CHAPTER

THREE

ROBOT ARM DYNAMICS

The inevitable comes to pass by effort.
Oliver Wendell Holmes

3.1 INTRODUCTION

Robot arm dynamics deals with the mathematical formulations of the equations of robot arm motion. The dynamic equations of motion of a manipulator are a set of mathematical equations describing the dynamic behavior of the manipulator. Such equations of motion are useful for computer simulation of the robot arm motion, the design of suitable control equations for a robot arm, and the evaluation of the kinematic design and structure of a robot arm. In this chapter, we shall concentrate on the formulation, characteristics, and properties of the dynamic equations of motion that are suitable for control purposes. The purpose of manipulator control is to maintain the dynamic response of a computer-based manipulator in accordance with some prespecified system performance and desired goals. In general, the dynamic performance of a manipulator directly depends on the efficiency of the control algorithms and the dynamic model of the manipulator. The control problem consists of obtaining dynamic models of the physical robot arm system and then specifying corresponding control laws or strategies to achieve the desired system response and performance. This chapter deals mainly with the former part of the manipulator control problem; that is, modeling and evaluating the dynamical properties and behavior of computer-controlled robots.

The actual dynamic model of a robot arm can be obtained from known physical laws such as the laws of newtonian mechanics and lagrangian mechanics. This leads to the development of the dynamic equations of motion for the various articulated joints of the manipulator in terms of specified geometric and inertial parameters of the links. Conventional approaches like the Lagrange-Euler (L-E) and Newton-Euler (N-E) formulations could then be applied systematically to develop the actual robot arm motion equations. Various forms of robot arm motion equations describing the rigid-body robot arm dynamics are obtained from these two formulations, such as Uicker's Lagrange-Euler equations (Uicker [1965], Bejczy [1974]), Hollerbach's Recursive-Lagrange (R-L) equations (Hollerbach [1980]), Luh's Newton-Euler equations (Luh et al. [1980*a*]), and Lee's generalized d'Alembert (G-D) equations (Lee et al. [1983]). These motion equations are "equivalent" to each other in the sense that they describe the dynamic behavior of the same physical robot manipulator. However, the structure of these equations may differ as they are obtained for various reasons and purposes. Some are obtained to achieve fast computation time in evaluating the nominal joint torques in servoing a manipulator, others are obtained to facilitate control analysis and synthesis, and still others are obtained to improve computer simulation of robot motion.

The derivation of the dynamic model of a manipulator based on the L-E formulation is simple and systematic. Assuming rigid body motion, the resulting equations of motion, excluding the dynamics of electronic control devices, backlash, and gear friction, are a set of second-order coupled nonlinear differential equations. Bejczy [1974], using the 4×4 homogeneous transformation matrix representation of the kinematic chain and the lagrangian formulation, has shown that the dynamic motion equations for a six-joint Stanford robot arm are highly nonlinear and consist of inertia loading, coupling reaction forces between joints (Coriolis and centrifugal), and gravity loading effects. Furthermore, these torques/forces depend on the manipulator's physical parameters, instantaneous joint configuration, joint velocity and acceleration, and the load it is carrying. The L-E equations of motion provide explicit state equations for robot dynamics and can be utilized to analyze and design advanced joint-variable space control strategies. To a lesser extent, they are being used to solve for the *forward dynamics* problem, that is, given the desired torques/forces, the dynamic equations are used to solve for the joint accelerations which are then integrated to solve for the generalized coordinates and their velocities; or for the *inverse dynamics* problem, that is, given the desired generalized coordinates and their first two time derivatives, the generalized forces/torques are computed. In both cases, it may be required to compute the dynamic coefficients D_{ik}, h_{ikm}, and c_i defined in Eqs. (3.2-31), (3.2-33), and (3.2-34), respectively. Unfortunately, the computation of these coefficients requires a fair amount of arithmetic operations. Thus, the L-E equations are very difficult to utilize for real-time control purposes unless they are simplified.

As an alternative to deriving more efficient equations of motion, attention was turned to develop efficient algorithms for computing the generalized forces/torques based on the N-E equations of motion (Armstrong [1979], Orin et al. [1979], Luh et al. [1980*a*]). The derivation is simple, but messy, and involves vector cross-product terms. The resulting dynamic equations, excluding the dynamics of the control device, backlash, and gear friction, are a set of forward and backward recursive equations. This set of recursive equations can be applied to the robot links sequentially. The forward recursion propagates kinematics information—such as linear velocities, angular velocities, angular accelerations, and linear accelerations at the center of mass of each link—from the inertial coordinate frame to the hand coordinate frame. The backward recursion propagates the forces and moments exerted on each link from the end-effector of the manipulator to the base reference frame. The most significant result of this formulation is that the computation time of the generalized forces/torques is found linearly proportional to the number of joints of the robot arm and independent of the robot arm configuration. With this algorithm, one can implement simple real-time control of a robot arm in the joint-variable space.

The inefficiency of the L-E equations of motion arises partly from the 4×4 homogeneous matrices describing the kinematic chain, while the efficiency of the N-E formulation is based on the vector formulation and its recursive nature. To further improve the computation time of the lagrangian formulation, Hollerbach [1980] has exploited the recursive nature of the lagrangian formulation. However, the recursive equations destroy the "structure" of the dynamic model which is quite useful in providing insight for designing the controller in state space. For state-space control analysis, one would like to obtain an explicit set of closed-form differential equations (state equations) that describe the dynamic behavior of a manipulator. In addition, the interaction and coupling reaction forces in the equations should be easily identified so that an appropriate controller can be designed to compensate for their effects (Huston and Kelly [1982]). Another approach for obtaining an efficient set of explicit equations of motion is based on the generalized d'Alembert principle to derive the equations of motion which are expressed explicitly in vector-matrix form suitable for control analysis. In addition to allowing faster computation of the dynamic coefficients than the L-E equations of motion, the G-D equations of motion explicitly identify the contributions of the *translational* and *rotational* effects of the links. Such information is useful for designing a controller in state space. The computational efficiency is achieved from a compact formulation using Euler transformation matrices (or rotation matrices) and relative position vectors between joints.

In this chapter, the L-E, N-E, and G-D equations of robot arm motion are derived and discussed, and the motion equations of a two-link manipulator are worked out to illustrate the use of these equations. Since the computation of the dynamic coefficients of the equations of motion is important both in control analysis and computer simulation, the mathematical operations and their computational issues for these motion equations are tabulated. The computation of the applied forces/torques from the generalized d'Alembert equations of motion is of order $O(n^3)$, while the L-E equations are of order $O(n^4)$ [or of order $O(n^3)$ if optimized] and the N-E equations are of order $O(n)$, where n is the number of degrees of freedom of the robot arm.

3.2 LAGRANGE-EULER FORMULATION

The general motion equations of a manipulator can conveniently be expressed through the direct application of the Lagrange-Euler formulation to nonconservative systems. Many investigators utilize the Denavit-Hartenberg matrix representation to describe the spatial displacement between the neighboring link coordinate frames to obtain the link kinematic information, and they employ the lagrangian dynamics technique to derive the dynamic equations of a manipulator. The direct application of the lagrangian dynamics formulation, together with the Denavit-Hartenberg link coordinate representation, results in a convenient and compact algorithmic description of the manipulator equations of motion. The algorithm is expressed by matrix operations and facilitates both analysis and computer implementation. The evaluation of the dynamic and control equations in functionally explicit terms will be based on the compact matrix algorithm derived in this section.

The derivation of the dynamic equations of an n degrees of freedom manipulator is based on the understanding of:

1. The 4×4 homogeneous coordinate transformation matrix, ${}^{i-1}\mathbf{A}_i$, which describes the spatial relationship between the ith and the $(i-1)$th link coordinate frames. It relates a point fixed in link i expressed in homogeneous coordinates with respect to the ith coordinate system to the $(i-1)$th coordinate system.
2. The Lagrange-Euler equation

$$\frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i} = \tau_i \qquad i = 1, 2, \ldots, n \tag{3.2-1}$$

where

L = lagrangian function = kinetic energy K − potential energy P
K = total kinetic energy of the robot arm
P = total potential energy of the robot arm
q_i = generalized coordinates of the robot arm
$\dot{q}_i$ = first time derivative of the generalized coordinate, q_i
τ_i = generalized force (or torque) applied to the system at joint i to drive link i

From the above Lagrange-Euler equation, one is required to properly choose a set of *generalized coordinates* to describe the system. Generalized coordinates are used as a convenient set of coordinates which completely describe the location (position and orientation) of a system with respect to a reference coordinate frame. For a simple manipulator with rotary-prismatic joints, various sets of generalized coordinates are available to describe the manipulator. However, since the angular positions of the joints are readily available because they can be measured by potentiometers or encoders or other sensing devices, they provide a natural correspondence with the generalized coordinates. This, in effect, corresponds to the generalized coordinates with the joint variable defined in each of the 4×4 link coordinate transformation matrices. Thus, in the case of a rotary joint, $q_i \equiv \theta_i$, the joint angle span of the joint; whereas for a prismatic joint, $q_i \equiv d_i$, the distance traveled by the joint.

The following derivation of the equations of motion of an n degrees of freedom manipulator is based on the homogeneous coordinate transformation matrices developed in Chap. 2.

3.2.1 Joint Velocities of a Robot Manipulator

The Lagrange-Euler formulation requires knowledge of the kinetic energy of the physical system, which in turn requires knowledge of the velocity of each joint. In this section, the velocity of a point fixed in link i will be derived and the effects of the motion of other joints on all the points in this link will be explored.

