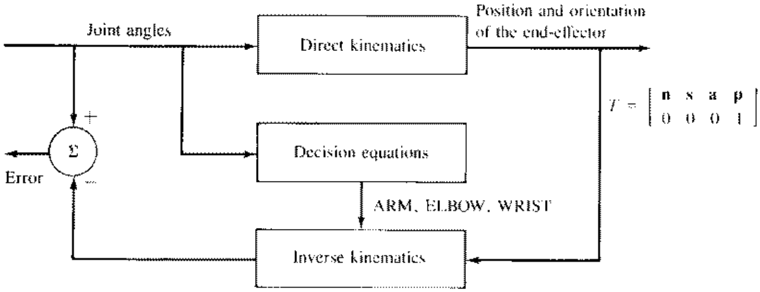


Figure 2.26 Computer simulation of joint solution.

$$\text{ELBOW} = \text{ARM} \cdot \text{sign}\,(d_4 C_3 - a_3 S_3) = \begin{cases} +1 & \Rightarrow \text{ELBOW above wrist} \\ -1 & \Rightarrow \text{ELBOW below wrist} \end{cases} \quad (2.3\text{-}80)$$

For the WRIST indicator, we follow the definition of DOWN/UP wrist to obtain a positive dot product of the $\mathbf{s}$ and $\mathbf{y}_5$ (or $\mathbf{z}_4$) unit vectors:

$$\text{WRIST} = \begin{cases} +1 & \text{if } \mathbf{s} \cdot \mathbf{z}_4 > 0 \\ -1 & \text{if } \mathbf{s} \cdot \mathbf{z}_4 < 0 \end{cases} = \text{sign}\,(\mathbf{s} \cdot \mathbf{z}_4) \quad (2.3\text{-}81)$$

If $\mathbf{s} \cdot \mathbf{z}_4 = 0$, then the WRIST indicator can be found from:

$$\text{WRIST} = \begin{cases} +1 & \text{if } \mathbf{n} \cdot \mathbf{z}_4 > 0 \\ -1 & \text{if } \mathbf{n} \cdot \mathbf{z}_4 < 0 \end{cases} = \text{sign}\,(\mathbf{n} \cdot \mathbf{z}_4) \quad (2.3\text{-}82)$$

Combining Eqs. (2.3-81) and (2.3-82), we have

$$\text{WRIST} = \begin{cases} \text{sign}\,(\mathbf{s} \cdot \mathbf{z}_4) & \text{if } \mathbf{s} \cdot \mathbf{z}_4 \neq 0 \\ \text{sign}\,(\mathbf{n} \cdot \mathbf{z}_4) & \text{if } \mathbf{s} \cdot \mathbf{z}_4 = 0 \end{cases} = \begin{cases} +1 & \Rightarrow \text{WRIST DOWN} \\ -1 & \Rightarrow \text{WRIST UP} \end{cases} \quad (2.3\text{-}83)$$

These decision equations provide a verification of the arm solution. We use them to preset the arm configuration in the direct kinematics and then use the arm configuration indicators to find the inverse kinematics solution (see Fig. 2.26).

Computer Simulation. A computer program can be written to verify the validity of the inverse solution of the PUMA robot arm shown in Fig. 2.11. The software initially generates all the locations in the workspace of the robot within the joint angles limits. They are inputed into the direct kinematics routine to obtain the arm matrix **T**. These joint angles are also used to compute the decision equations to obtain the three arm configuration indicators. These indicators together with the arm matrix **T** are fed into the inverse solution routine to obtain the joint angle solution which should agree to the joint angles fed into the direct kinematics routine previously. A computer simulation block diagram is shown in Fig. 2.26.

2.4 CONCLUDING REMARKS

We have discussed both direct and inverse kinematics in this chapter. The parameters of robot arm links and joints are defined and a 4×4 homogeneous transformation matrix is introduced to describe the location of a link with respect to a fixed coordinate frame. The forward kinematic equations for a six-axis PUMA-like robot arm are derived.