With reference to Fig. 3.1, let ${}^i\mathbf{r}_i$ be a point fixed and at rest in a link i and expressed in homogeneous coordinates with respect to the ith link coordinate frame,

$$ {}^i\mathbf{r}_i = \begin{bmatrix} x_i \\ y_i \\ z_i \\ 1 \end{bmatrix} = (x_i, y_i, z_i, 1)^T \tag{3.2-2} $$

Let ${}^0\mathbf{r}_i$ be the same point ${}^i\mathbf{r}_i$ with respect to the base coordinate frame, ${}^{i-1}\mathbf{A}_i$ the homogeneous coordinate transformation matrix which relates the spatial displacement of the ith link coordinate frame to the $(i-1)$th link coordinate frame, and ${}^0\mathbf{A}_i$ the coordinate transformation matrix which relates the ith coordinate frame to the base coordinate frame; then ${}^0\mathbf{r}_i$ is related to the point ${}^i\mathbf{r}_i$ by

$$ {}^0\mathbf{r}_i = {}^0\mathbf{A}_i \, {}^i\mathbf{r}_i \tag{3.2-3} $$

where

$$ {}^0\mathbf{A}_i = {}^0\mathbf{A}_1 \, {}^1\mathbf{A}_2 \cdots {}^{i-1}\mathbf{A}_i \tag{3.2-4} $$

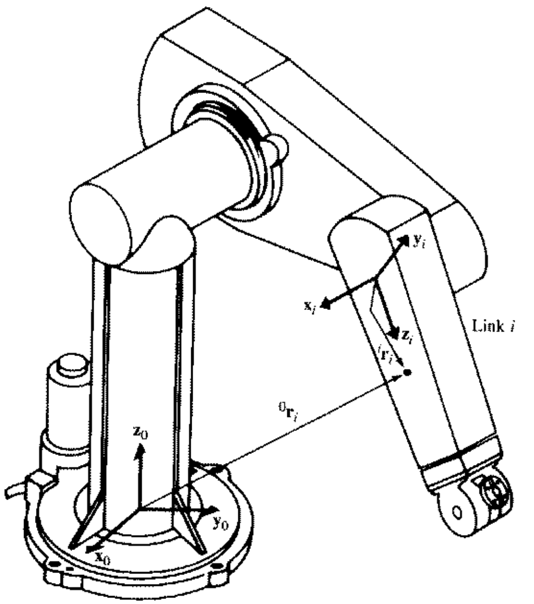


Figure 3.1 A point ${}^i\mathbf{r}_i$ in link i.

If joint i is revolute, it follows from Eq. (2.2-29) that the general form of ${}^{i-1}\mathbf{A}_i$ is given by

$$
{}^{i-1}\mathbf{A}_i = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & a_i \cos\theta_i \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & a_i \sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.2-5}
$$

or, if joint i is prismatic, from Eq. (2.2-31), the general form of ${}^{i-1}\mathbf{A}_i$ is

$$
{}^{i-1}\mathbf{A}_i = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i \sin\theta_i & \sin\alpha_i \sin\theta_i & 0 \\ \sin\theta_i & \cos\alpha_i \cos\theta_i & -\sin\alpha_i \cos\theta_i & 0 \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3.2-6}
$$

In general, all the nonzero elements in the matrix ${}^0\mathbf{A}_i$ are a function of $(\theta_1, \theta_2, \ldots, \theta_i)$, and α_i, a_i, d_i are known parameters from the kinematic structure of the arm and θ_i or d_i is the joint variable of joint i. In order to derive the equations of motion that are applicable to both revolute and prismatic joints, we shall use the variable q_i to represent the generalized coordinate of joint i which is either θ_i (for a rotary joint) or d_i (for a prismatic joint).

Since the point ${}^i\mathbf{r}_i$ is at rest in link i, and assuming rigid body motion, other points as well as the point ${}^i\mathbf{r}_i$ fixed in the link i and expressed with respect to the ith coordinate frame will have zero velocity with respect to the ith coordinate frame (which is not an inertial frame). The velocity of ${}^i\mathbf{r}_i$ expressed in the base coordinate frame (which is an inertial frame) can be expressed as

$$
\begin{aligned}
{}^0\mathbf{v}_i \equiv \mathbf{v}_i &= \frac{d}{dt}({}^0\mathbf{r}_i) = \frac{d}{dt}({}^0\mathbf{A}_i\, {}^i\mathbf{r}_i) \\
&= {}^0\dot{\mathbf{A}}_1\, {}^1\mathbf{A}_2 \cdots {}^{i-1}\mathbf{A}_i\, {}^i\mathbf{r}_i + {}^0\mathbf{A}_1\, {}^1\dot{\mathbf{A}}_2 \cdots {}^{i-1}\mathbf{A}_i\, {}^i\mathbf{r}_i + \cdots \\
&\quad + {}^0\mathbf{A}_1 \cdots {}^{i-1}\dot{\mathbf{A}}_i\, {}^i\mathbf{r}_i + {}^0\mathbf{A}_i\, {}^i\dot{\mathbf{r}}_i = \left(\sum_{j=1}^{i} \frac{\partial\, {}^0\mathbf{A}_i}{\partial q_j} \dot{q}_j \right) {}^i\mathbf{r}_i
\end{aligned} \tag{3.2-7}
$$

The above compact form is obtained because ${}^i\dot{\mathbf{r}}_i = 0$. The partial derivative of ${}^0\mathbf{A}_i$ with respect to q_j can be easily calculated with the help of a matrix $\mathbf{Q}_i$ which, for a revolute joint, is defined as

$$
\mathbf{Q}_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.2-8a}
$$

and, for a prismatic joint, as

$$\mathbf{Q}_i = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \tag{3.2-8b}$$

It then follows that

$$\frac{\partial\, {}^{i-1}\mathbf{A}_i}{\partial q_i} = \mathbf{Q}_i\, {}^{i-1}\mathbf{A}_i \tag{3.2-9}$$

For example, for a robot arm with all rotary joints, $q_i = \theta_i$, and using Eq. (3.2-5),

$$\frac{\partial\, {}^{i-1}\mathbf{A}_i}{\partial \theta_i} = \begin{bmatrix} -\sin\theta_i & -\cos\alpha_i\cos\theta_i & \sin\alpha_i\cos\theta_i & -a_i\sin\theta_i \\ \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i & a_i\cos\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i & a_i\sin\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i & d_i \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \mathbf{Q}_i\, {}^{i-1}\mathbf{A}_i$$

Hence, for $i = 1, 2, \ldots, n$,

$$\frac{\partial\, {}^0\mathbf{A}_i}{\partial q_j} = \begin{cases} {}^0\mathbf{A}_1\, {}^1\mathbf{A}_2 \cdots {}^{j-2}\mathbf{A}_{j-1}\mathbf{Q}_j\, {}^{j-1}\mathbf{A}_j \cdots {}^{i-1}\mathbf{A}_i & \text{for } j \leq i \\ 0 & \text{for } j > i \end{cases} \tag{3.2-10}$$

Eq. (3.2-10) can be interpreted as the effect of the motion of joint j on all the points on link i. In order to simplify notations, let us define $\mathbf{U}_{ij} \triangleq \partial\, {}^0\mathbf{A}_i / \partial q_j$, then Eq. (3.2-10) can be written as follows for $i = 1, 2, \ldots, n$,

$$\mathbf{U}_{ij} = \begin{cases} {}^0\mathbf{A}_{j-1}\, \mathbf{Q}_j\, {}^{j-1}\mathbf{A}_i & \text{for } j \leq i \\ 0 & \text{for } j > i \end{cases} \tag{3.2-11}$$

Using this notation, $\mathbf{v}_i$ can be expressed as

$$\mathbf{v}_i = \left(\sum_{j=1}^{i} \mathbf{U}_{ij}\dot{q}_j \right) {}^i\mathbf{r}_i \tag{3.2-12}$$

It is worth pointing out that the partial derivative of ${}^{i-1}\mathbf{A}_i$ with respect to q_i results in a matrix that does not retain the structure of a homogeneous coordinate transformation matrix. For a rotary joint, the effect of premultiplying ${}^{i-1}\mathbf{A}_i$ by $\mathbf{Q}_i$ is equivalent to interchanging the elements of the first two rows of ${}^{i-1}\mathbf{A}_i$, negating all the elements of the first row, and zeroing out all the elements of the third and fourth rows. For a prismatic joint, the effect is to replace the elements of the third row with the fourth row of ${}^{i-1}\mathbf{A}_i$ and zeroing out the elements in the other rows. The advantage of using the $\mathbf{Q}_i$ matrices is that we can still use the ${}^{i-1}\mathbf{A}_i$ matrices and apply the above operations to ${}^{i-1}\mathbf{A}_i$ when premultiplying it with the $\mathbf{Q}_i$.

Next, we need to find the interaction effects between joints as

$$\frac{\partial \mathbf{U}_{ij}}{\partial q_k} \triangleq \mathbf{U}_{ijk} = \begin{cases} {}^0\mathbf{A}_{j-1}\mathbf{Q}_j\,{}^{j-1}\mathbf{A}_{k-1}\mathbf{Q}_k\,{}^{k-1}\mathbf{A}_i & i \geqslant k \geqslant j \\ {}^0\mathbf{A}_{k-1}\mathbf{Q}_k\,{}^{k-1}\mathbf{A}_{j-1}\mathbf{Q}_j\,{}^{j-1}\mathbf{A}_i & i \geqslant j \geqslant k \\ 0 & i < j \text{ or } i < k \end{cases} \tag{3.2-13}$$

For example, for a robot arm with all rotary joints, $i = j = k = 1$ and $q_1 = \theta_1$, so that

$$\frac{\partial \mathbf{U}_{11}}{\partial \theta_1} = \frac{\partial}{\partial \theta_1}(\mathbf{Q}_1\,{}^0\mathbf{A}_1) = \mathbf{Q}_1\mathbf{Q}_1\,{}^0\mathbf{A}_1$$

Eq. (3.2-13) can be interpreted as the interaction effects of the motion of joint j and joint k on all the points on link i.