The inverse kinematics problem is introduced and the inverse transform technique is used to determine the Euler angle solution. This technique can also be used to find the inverse solution of simple robots. However, it does not provide geometric insight to the problem. Thus, a geometric approach is introduced to find the inverse solution of a six-joint robot arm with rotary joints. The inverse solution is determined with the assistance of three arm configuration indicators (ARM, ELBOW, and WRIST). There are eight solutions to a six-joint PUMA-like robot arm—four solutions for the first three joints and for each arm configuration, two more solutions for the last three joints. The validity of the forward and inverse kinematics solution can be verified by computer simulation. The geometric approach, with appropriate modification and adjustment, can be generalized to other simple industrial robots with rotary joints. The kinematics concepts covered in this chapter will be used extensively in Chap. 3 for deriving the equations of motion that describe the dynamic behavior of a robot arm.

REFERENCES

Further reading on matrices can be found in Bellman [1970], Frazer et al. [1960], and Gantmacher [1959]. Utilization of matrices to describe the location of a rigid mechanical link can be found in the paper by Denavit and Hartenberg [1955] and in their book (Hartenberg and Denavit [1964]). Further reading about homogeneous coordinates can be found in Duda and Hart [1973] and Newman and Sproull [1979]. The discussion on kinematics is an extension of a paper by Lee [1982]. More discussion in kinematics can be found in Hartenberg and Denavit [1964] and Suh and Radcliffe [1978]. Although matrix representation of linkages presents a systematic approach to solving the forward kinematics problem, the vector approach to the kinematics problem presents a more concise representation of linkages. This is discussed in a paper by Chase [1963]. Other robotics books that discuss the kinematics problem are Paul [1981], Lee, Gonzalez and Fu [1986], and Snyder [1985].

Pieper [1968] in his doctoral dissertation utilized an algebraic approach to solve the inverse kinematics problem. The discussion of the inverse transform technique in finding the arm solution was based on the paper by Paul et al. [1981]. The geometric approach to solving the inverse kinematics for a six-link manipulator with rotary joints was based on the paper by Lee and Ziegler [1984]. The arm solution of a Stanford robot arm can be found in a report by Lewis [1974]. Other techniques in solving the inverse kinematics can be found in articles by Denavit [1956], Kohli and Soni [1975], Yang and Freudenstein [1964], Yang [1969], Yuan and Freudenstein [1971], Duffy and Rooney [1975], Uicker et al. [1964]. Finally, the tutorial book edited by Lee, Gonzalez, and Fu [1986] contains numerous recent papers on robotics.

PROBLEMS

2.1 What is the rotation matrix for a rotation of 30° about the *OZ* axis, followed by a rotation of 60° about the *OX* axis, followed by a rotation of 90° about the *OY* axis?

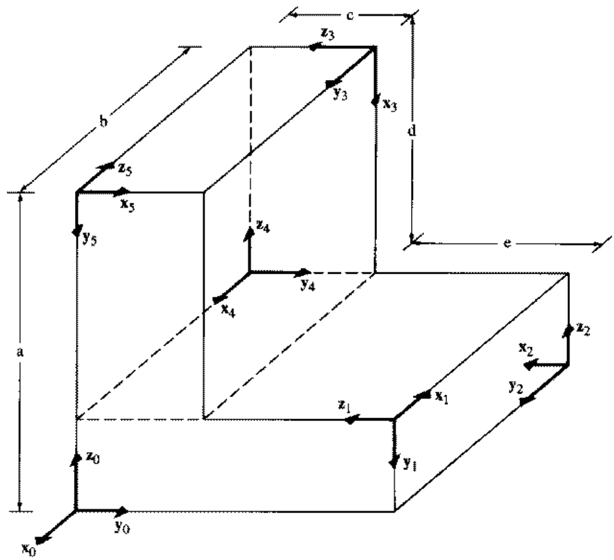
2.2 What is the rotation matrix for a rotation of ϕ angle about the *OX* axis, followed by a rotation of ψ angle about the *OW* axis, followed by a rotation of θ angle about the *OY* axis?