3.2.2 Kinetic Energy of a Robot Manipulator

After obtaining the joint velocity of each link, we need to find the kinetic energy of link i. Let K_i be the kinetic energy of link i, $i = 1, 2, \ldots, n$, as expressed in the base coordinate system, and let dK_i be the kinetic energy of a particle with differential mass dm in link i; then

$$\begin{aligned} dK_i &= \tfrac{1}{2}(\dot{x}_i^2 + \dot{y}_i^2 + \dot{z}_i^2)\,dm \\ &= \tfrac{1}{2}\,\text{trace}\,(\mathbf{v}_i\mathbf{v}_i^T)\,dm = \tfrac{1}{2}\,\text{Tr}\,(\mathbf{v}_i\mathbf{v}_i^T)\,dm \end{aligned} \tag{3.2-14}$$

where a trace operator† instead of a vector dot product is used in the above equation to form the tensor from which the link inertia matrix (or pseudo-inertia matrix) $\mathbf{J}_i$ can be obtained. Substituting $\mathbf{v}_i$ from Eq. (3.2-12), the kinetic energy of the differential mass is

† $\text{Tr}\,\mathbf{A} \triangleq \sum_{i=1}^{n} a_{ii}$.

$$
\begin{aligned}
dK_i &= \tfrac{1}{2}\,\mathrm{Tr}\left[\sum_{p=1}^{i} \mathbf{U}_{ip}\dot{q}_p\,{}^i\mathbf{r}_i \left(\sum_{r=1}^{i} \mathbf{U}_{ir}\dot{q}_r\,{}^i\mathbf{r}_i\right)^T\right] dm \\
&= \tfrac{1}{2}\,\mathrm{Tr}\left[\sum_{p=1}^{i}\sum_{r=1}^{i} \mathbf{U}_{ip}\,{}^i\mathbf{r}_i\,{}^i\mathbf{r}_i^T\mathbf{U}_{ir}^T\dot{q}_p\dot{q}_r\right] dm \\
&= \tfrac{1}{2}\,\mathrm{Tr}\left[\sum_{p=1}^{i}\sum_{r=1}^{i} \mathbf{U}_{ip}({}^i\mathbf{r}_i\,dm\,{}^i\mathbf{r}_i^T)\,\mathbf{U}_{ir}^T\dot{q}_p\dot{q}_r\right] \qquad (3.2\text{-}15)
\end{aligned}
$$

The matrix $\mathbf{U}_{ij}$ is the rate of change of the points $({}^i\mathbf{r}_i)$ on link i relative to the base coordinate frame as q_j changes. It is constant for all points on link i and independent of the mass distribution of the link i. Also $\dot{q}_i$ are independent of the mass distribution of link i, so summing all the kinetic energies of all links and putting the integral inside the bracket,

$$
K_i = \int dK_i = \tfrac{1}{2}\mathrm{Tr}\left[\sum_{p=1}^{i}\sum_{r=1}^{i} \mathbf{U}_{ip}\left(\int {}^i\mathbf{r}_i\,{}^i\mathbf{r}_i^T dm\right)\mathbf{U}_{ir}^T\dot{q}_p\dot{q}_r\right] \qquad (3.2\text{-}16)
$$

The integral term inside the bracket is the inertia of all the points on link i, hence,

$$
\mathbf{J}_i = \int {}^i\mathbf{r}_i\,{}^i\mathbf{r}_i^T dm = \begin{bmatrix} \int x_i^2\,dm & \int x_iy_i\,dm & \int x_iz_i\,dm & \int x_i\,dm \\ \int x_iy_i\,dm & \int y_i^2\,dm & \int y_iz_i\,dm & \int y_i\,dm \\ \int x_iz_i\,dm & \int y_iz_i\,dm & \int z_i^2\,dm & \int z_i\,dm \\ \int x_i\,dm & \int y_i\,dm & \int z_i\,dm & \int dm \end{bmatrix} \qquad (3.2\text{-}17)
$$

where ${}^i\mathbf{r}_i = (x_i, y_i, z_i, 1)^T$ as defined before. If we use inertia tensor I_{ij} which is defined as

$$
I_{ij} = \int \left[\delta_{ij}\left(\sum_k x_k^2\right) - x_ix_j\right] dm
$$

where the indices i, j, k indicate principal axes of the ith coordinate frame and δ_{ij} is the so-called Kronecker delta, then $\mathbf{J}_i$ can be expressed in inertia tensor as

$$
\mathbf{J}_i = \begin{bmatrix} \dfrac{-I_{xx}+I_{yy}+I_{zz}}{2} & I_{xy} & I_{xz} & m_i\bar{x}_i \\ I_{xy} & \dfrac{I_{xx}-I_{yy}+I_{zz}}{2} & I_{yz} & m_i\bar{y}_i \\ I_{xz} & I_{yz} & \dfrac{I_{xx}+I_{yy}-I_{zz}}{2} & m_i\bar{z}_i \\ m_i\bar{x}_i & m_i\bar{y}_i & m_i\bar{z}_i & m_i \end{bmatrix} \qquad (3.2\text{-}18)
$$

or using the radius of gyration of the rigid body m_i in the $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ coordinate system, $\mathbf{J}_i$ can be expressed as

$$\mathbf{J}_i = m_i \begin{bmatrix} \dfrac{-k_{i11}^2 + k_{i22}^2 + k_{i33}^2}{2} & k_{i12}^2 & k_{i13}^2 & \bar{x}_i \\ k_{i12}^2 & \dfrac{k_{i11}^2 - k_{i22}^2 + k_{i33}^2}{2} & k_{i23}^2 & \bar{y}_i \\ k_{i13}^2 & k_{i23}^2 & \dfrac{k_{i11}^2 + k_{i22}^2 - k_{i33}^2}{2} & \bar{z}_i \\ \bar{x}_i & \bar{y}_i & \bar{z}_i & 1 \end{bmatrix} \tag{3.2-19}$$

where k_{i23} is the radius of gyration of link i about the **yz** axes and ${}^i\bar{\mathbf{r}}_i = (\bar{x}_i, \bar{y}_i, \bar{z}_i, 1)^T$ is the center of mass vector of link i from the ith link coordinate frame and expressed in the ith link coordinate frame. Hence, the total kinetic energy K of a robot arm is

$$K = \sum_{i=1}^{n} K_i = \tfrac{1}{2} \sum_{i=1}^{n} \mathrm{Tr} \left[\sum_{p=1}^{i} \sum_{r=1}^{i} \mathbf{U}_{ip} \mathbf{J}_i \mathbf{U}_{ir}^T \dot{q}_p \, \dot{q}_r \right]$$

$$= \tfrac{1}{2} \sum_{i=1}^{n} \sum_{p=1}^{i} \sum_{r=1}^{i} [\mathrm{Tr}\,(\mathbf{U}_{ip} \, \mathbf{J}_i \, \mathbf{U}_{ir}^T) \dot{q}_p \, \dot{q}_r \,] \tag{3.2-20}$$

which is a *scalar* quantity. Note that the $\mathbf{J}_i$ are dependent on the mass distribution of link i and not their position or rate of motion and are expressed with respect to the ith coordinate frame. Hence, the $\mathbf{J}_i$ need be computed only *once* for evaluating the kinetic energy of a robot arm.

3.2.3 Potential Energy of a Robot Manipulator

Let the total potential energy of a robot arm be P and let each of its link's potential energy be P_i:

$$P_i = -\, m_i \mathbf{g}\, {}^0\bar{\mathbf{r}}_i = -m_i \mathbf{g} ({}^0\mathbf{A}_i \, {}^i\bar{\mathbf{r}}_i) \qquad i = 1, 2, \ldots, n \tag{3.2-21}$$

and the total potential energy of the robot arm can be obtained by summing all the potential energies in each link,

$$P = \sum_{i=1}^{n} P_i = \sum_{i=1}^{n} -\, m_i \mathbf{g}\, ({}^0\mathbf{A}_i \, {}^i\bar{\mathbf{r}}_i) \tag{3.2-22}$$

where $\mathbf{g} = (g_x, g_y, g_z, 0)$ is a gravity row vector expressed in the base coordinate system. For a level system, $\mathbf{g} = (0, 0, -|g|, 0)$ and g is the gravitational constant ($g = 9.8062$ m/sec^2).

3.2.4 Motion Equations of a Manipulator

From Eqs. (3.2-20) and (3.2-22), the lagrangian function $L = K - P$ is given by

$$L = \tfrac{1}{2} \sum_{i=1}^{n} \sum_{j=1}^{i} \sum_{k=1}^{i} [\mathrm{Tr}\,(\mathbf{U}_{ij}\,\mathbf{J}_i\,\mathbf{U}_{ik}^T)\dot{q}_j\,\dot{q}_k] + \sum_{i=1}^{n} m_i \mathbf{g}({}^0\mathbf{A}_i\,{}^i\bar{\mathbf{r}}_i) \qquad (3.2\text{-}23)$$

Applying the Lagrange-Euler formulation to the lagrangian function of the robot arm [Eq. (3.2-23)] yields the necessary generalized torque τ_i for joint i actuator to drive the ith link of the manipulator,

$$\tau_i = \frac{d}{dt}\left[\frac{\partial L}{\partial \dot{q}_i}\right] - \frac{\partial L}{\partial q_i}$$

$$= \sum_{j=i}^{n} \sum_{k=1}^{j} \mathrm{Tr}\,(\mathbf{U}_{jk}\,\mathbf{J}_j\,\mathbf{U}_{ji}^T)\,\ddot{q}_k + \sum_{j=i}^{n} \sum_{k=1}^{j} \sum_{m=1}^{j} \mathrm{Tr}\,(\mathbf{U}_{jkm}\,\mathbf{J}_j\,\mathbf{U}_{ji}^T)\,\dot{q}_k\dot{q}_m - \sum_{j=i}^{n} m_j \mathbf{g}\mathbf{U}_{ji}\,{}^j\bar{\mathbf{r}}_j \qquad (3.2\text{-}24)$$

for $i = 1, 2, \ldots, n$. The above equation can be expressed in a much simpler matrix *notation* form as

$$\tau_i = \sum_{k=1}^{n} D_{ik}\ddot{q}_k + \sum_{k=1}^{n} \sum_{m=1}^{n} h_{ikm}\dot{q}_k\dot{q}_m + c_i \qquad i = 1, 2, \ldots, n \quad (3.2\text{-}25)$$

or in a matrix form as

$$\boldsymbol{\tau}(t) = \mathbf{D}(\mathbf{q}(t))\,\ddot{\mathbf{q}}(t) + \mathbf{h}(\mathbf{q}(t), \dot{\mathbf{q}}(t)) + \mathbf{c}(\mathbf{q}(t)) \qquad (3.2\text{-}26)$$

where

$\boldsymbol{\tau}(t)$ = $n \times 1$ generalized torque vector applied at joints $i = 1, 2, \ldots, n$; that is,

$$\boldsymbol{\tau}(t) = (\tau_1(t), \tau_2(t), \ldots, \tau_n(t))^T \qquad (3.2\text{-}27)$$

$\mathbf{q}(t)$ = an $n \times 1$ vector of the joint variables of the robot arm and can be expressed as

$$\mathbf{q}(t) = (q_1(t), q_2(t), \ldots, q_n(t))^T \qquad (3.2\text{-}28)$$

$\dot{\mathbf{q}}(t)$ = an $n \times 1$ vector of the joint velocity of the robot arm and can be expressed as

$$\dot{\mathbf{q}}(t) = (\dot{q}_1(t), \dot{q}_2(t), \ldots, \dot{q}_n(t))^T \qquad (3.2\text{-}29)$$

$\ddot{\mathbf{q}}(t)$ = an $n \times 1$ vector of the acceleration of the joint variables $\mathbf{q}(t)$ and can be expressed as

$$\ddot{\mathbf{q}}(t) = (\ddot{q}_1(t), \ddot{q}_2(t), \ldots, \ddot{q}_n(t))^T \qquad (3.2\text{-}30)$$

$\mathbf{D}(\mathbf{q})$ = an $n \times n$ inertial acceleration-related symmetric matrix whose elements are

$$D_{ik} = \sum_{j=\max(i,k)}^{n} \operatorname{Tr}\,(\mathbf{U}_{jk}\,\mathbf{J}_j\,\mathbf{U}_{ji}^T) \qquad i, k = 1, 2, \ldots, n \tag{3.2-31}$$

$\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}})$ = an $n \times 1$ nonlinear Coriolis and centrifugal force vector whose elements are

$$\mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) = (h_1, h_2, \ldots, h_n)^T$$

where
$$h_i = \sum_{k=1}^{n} \sum_{m=1}^{n} h_{ikm}\dot{q}_k\dot{q}_m \qquad i = 1, 2, \ldots, n \tag{3.2-32}$$

and
$$h_{ikm} = \sum_{j=\max(i,k,m)}^{n} \operatorname{Tr}\,(\mathbf{U}_{jkm}\,\mathbf{J}_j\,\mathbf{U}_{ji}^T) \qquad i, k, m = 1, 2, \ldots, n \tag{3.2-33}$$

$\mathbf{c}(\mathbf{q})$ = an $n \times 1$ gravity loading force vector whose elements are

$$\mathbf{c}(\mathbf{q}) = (c_1, c_2, \ldots, c_n)^T$$

where
$$c_i = \sum_{j=i}^{n} (-m_j\,\mathbf{g}\,\mathbf{U}_{ji}\,{}^j\bar{\mathbf{r}}_j) \qquad i = 1, 2, \ldots, n \tag{3.2-34}$$

3.2.5 Motion Equations of a Robot Arm with Rotary Joints

If the equations given by Eqs. (3.2-26) to (3.2-34) are expanded for a six-axis robot arm with rotary joints, then the following terms that form the dynamic motion equations are obtained:

The Acceleration-Related Symmetric Matrix, D(θ). From Eq. (3.2-31), we have

$$\mathbf{D}(\theta) = \begin{bmatrix} D_{11} & D_{12} & D_{13} & D_{14} & D_{15} & D_{16} \\ D_{12} & D_{22} & D_{23} & D_{24} & D_{25} & D_{26} \\ D_{13} & D_{23} & D_{33} & D_{34} & D_{35} & D_{36} \\ D_{14} & D_{24} & D_{34} & D_{44} & D_{45} & D_{46} \\ D_{15} & D_{25} & D_{35} & D_{45} & D_{55} & D_{56} \\ D_{16} & D_{26} & D_{36} & D_{46} & D_{56} & D_{66} \end{bmatrix} \tag{3.2-35}$$

where

$$D_{11} = \operatorname{Tr}\,(\mathbf{U}_{11}\mathbf{J}_1\mathbf{U}_{11}^T) + \operatorname{Tr}\,(\mathbf{U}_{21}\mathbf{J}_2\mathbf{U}_{21}^T) + \operatorname{Tr}\,(\mathbf{U}_{31}\mathbf{J}_3\mathbf{U}_{31}^T) + \operatorname{Tr}\,(\mathbf{U}_{41}\mathbf{J}_4\mathbf{U}_{41}^T)$$
$$+ \operatorname{Tr}\,(\mathbf{U}_{51}\mathbf{J}_5\mathbf{U}_{51}^T) + \operatorname{Tr}\,(\mathbf{U}_{61}\mathbf{J}_6\mathbf{U}_{61}^T)$$

$$D_{12} = D_{21} = \text{Tr}\,(\mathbf{U}_{22}\mathbf{J}_2\mathbf{U}_{21}^T) + \text{Tr}\,(\mathbf{U}_{32}\mathbf{J}_3\mathbf{U}_{31}^T) + \text{Tr}\,(\mathbf{U}_{42}\mathbf{J}_4\mathbf{U}_{41}^T)$$

$$+ \text{Tr}\,(\mathbf{U}_{52}\mathbf{J}_5\mathbf{U}_{51}^T) + \text{Tr}\,(\mathbf{U}_{62}\mathbf{J}_6\mathbf{U}_{61}^T)$$

$$D_{13} = D_{31} = \text{Tr}\,(\mathbf{U}_{33}\mathbf{J}_3\mathbf{U}_{31}^T) + \text{Tr}\,(\mathbf{U}_{43}\mathbf{J}_4\mathbf{U}_{41}^T) + \text{Tr}\,(\mathbf{U}_{53}\mathbf{J}_5\mathbf{U}_{51}^T) + \text{Tr}\,(\mathbf{U}_{63}\mathbf{J}_6\mathbf{U}_{61}^T)$$

$$D_{14} = D_{41} = \text{Tr}\,(\mathbf{U}_{44}\mathbf{J}_4\mathbf{U}_{41}^T) + \text{Tr}\,(\mathbf{U}_{54}\mathbf{J}_5\mathbf{U}_{51}^T) + \text{Tr}\,(\mathbf{U}_{64}\mathbf{J}_6\mathbf{U}_{61}^T)$$

$$D_{15} = D_{51} = \text{Tr}\,(\mathbf{U}_{55}\mathbf{J}_5\mathbf{U}_{51}^T) + \text{Tr}\,(\mathbf{U}_{65}\mathbf{J}_6\mathbf{U}_{61}^T)$$

$$D_{16} = D_{61} = \text{Tr}\,(\mathbf{U}_{66}\mathbf{J}_6\mathbf{U}_{61}^T)$$

$$D_{22} = \text{Tr}\,(\mathbf{U}_{22}\mathbf{J}_2\mathbf{U}_{22}^T) + \text{Tr}\,(\mathbf{U}_{32}\mathbf{J}_3\mathbf{U}_{32}^T) + \text{Tr}\,(\mathbf{U}_{42}\mathbf{J}_4\mathbf{U}_{42}^T)$$

$$+ \text{Tr}\,(\mathbf{U}_{52}\mathbf{J}_5\mathbf{U}_{52}^T) + \text{Tr}\,(\mathbf{U}_{62}\mathbf{J}_6\mathbf{U}_{62}^T)$$

$$D_{23} = D_{32} = \text{Tr}\,(\mathbf{U}_{33}\mathbf{J}_3\mathbf{U}_{32}^T) + \text{Tr}\,(\mathbf{U}_{43}\mathbf{J}_4\mathbf{U}_{42}^T) + \text{Tr}\,(\mathbf{U}_{53}\mathbf{J}_5\mathbf{U}_{52}^T) + \text{Tr}\,(\mathbf{U}_{63}\mathbf{J}_6\mathbf{U}_{62}^T)$$

$$D_{24} = D_{42} = \text{Tr}\,(\mathbf{U}_{44}\mathbf{J}_4\mathbf{U}_{42}^T) + \text{Tr}\,(\mathbf{U}_{54}\mathbf{J}_5\mathbf{U}_{52}^T) + \text{Tr}\,(\mathbf{U}_{64}\mathbf{J}_6\mathbf{U}_{62}^T)$$

$$D_{25} = D_{52} = \text{Tr}\,(\mathbf{U}_{55}\mathbf{J}_5\mathbf{U}_{52}^T) + \text{Tr}\,(\mathbf{U}_{65}\mathbf{J}_6\mathbf{U}_{62}^T)$$

$$D_{26} = D_{62} = \text{Tr}\,(\mathbf{U}_{66}\mathbf{J}_6\mathbf{U}_{62}^T)$$

$$D_{33} = \text{Tr}\,(\mathbf{U}_{33}\mathbf{J}_3\mathbf{U}_{33}^T) + \text{Tr}\,(\mathbf{U}_{43}\mathbf{J}_4\mathbf{U}_{43}^T) + \text{Tr}\,(\mathbf{U}_{53}\mathbf{J}_5\mathbf{U}_{53}^T) + \text{Tr}\,(\mathbf{U}_{63}\mathbf{J}_6\mathbf{U}_{63}^T)$$