2.3 Find another sequence of rotations that is different from Prob. 2.2, but which results in the same rotation matrix.

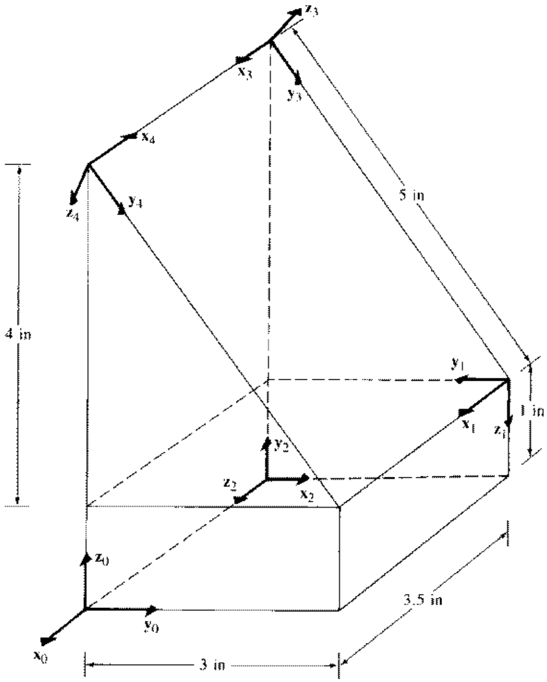
2.4 Derive the formula for $\sin(\phi + \theta)$ and $\cos(\phi + \theta)$ by expanding symbolically two rotations of ϕ and θ using the rotation matrix concepts discussed in this chapter.

2.5 Determine a **T** matrix that represents a rotation of α angle about the *OX* axis, followed by a translation of b unit of distance along the *OZ* axis, followed by a rotation of ϕ angle about the *OV* axis.

2.6 For the figure shown below, find the 4 × 4 homogeneous transformation matrices ${}^{i-1}\mathbf{A}_i$ and ${}^{0}\mathbf{A}_i$ for $i = 1, 2, 3, 4, 5$.



2.7 For the figure shown below, find the 4 × 4 homogeneous transformation matrices ${}^{i-1}\mathbf{A}_i$ and ${}^{0}\mathbf{A}_i$ for $i = 1, 2, 3, 4$.



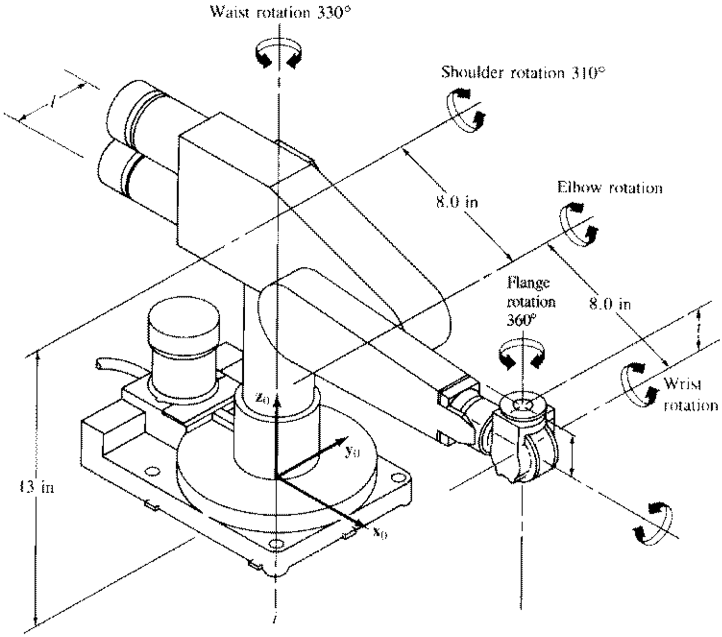
2.8 A robot workstation has been set up with a TV camera, as shown in the example in Sec. 2.2.11. The camera can see the origin of the base coordinate system where a six-link robot arm is attached, and also the center of a cube to be manipulated by the robot. If a local coordinate system has been established at the center of the cube, then this object, as seen by the camera, can be represented by a homogeneous transformation matrix $\mathbf{T}_1$. Also, the origin of the base coordinate system as seen by the camera can be expressed by a homogeneous transformation matrix $\mathbf{T}_2$, where

$$\mathbf{T}_1 = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 10 \\ 0 & 0 & -1 & 9 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \mathbf{T}_2 = \begin{bmatrix} 1 & 0 & 0 & -10 \\ 0 & -1 & 0 & 20 \\ 0 & 0 & -1 & 10 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

(*a*) Unfortunately, after the equipment has been set up and these coordinate systems have been taken, someone rotates the camera 90° about the **z** axis of the camera. What is the position/orientation of the camera with respect to the robot's base coordinate system? (*b*) After you have calculated the answer for question (*a*), the same person rotated the object 90° about the **x** axis of the object and translated it 4 units of distance along the rotated **y** axis. What is the position/orientation of the object with respect to the robot's base coordinate system? To the rotated camera coordinate system?