$$D_{34} = D_{43} = \text{Tr}\,(\mathbf{U}_{44}\mathbf{J}_4\mathbf{U}_{43}^T) + \text{Tr}\,(\mathbf{U}_{54}\mathbf{J}_5\mathbf{U}_{53}^T) + \text{Tr}\,(\mathbf{U}_{64}\mathbf{J}_6\mathbf{U}_{63}^T)$$

$$D_{35} = D_{53} = \text{Tr}\,(\mathbf{U}_{55}\mathbf{J}_5\mathbf{U}_{53}^T) + \text{Tr}\,(\mathbf{U}_{65}\mathbf{J}_6\mathbf{U}_{63}^T)$$

$$D_{36} = D_{63} = \text{Tr}\,(\mathbf{U}_{66}\mathbf{J}_6\mathbf{U}_{63}^T)$$

$$D_{44} = \text{Tr}\,(\mathbf{U}_{44}\mathbf{J}_4\mathbf{U}_{44}^T) + \text{Tr}\,(\mathbf{U}_{54}\mathbf{J}_5\mathbf{U}_{54}^T) + \text{Tr}\,(\mathbf{U}_{64}\mathbf{J}_6\mathbf{U}_{64}^T)$$

$$D_{45} = D_{54} = \text{Tr}\,(\mathbf{U}_{55}\mathbf{J}_5\mathbf{U}_{54}^T) + \text{Tr}\,(\mathbf{U}_{65}\mathbf{J}_6\mathbf{U}_{64}^T)$$

$$D_{46} = D_{64} = \text{Tr}\,(\mathbf{U}_{66}\mathbf{J}_6\mathbf{U}_{64}^T)$$

$$D_{55} = \text{Tr}\,(\mathbf{U}_{55}\mathbf{J}_5\mathbf{U}_{55}^T) + \text{Tr}\,(\mathbf{U}_{65}\mathbf{J}_6\mathbf{U}_{65}^T)$$

$$D_{56} = D_{65} = \text{Tr}\,(\mathbf{U}_{66}\mathbf{J}_6\mathbf{U}_{65}^T)$$

$$D_{66} = \text{Tr}\,(\mathbf{U}_{66}\mathbf{J}_6\mathbf{U}_{66}^T)$$

The Coriolis and Centrifugal Terms, $\mathbf{h}(\theta, \dot{\theta})$. The velocity-related coefficients in the Coriolis and centrifugal terms in Eqs. (3.2-32) and (3.2-33) can be expressed separately by a 6×6 symmetric matrix denoted by $\mathbf{H}_{i,v}$ and defined in the following way:

$$\mathbf{H}_{i,v} = \begin{bmatrix} h_{i11} & h_{i12} & h_{i13} & h_{i14} & h_{i15} & h_{i16} \\ h_{i12} & h_{i22} & h_{i23} & h_{i24} & h_{i25} & h_{i26} \\ h_{i13} & h_{i23} & h_{i33} & h_{i34} & h_{i35} & h_{i36} \\ h_{i14} & h_{i24} & h_{i34} & h_{i44} & h_{i45} & h_{i46} \\ h_{i15} & h_{i25} & h_{i35} & h_{i45} & h_{i55} & h_{i56} \\ h_{i16} & h_{i26} & h_{i36} & h_{i46} & h_{i56} & h_{i66} \end{bmatrix} \qquad i = 1, 2, \ldots, 6 \tag{3.2-36}$$

Let the velocity of the six joint variables be expressed by a six-dimensional column vector denoted by $\dot{\boldsymbol{\theta}}$:

$$\dot{\boldsymbol{\theta}}(t) = [\,\dot{\theta}_1(t),\ \dot{\theta}_2(t),\ \dot{\theta}_3(t),\ \dot{\theta}_4(t),\ \dot{\theta}_5(t),\ \dot{\theta}_6(t)\,]^T \tag{3.2-37}$$

Then, Eq. (3.2-32) can be expressed in the following compact matrix-vector product form:

$$h_i = \dot{\boldsymbol{\theta}}^T \mathbf{H}_{i,v} \dot{\boldsymbol{\theta}} \tag{3.2-38}$$

where the subscript i refers to the joint $(i = 1, \ldots, 6)$ at which the velocity-induced torques or forces are "felt."

The expression given by Eq. (3.2-38) is a component in a six-dimensional column vector denoted by $\mathbf{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$:

$$\mathbf{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}}) = \begin{bmatrix} h_1 \\ h_2 \\ h_3 \\ h_4 \\ h_5 \\ h_6 \end{bmatrix} = \begin{bmatrix} \dot{\boldsymbol{\theta}}^T \mathbf{H}_{1,v} \dot{\boldsymbol{\theta}} \\ \dot{\boldsymbol{\theta}}^T \mathbf{H}_{2,v} \dot{\boldsymbol{\theta}} \\ \dot{\boldsymbol{\theta}}^T \mathbf{H}_{3,v} \dot{\boldsymbol{\theta}} \\ \dot{\boldsymbol{\theta}}^T \mathbf{H}_{4,v} \dot{\boldsymbol{\theta}} \\ \dot{\boldsymbol{\theta}}^T \mathbf{H}_{5,v} \dot{\boldsymbol{\theta}} \\ \dot{\boldsymbol{\theta}}^T \mathbf{H}_{6,v} \dot{\boldsymbol{\theta}} \end{bmatrix} \tag{3.2-39}$$

The Gravity Terms, $\mathbf{c}(\boldsymbol{\theta})$. From Eq. (3.2-34) we have

$$\mathbf{c}(\boldsymbol{\theta}) = (c_1, c_2, c_3, c_4, c_5, c_6)^T \tag{3.2-40}$$

where

$$c_1 = -(m_1 \mathbf{g} \mathbf{U}_{11}\, {}^1\bar{\mathbf{r}}_1 + m_2 \mathbf{g} \mathbf{U}_{21}\, {}^2\bar{\mathbf{r}}_2 + m_3 \mathbf{g} \mathbf{U}_{31}\, {}^3\bar{\mathbf{r}}_3 + m_4 \mathbf{g} \mathbf{U}_{41}\, {}^4\bar{\mathbf{r}}_4 + m_5 \mathbf{g} \mathbf{U}_{51}\, {}^5\bar{\mathbf{r}}_5 + m_6 \mathbf{g} \mathbf{U}_{61}\, {}^6\bar{\mathbf{r}}_6)$$

$$c_2 = -(m_2 \mathbf{g} \mathbf{U}_{22}\, {}^2\bar{\mathbf{r}}_2 + m_3 \mathbf{g} \mathbf{U}_{32}\, {}^3\bar{\mathbf{r}}_3 + m_4 \mathbf{g} \mathbf{U}_{42}\, {}^4\bar{\mathbf{r}}_4 + m_5 \mathbf{g} \mathbf{U}_{52}\, {}^5\bar{\mathbf{r}}_5 + m_6 \mathbf{g} \mathbf{U}_{62}\, {}^6\bar{\mathbf{r}}_6)$$

$$c_3 = -(m_3\mathbf{g}\mathbf{U}_{33}{}^3\bar{\mathbf{r}}_3 + m_4\mathbf{g}\mathbf{U}_{43}{}^4\bar{\mathbf{r}}_4 + m_5\mathbf{g}\mathbf{U}_{53}{}^5\bar{\mathbf{r}}_5 + m_6\mathbf{g}\mathbf{U}_{63}{}^6\bar{\mathbf{r}}_6)$$

$$c_4 = -(m_4\mathbf{g}\mathbf{U}_{44}{}^4\bar{\mathbf{r}}_4 + m_5\mathbf{g}\mathbf{U}_{54}{}^5\bar{\mathbf{r}}_5 + m_6\mathbf{g}\mathbf{U}_{64}{}^6\bar{\mathbf{r}}_6)$$

$$c_5 = -(m_5\mathbf{g}\mathbf{U}_{55}{}^5\bar{\mathbf{r}}_5 + m_6\mathbf{g}\mathbf{U}_{65}{}^6\bar{\mathbf{r}}_6)$$

$$c_6 = -m_6\mathbf{g}\mathbf{U}_{66}{}^6\bar{\mathbf{r}}_6$$

The coefficients c_i, D_{ik}, and h_{ikm} in Eqs. (3.2-31) to (3.2-34) are functions of both the joint variables and inertial parameters of the manipulator, and sometimes are called the *dynamic coefficients* of the manipulator. The physical meaning of these dynamic coefficients can easily be seen from the Lagrange-Euler equations of motion given by Eqs. (3.2-26) to (3.2-34):

1. The coefficient c_i represents the gravity loading terms due to the links and is defined by Eq. (3.2-34).
2. The coefficient D_{ik} is related to the acceleration of the joint variables and is defined by Eq. (3.2-31). In particular, for $i = k$, D_{ii} is related to the acceleration of joint i where the driving torque τ_i acts, while for $i \neq k$, D_{ik} is related to the reaction torque (or force) induced by the acceleration of joint k and acting at joint i, or vice versa. Since the inertia matrix is symmetric and $\mathrm{Tr}(\mathbf{A}) = \mathrm{Tr}(\mathbf{A}^T)$, it can be shown that $D_{ik} = D_{ki}$.
3. The coefficient h_{ikm} is related to the velocity of the joint variables and is defined by Eqs. (3.2-32) and (3.2-33). The last two indices, km, are related to the velocities of joints k and m, whose dynamic interplay induces a reaction torque (or force) at joint i. Thus, the first index i is always related to the joint where the velocity-induced reaction torques (or forces) are "felt." In particular, for $k = m$, h_{ikk} is related to the centrifugal force generated by the angular velocity of joint k and "felt" at joint i, while for $k \neq m$, h_{ikm} is related to the Coriolis force generated by the velocities of joints k and m and "felt" at joint i. It is noted that, for a given i, we have $h_{ikm} = h_{imk}$ which is apparent by physical reasoning.

In evaluating these coefficients, it is worth noting that some of the coefficients may be zero for the following reasons:

1. The particular kinematic design of a manipulator can eliminate some dynamic coupling (D_{ij} and h_{ikm} coefficients) between joint motions.
2. Some of the velocity-related dynamic coefficients have only a dummy existence in Eqs. (3.2-32) and (3.2-33); that is, they are physically nonexistent. (For instance, the centrifugal force will not interact with the motion of that joint which generates it, that is, $h_{iii} = 0$ always; however, it can interact with motions at the other joints in the chain, that is, we can have $h_{jii} \neq 0$.)
3. Due to particular variations in the link configuration during motion, some dynamic coefficients may become zero at particular instants of time.

Table 3.1 Computational complexity of Lagrange-Euler equations of motion†

Lagrange-Euler formulation	Multiplications†	Additions
$\mathbf{A}_j^i$	$32n(n-1)$	$24n(n-1)$
$-m_j\mathbf{g}\mathbf{U}_{ji}\,{}^j\bar{\mathbf{r}}_j$	$4n(9n-7)$	$n\frac{51n-45}{2}$
$\sum_{j=i}^{n} m_j\mathbf{g}\mathbf{U}_{ji}\,{}^j\bar{\mathbf{r}}_j$	0	$\frac{1}{2}n(n-1)$
$\mathrm{Tr}\,[\mathbf{U}_{kj}\mathbf{J}_k(\mathbf{U}_{ki})^T]$	$(128/3)\,n(n+1)(n+2)$	$(65/2)\,n(n+1)(n+2)$
$\sum_{k=\max(i,j)}^{n} \mathrm{Tr}\,[\mathbf{U}_{kj}\mathbf{J}_k(\mathbf{U}_{ki})^T]$	0	$(1/6)\,n(n-1)(n+1)$
$\mathrm{Tr}\,[\mathbf{U}_{mjk}\mathbf{J}_m(\mathbf{U}_{mi})^T]$	$(128/3)\,n^2(n+1)(n+2)$	$(65/2)\,n^2(n+1)(n+2)$
$\sum_{m=\max(i,j,k)}^{n} \mathrm{Tr}\,[\mathbf{U}_{mjk}\mathbf{J}_m(\mathbf{U}_{mi})^T]$	0	$(1/6)\,n^2(n-1)(n+1)$
$\tau = \mathbf{D}(\mathbf{q})\ddot{\mathbf{q}} + \mathbf{h}(\mathbf{q}, \dot{\mathbf{q}}) + \mathbf{c}(\mathbf{q})$	$(128/3)\,n^4 + (512/3)\,n^3 + (844/3)\,n^2 + (76/3)\,n$	$(98/3)\,n^4 + (781/6)\,n^3 + (637/3)\,n^2 + (107/6)\,n$

† n = number of degrees of freedom of the robot arm.

The motion equations of a manipulator as given by Eqs. (3.2-26) to (3.2-34) are coupled, nonlinear, second-order ordinary differential equations. These equations are in symbolic differential equation form and they include all inertial, centrifugal and Coriolis, and gravitational effects of the links. For a given set of applied torques $\tau_i (i = 1, 2, \ldots, n)$ as a function of time, Eq. (3.2-26) should be integrated simultaneously to obtain the actual motion of the manipulator in terms of the time history of the joint variables $\mathbf{q}(t)$. Then, the time history of the joint variables can be transformed to obtain the time history of the hand motion (hand trajectory) by using the appropriate homogeneous transformation matrices. Or, if the time history of the joint variables, the joint velocities, and the joint accelerations is known ahead of time from a trajectory planning program, then Eqs. (3.2-26) to (3.2-34) can be utilized to compute the applied torques $\tau(t)$ as a function of time which is required to produce the particular planned manipulator motion. This is known as *open-loop control*. However, closed-loop control is more desirable for an autonomous robotic system. This subject will be discussed in Chap. 5.

Because of its matrix structure, the L-E equations of motion are appealing from the closed-loop control viewpoint in that they give a set of state equations as in Eq. (3.2-26). This form allows design of a control law that easily compensates for all the nonlinear effects. Quite often in designing a feedback controller for a manipulator, the dynamic coefficients are used to minimize the nonlinear effects of the reaction forces (Markiewicz [1973]).

It is of interest to evaluate the computational complexities inherent in obtaining the coefficients in Eqs. (3.2-31) to (3.2-34). Table 3.1 summarizes the computational complexities of the L-E equations of motion in terms of required mathematical operations (multiplications and additions) that are required to compute Eq. (3.2-26) for every set point in the trajectory. Computationally, these equations of motion are extremely inefficient as compared with other formulations. In the next section, we shall develop the motion equations of a robot arm which will prove to be more efficient in computing the nominal torques.

3.2.6 A Two-Link Manipulator Example

To show how to use the L-E equations of motion in Eqs. (3.2-26) to (3.2-34), an example is worked out in this section for a two-link manipulator with revolute joints, as shown in Fig. 3.2. All the rotation axes at the joints are along the z axis normal to the paper surface. The physical dimensions such as location of center of mass, mass of each link, and coordinate systems are shown below. We would like to derive the motion equations for the above two-link robot arm using Eqs. (3.2-26) to (3.2-34).

We assume the following: joint variables $= \theta_1, \theta_2$; mass of the links $= m_1, m_2$; link parameters $= \alpha_1 = \alpha_2 = 0$; $d_1 = d_2 = 0$; and $a_1 = a_2 = l$. Then, from Fig. 3.2, and the discussion in the previous section, the homogeneous coordinate transformation matrices ${}^{i-1}\mathbf{A}_i$ $(i = 1, 2)$ are obtained as

$$
{}^0\mathbf{A}_1 = \begin{bmatrix} C_1 & -S_1 & 0 & lC_1 \\ S_1 & C_1 & 0 & lS_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \qquad {}^1\mathbf{A}_2 = \begin{bmatrix} C_2 & -S_2 & 0 & lC_2 \\ S_2 & C_2 & 0 & lS_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^0\mathbf{A}_2 = {}^0\mathbf{A}_1\,{}^1\mathbf{A}_2 = \begin{bmatrix} C_{12} & -S_{12} & 0 & l(C_{12} + C_1) \\ S_{12} & C_{12} & 0 & l(S_{12} + S_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}
$$

where $C_i = \cos\theta_i$; $S_i = \sin\theta_i$; $C_{ij} = \cos(\theta_i + \theta_j)$; $S_{ij} = \sin(\theta_i + \theta_j)$. From the definition of the $\mathbf{Q}_i$ matrix, for a rotary joint i, we have:

$$
\mathbf{Q}_i = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}
$$

Then, using Eq. (3.2-11), we have

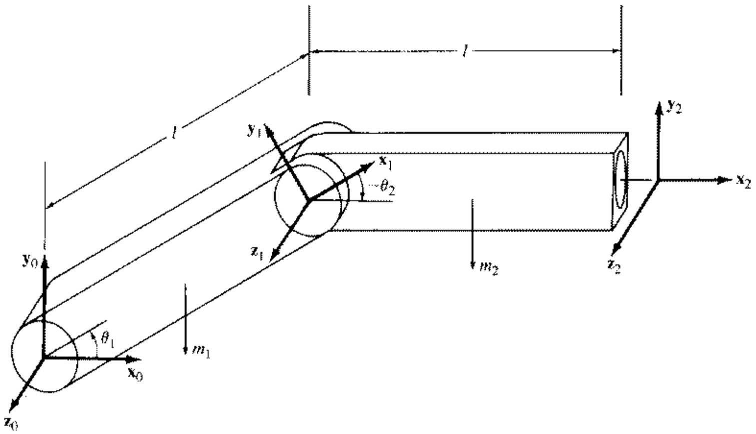


Figure 3.2 A two-link manipulator.

$$\mathbf{U}_{11} = \frac{\partial\, {}^0\mathbf{A}_1}{\partial \theta_1} = \mathbf{Q}_1\, {}^0\mathbf{A}_1 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_1 & -S_1 & 0 & lC_1 \\ S_1 & C_1 & 0 & lS_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -S_1 & -C_1 & 0 & -lS_1 \\ C_1 & -S_1 & 0 & lC_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Similarly, for $\mathbf{U}_{21}$ and $\mathbf{U}_{22}$ we have

$$\mathbf{U}_{21} = \frac{\partial\, {}^0\mathbf{A}_2}{\partial \theta_1} = \mathbf{Q}_1\, {}^0\mathbf{A}_2 = \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_{12} & -S_{12} & 0 & l(C_{12} + C_1) \\ S_{12} & C_{12} & 0 & l(S_{12} + S_1) \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -S_{12} & -C_{12} & 0 & -l(S_{12} + S_1) \\ C_{12} & -S_{12} & 0 & l(C_{12} + C_1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{U}_{22} = \frac{\partial\, {}^0\mathbf{A}_2}{\partial \theta_2}$$

$$= {}^0\mathbf{A}_1 \mathbf{Q}_2\, {}^1\mathbf{A}_2$$

$$= \begin{bmatrix} C_1 & -S_1 & 0 & lC_1 \\ S_1 & C_1 & 0 & lS_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} C_2 & -S_2 & 0 & lC_2 \\ S_2 & C_2 & 0 & lS_2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -S_{12} & -C_{12} & 0 & -lS_{12} \\ C_{12} & -S_{12} & 0 & lC_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