2.9 We have discussed a geometric approach for finding the inverse kinematic solution of a PUMA robot arm. Find the computational requirements of the joint solution in terms of multiplication and addition operations and the number of transcendental calls (if the same term appears twice, the computation should be counted once only).

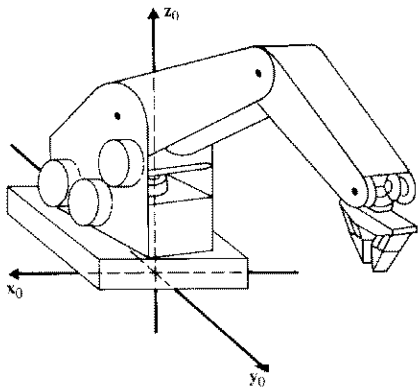
2.10 Establish orthonormal link coordinate systems $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ for $i = 1, 2, \ldots, 6$ for the PUMA 260 robot arm shown in the figure below and complete the table.



PUMA robot arm link coordinate parameters

Joint i	θ_i	α_i	a_i	d_i
1				
2				
3				
4				
5				
6				

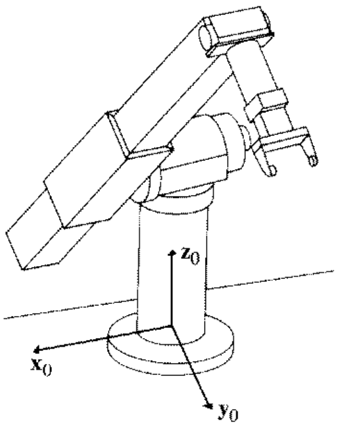
2.11 Establish orthonormal link coordinate systems ($\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$) for i = 1, 2, . . . , 5 for the MINIMOVER robot arm shown in the figure below and complete the table.



MINIMOVER robot arm link coordinate parameters

Joint i	θ_i	α_i	a_i	d_i
1				
2				
3				
4				
5				

2.12 A Stanford robot arm has moved to the position shown in the figure below. The joint variables at this position are: $\mathbf{q}$ = (90°, −120°, 22 cm, 0°, 70°, 90°)^T. Establish the orthonormal link coordinate systems ($\mathbf{x}_i$, $\mathbf{y}_i$, $\mathbf{z}_i$) for i = 1, 2, . . . , 6, for this arm and complete the table.



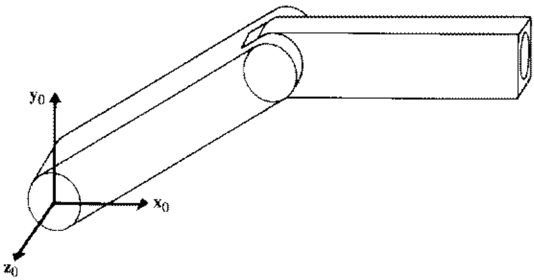
Stanford arm link coordinate parameters

Joint i	θ_i	α_i	a_i	d_i
1				
2				
3				
4				
5				
6				

2.13 Using the six $^{i-1}\mathbf{A}_i$ matrices (i = 1, 2, . . . , 6) of the PUMA robot arm in Fig. 2.13, find its *position* error at the end of link 3 due to the measurement error of the first three joint angles ($\Delta\theta_1$, $\Delta\theta_2$, $\Delta\theta_3$). A first-order approximation solution is adequate.

2.14 Repeat Prob. 2.13 for the Stanford arm shown in Fig. 2.12.

2.15 A two degree-of-freedom manipulator is shown in the figure below. Given that the length of each link is 1 m, establish its link coordinate frames and find ${}^0\mathbf{A}_1$ and ${}^1\mathbf{A}_2$. Find the inverse kinematics solution for this manipulator.



2.16 For the PUMA robot arm shown in Fig. 2.11, assume that we have found the first three joint solution $(\theta_1, \theta_2, \theta_3)$ correctly and that we are given ${}^{i-1}\mathbf{A}_i$, $i = 1, 2, \ldots, 6$ and ${}^0\mathbf{T}_6$. Use the inverse transformation technique to find the solution for the last three joint angles $(\theta_4, \theta_5, \theta_6)$. Compare your solution with the one given in Eqs. (2.3-71), (2.3-73), and (2.3-75).

2.17 For the Stanford robot arm shown in Fig. 2.12, derive the solution of the first three joint angles. You may use any method that you feel comfortable with.

2.18 Repeat Prob. 2.16 for the Stanford arm shown in Fig. 2.12.