From Eq. (3.2-18), assuming all the products of inertia are zero, we can derive the pseudo-inertia matrix $\mathbf{J}_i$:

$$\mathbf{J}_1 = \begin{bmatrix} \frac{1}{3}m_1 l^2 & 0 & 0 & -\frac{1}{2}m_1 l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}m_1 l & 0 & 0 & m_1 \end{bmatrix} \qquad \mathbf{J}_2 = \begin{bmatrix} \frac{1}{3}m_2 l^2 & 0 & 0 & -\frac{1}{2}m_2 l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}m_2 l & 0 & 0 & m_2 \end{bmatrix}$$

Then, using Eq. (3.2-31), we have

$$D_{11} = \mathrm{Tr}\,(\mathbf{U}_{11}\mathbf{J}_1\mathbf{U}_{11}^T) + \mathrm{Tr}\,(\mathbf{U}_{21}\mathbf{J}_2\mathbf{U}_{21}^T)$$

$$= \mathrm{Tr}\left\{ \begin{bmatrix} -S_1 & -C_1 & 0 & -lS_1 \\ C_1 & -S_1 & 0 & lC_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3}m_1 l^2 & 0 & 0 & -\frac{1}{2}m_1 l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}m_1 l & 0 & 0 & m_1 \end{bmatrix} \mathbf{U}_{11}^T \right\}$$

$$+ \mathrm{Tr}\left\{ \begin{bmatrix} -S_{12} & -C_{12} & 0 & -l(S_{12} + S_1) \\ C_{12} & -S_{12} & 0 & l(C_{12} + C_1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3}m_2 l^2 & 0 & 0 & -\frac{1}{2}m_2 l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}m_2 l & 0 & 0 & m_2 \end{bmatrix} \mathbf{U}_{21}^T \right\}$$

$$= \tfrac{1}{3}m_1 l^2 + \tfrac{4}{3}m_2 l^2 + m_2 C_2 l^2$$

For D_{12} we have

$$D_{12} = D_{21} = \mathrm{Tr}(\mathbf{U}_{22}\mathbf{J}_2\mathbf{U}_{21}^T)$$

$$= \mathrm{Tr}\left\{ \begin{bmatrix} -S_{12} & -C_{12} & 0 & -lS_{12} \\ C_{12} & -S_{12} & 0 & lC_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{3}m_2 l^2 & 0 & 0 & -\frac{1}{2}m_2 l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}m_2 l & 0 & 0 & m_2 \end{bmatrix} \mathbf{U}_{21}^T \right\}$$

$$= m_2 l^2(-\tfrac{1}{6} + \tfrac{1}{2} + \tfrac{1}{2}C_2) = \tfrac{1}{3}m_2 l^2 + \tfrac{1}{2}m_2 l^2 C_2$$

For D_{22} we have

$$D_{22} = \mathrm{Tr}\,(\mathbf{U}_{22}\mathbf{J}_2\mathbf{U}_{22}^T)$$

$$= \mathrm{Tr}\left\{\begin{bmatrix} -S_{12} & -C_{12} & 0 & -lS_{12} \\ C_{12} & -S_{12} & 0 & lC_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} \frac{1}{3}m_2 l^2 & 0 & 0 & -\frac{1}{2}m_2 l \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -\frac{1}{2}m_2 l & 0 & 0 & m_2 \end{bmatrix}\mathbf{U}_{22}^T\right\}$$

$$= \tfrac{1}{3}m_2 l^2 S_{12}^2 + \tfrac{1}{3}m_2 l^2 C_{12}^2 = \tfrac{1}{3}m_2 l^2$$

To derive the Coriolis and centrifugal terms, we use Eq. (3.2-32). For $i = 1$, using Eq. (3.2-32), we have

$$h_1 = \sum_{k=1}^{2}\sum_{m=1}^{2} h_{1km}\dot{\theta}_k\dot{\theta}_m = h_{111}\dot{\theta}_1^2 + h_{112}\dot{\theta}_1\dot{\theta}_2 + h_{121}\dot{\theta}_1\dot{\theta}_2 + h_{122}\dot{\theta}_2^2$$

Using Eq. (3.2-33), we can obtain the value of h_{ikm}. Therefore, the above value which corresponds to joint 1 is

$$h_1 = -\tfrac{1}{2}m_2 S_2 l^2\dot{\theta}_2^2 - m_2 S_2 l^2\dot{\theta}_1\dot{\theta}_2$$

Similarly, for $i = 2$ we have

$$h_2 = \sum_{k=1}^{2}\sum_{m=1}^{2} h_{2km}\dot{\theta}_k\dot{\theta}_m = h_{211}\dot{\theta}_1^2 + h_{212}\dot{\theta}_1\dot{\theta}_2 + h_{221}\dot{\theta}_1 + h_{222}\dot{\theta}_2^2$$

$$= \tfrac{1}{2}m_2 S_2 l^2\dot{\theta}_1^2$$

Therefore,

$$\mathbf{h}(\theta, \dot{\theta}) = \begin{bmatrix} -\frac{1}{2}m_2 S_2 l^2\dot{\theta}_2^2 - m_2 S_2 l^2\dot{\theta}_1\dot{\theta}_2 \\ \frac{1}{2}m_2 S_2 l^2\dot{\theta}_1^2 \end{bmatrix}$$

Next, we need to derive the gravity-related terms, $\mathbf{c} = (c_1, c_2)^T$. Using Eq. (3.2-34), we have:

$$c_1 = -(m_1\mathbf{g}\mathbf{U}_{11}\,{}^1\bar{\mathbf{r}}_1 + m_2\mathbf{g}\mathbf{U}_{21}\,{}^2\bar{\mathbf{r}}_2)$$

$$= -m_1(0, -g, 0, 0)\begin{bmatrix} -S_1 & -C_1 & 0 & -lS_1 \\ C_1 & -S_1 & 0 & lC_1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$- m_2(0, -g, 0, 0) \begin{bmatrix} -S_{12} & -C_{12} & 0 & -l(S_{12} + S_1) \\ C_{12} & -S_{12} & 0 & l(C_{12} + C_1) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= \tfrac{1}{2} m_1 g l C_1 + \tfrac{1}{2} m_2 g l C_{12} + m_2 g l C_1$$

$$c_2 = -m_2 \mathbf{g} \mathbf{U}_{22} {}^2\bar{\mathbf{r}}_2$$

$$= -m_2 (0, -g, 0, 0) \begin{bmatrix} -S_{12} & -C_{12} & 0 & -lS_{12} \\ C_{12} & -S_{12} & 0 & lC_{12} \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

$$= -m_2 (\tfrac{1}{2} g l C_{12} - g l C_{12})$$

Hence, we obtain the gravity matrix terms:

$$\mathbf{c}(\theta) = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{2} m_1 g l C_1 + \tfrac{1}{2} m_2 g l C_{12} + m_2 g l C_1 \\ \tfrac{1}{2} m_2 g l C_{12} \end{bmatrix}$$

Finally, the Lagrange-Euler equations of motion for the two-link manipulator are found to be

$$\tau(\mathbf{t}) = \mathbf{D}(\theta)\ddot{\theta}(\mathbf{t}) + \mathbf{h}(\theta, \dot{\theta}) + \mathbf{c}(\theta)$$

$$\begin{bmatrix} \tau_1 \\ \tau_2 \end{bmatrix} = \begin{bmatrix} \tfrac{1}{3} m_1 l^2 + \tfrac{4}{3} m_2 l^2 + m_2 C_2 l^2 & \tfrac{1}{3} m_2 l^2 + \tfrac{1}{2} m_2 l^2 C_2 \\ \tfrac{1}{3} m_2 l^2 + \tfrac{1}{2} m_2 l^2 C_2 & \tfrac{1}{3} m_2 l^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1 \\ \ddot{\theta}_2 \end{bmatrix}$$

$$+ \begin{bmatrix} -\tfrac{1}{2} m_2 S_2 l^2 \dot{\theta}_2^2 - m_2 S_2 l^2 \dot{\theta}_1 \dot{\theta}_2 \\ \tfrac{1}{2} m_2 S_2 l^2 \dot{\theta}_1^2 \end{bmatrix}$$

$$+ \begin{bmatrix} \tfrac{1}{2} m_1 g l C_1 + \tfrac{1}{2} m_2 g l C_{12} + m_2 g l C_1 \\ \tfrac{1}{2} m_2 g l C_{12} \end{bmatrix}$$

3.3 NEWTON-EULER FORMULATION

In the previous sections, we have derived a set of nonlinear second-order differential equations from the Lagrange-Euler formulation that describe the dynamic behavior of a robot arm. The use of these equations to compute the nominal joint torques from the given joint positions, velocities, and accelerations for each trajectory set point in real time has been a computational bottleneck in open-loop control. The problem is due mainly to the inefficiency of the Lagrange-Euler equations of motion, which use the 4×4 homogeneous transformation matrices. In order to perform real-time control, a simplified robot arm dynamic model has been proposed which ignores the Coriolis and centrifugal forces. This reduces the computation time for the joint torques to an affordable limit (e.g., less than 10 ms for each trajectory point using a PDP 11/45 computer). However, the Coriolis and centrifugal forces are significant in the joint torques when the arm is moving at fast speeds. Thus, the simplified robot arm dynamics restricts a robot arm motion to slow speeds which are not desirable in the typical manufacturing environment. Furthermore, the errors in the joint torques resulting from ignoring the Coriolis and centrifugal forces cannot be corrected with feedback control when the arm is moving at fast speeds because of excessive requirements on the corrective torques.

As an alternative to deriving more efficient equations of motion, several investigators turned to Newton's second law and developed various forms of Newton-Euler equations of motion for an open kinematic chain (Armstrong [1979], Orin et al. [1979], Luh et al. [1980*a*], Walker and Orin [1982]). This formulation when applied to a robot arm results in a set of forward and backward recursive equations with "messy" vector cross-product terms. The most significant aspect of this formulation is that the computation time of the applied torques can be reduced significantly to allow real-time control. The derivation is based on the d'Alembert principle and a set of mathematical equations that describe the kinematic relation of the moving links of a robot arm with respect to the base coordinate system. In order to understand the Newton-Euler formulation, we need to review some concepts in moving and rotating coordinate systems.

3.3.1 Rotating Coordinate Systems

In this section, we shall develop the necessary mathematical relation between a rotating coordinate system and a fixed inertial coordinate frame, and then extend the concept to include a discussion of the relationship between a moving coordinate system (rotating and translating) and an inertial frame. From Fig. 3.3, two right-handed coordinate systems, an unstarred coordinate system $OXYZ$ (inertial frame) and a starred coordinate system $OX^*Y^*Z^*$ (rotating frame), whose origins are coincident at a point O, and the axes OX^*, OY^*, OZ^* are rotating relative to the axes OX, OY, OZ, respectively. Let $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ and $(\mathbf{i}^*, \mathbf{j}^*, \mathbf{k}^*)$ be their respective unit vectors along the principal axes. A point $\mathbf{r}$ fixed and at rest in the starred coordinate system can be expressed in terms of its components on either set of axes:

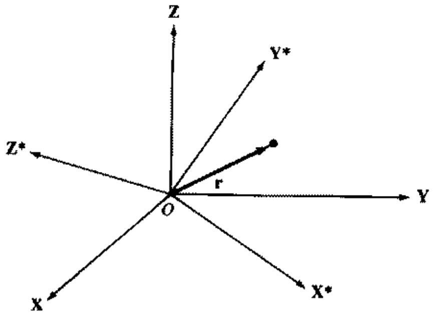


Figure 3.3 The rotating coordinate system.

$$\mathbf{r} = x\mathbf{i} + y\mathbf{j} + z\mathbf{k} \tag{3.3-1}$$

or

$$\mathbf{r} = x^*\mathbf{i}^* + y^*\mathbf{j}^* + z^*\mathbf{k}^* \tag{3.3-2}$$

We would like to evaluate the time derivative of the point $\mathbf{r}$, and because the coordinate systems are rotating with respect to each other, the time derivative of $\mathbf{r}(t)$ can be taken with respect to two different coordinate systems. Let us distinguish these two time derivatives by noting the following notation:

$$\frac{d(\ )}{dt} \triangleq \text{time derivative with respect to the fixed reference coordinate system which is fixed} = \text{time derivative of } \mathbf{r}(t) \tag{3.3-3}$$

$$\frac{d^*(\ )}{dt} \triangleq \text{time derivative with respect to the starred coordinate system which is rotating} = \text{starred derivative of } \mathbf{r}(t) \tag{3.3-4}$$

Then, using Eq. (3.3-1), the time derivative of $\mathbf{r}(t)$ can be expressed as

$$\frac{d\mathbf{r}}{dt} = \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} + x\frac{d\mathbf{i}}{dt} + y\frac{d\mathbf{j}}{dt} + z\frac{d\mathbf{k}}{dt}$$

$$= \dot{x}\mathbf{i} + \dot{y}\mathbf{j} + \dot{z}\mathbf{k} \tag{3.3-5}$$

and, using Eq. (3.3-2), the starred derivative of $\mathbf{r}(t)$ is

$$\frac{d^*\mathbf{r}}{dt} = \dot{x}^*\mathbf{i}^* + \dot{y}^*\mathbf{j}^* + \dot{z}^*\mathbf{k}^* + x^*\frac{d^*\mathbf{i}^*}{dt} + y^*\frac{d^*\mathbf{j}^*}{dt} + z^*\frac{d^*\mathbf{k}^*}{dt}$$

$$= \dot{x}^*\mathbf{i}^* + \dot{y}^*\mathbf{j}^* + \dot{z}^*\mathbf{k}^* \tag{3.3-6}$$

Using Eqs. (3.3-2) and (3.3-6), the time derivative of $\mathbf{r}(t)$ can be expressed as

$$\frac{d\mathbf{r}}{dt} = \dot{x}^*\mathbf{i}^* + \dot{y}^*\mathbf{j}^* + \dot{z}^*\mathbf{k}^* + x^*\frac{d\mathbf{i}^*}{dt} + y^*\frac{d\mathbf{j}^*}{dt} + z^*\frac{d\mathbf{k}^*}{dt}$$

$$= \frac{d^*\mathbf{r}}{dt} + x^*\frac{d\mathbf{i}^*}{dt} + y^*\frac{d\mathbf{j}^*}{dt} + z^*\frac{d\mathbf{k}^*}{dt} \tag{3.3-7}$$

In evaluating this derivative, we encounter the difficulty of finding $d\mathbf{i}^*/dt$, $d\mathbf{j}^*/dt$, and $d\mathbf{k}^*/dt$ because the unit vectors, $\mathbf{i}^*$, $\mathbf{j}^*$, and $\mathbf{k}^*$ are rotating with respect to the unit vectors $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$.

In order to find a relationship between the starred and unstarred derivatives, let us suppose that the starred coordinate system is rotating about some axis OQ passing through the origin O, with angular velocity $\boldsymbol{\omega}$ (see Fig. 3.4), then the angular velocity $\boldsymbol{\omega}$ is defined as a vector of magnitude ω directed along the axis OQ in the direction of a right-hand rotation with the starred coordinate system. Consider a vector $\mathbf{s}$ at rest in the starred coordinate system. Its starred derivative is zero, and we would like to show that its unstarred derivative is

$$\frac{d\mathbf{s}}{dt} = \boldsymbol{\omega} \times \mathbf{s} \tag{3.3-8}$$

Since the time derivative of a vector can be expressed as

$$\frac{d\mathbf{s}}{dt} = \lim_{\Delta t \to 0} \frac{\mathbf{s}(t + \Delta t) - \mathbf{s}(t)}{\Delta t} \tag{3.3-9}$$

we can verify the correctness of Eq. (3.3-8) by showing that

$$\frac{d\mathbf{s}}{dt} = \boldsymbol{\omega} \times \mathbf{s} = \lim_{\Delta t \to 0} \frac{\mathbf{s}(t + \Delta t) - \mathbf{s}(t)}{\Delta t} \tag{3.3-10}$$

With reference to Fig. 3.4, and recalling that a vector has magnitude and direction, we need to verify the correctness of Eq. (3.3-10) both in direction and magnitude. The magnitude of $d\mathbf{s}/dt$ is

$$\left|\frac{d\mathbf{s}}{dt}\right| = |\boldsymbol{\omega} \times \mathbf{s}| = \omega s \sin\theta \tag{3.3-11}$$

The above equation is correct because if Δt is small, then

$$|\Delta\mathbf{s}| = (s \sin\theta)(\omega\, \Delta t) \tag{3.3-12}$$

which is obvious in Fig. 3.4. The direction of $\boldsymbol{\omega} \times \mathbf{s}$ can be found from the definition of the vector cross product to be perpendicular to $\mathbf{s}$ and in the plane of the circle as shown in Fig. 3.4.

If Eq. (3.3-8) is applied to the unit vectors ($\mathbf{i}^*$, $\mathbf{j}^*$, $\mathbf{k}^*$), then Eq. (3.3-7) becomes

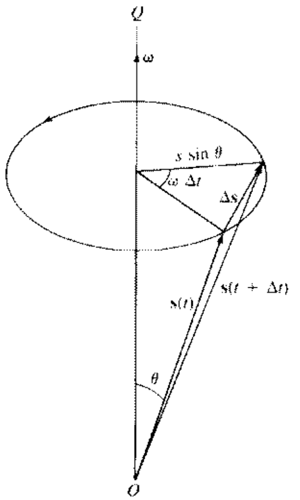


Figure 3.4 Time derivative of a rotating coordinate system.

$$\frac{d\mathbf{r}}{dt} = \frac{d^*\mathbf{r}}{dt} + x^*(\omega \times \mathbf{i}^*) + y^*(\omega \times \mathbf{j}^*) + z^*(\omega \times \mathbf{k}^*)$$

$$= \frac{d^*\mathbf{r}}{dt} + \omega \times \mathbf{r} \tag{3.3-13}$$

This is the fundamental equation establishing the relationship between time derivatives for rotating coordinate systems. Taking the derivative of right- and left-hand sides of Eq. (3.3-13) and applying Eq. (3.3-8) again to $\mathbf{r}$ and $d^*\mathbf{r}/dt$, we obtain the second time derivative of the vector $\mathbf{r}(t)$:

$$\frac{d^2\mathbf{r}}{dt^2} = \frac{d}{dt}\left[\frac{d^*\mathbf{r}}{dt}\right] + \omega \times \frac{d\mathbf{r}}{dt} + \frac{d\omega}{dt} \times \mathbf{r}$$

$$= \frac{d^{*2}\mathbf{r}}{dt^2} + \omega \times \frac{d^*\mathbf{r}}{dt} + \omega \times \left[\frac{d^*\mathbf{r}}{dt} + \omega \times \mathbf{r}\right] + \frac{d\omega}{dt} \times \mathbf{r}$$

$$= \frac{d^{*2}\mathbf{r}}{dt^2} + 2\omega \times \frac{d^*\mathbf{r}}{dt} + \omega \times (\omega \times \mathbf{r}) + \frac{d\omega}{dt} \times \mathbf{r} \tag{3.3-14}$$

Equation (3.3-14) is called the *Coriolis theorem*. The first term on the right-hand side of the equation is the acceleration relative to the starred coordinate system. The second term is called the *Coriolis acceleration*. The third term is called the *centripetal* (toward the center) *acceleration* of a point in rotation about an axis. One can verify that $\omega \times (\omega \times \mathbf{r})$ points directly toward and perpendicular to the axis of rotation. The last term vanishes for a constant angular velocity of rotation about a fixed axis.

3.3.2 Moving Coordinate Systems

Let us extend the above rotating coordinate systems concept further to include the translation motion of the starred coordinate system with respect to the unstarred coordinate system. From Fig. 3.5, the starred coordinate system $O^*X^*Y^*Z^*$ is rotating and translating with respect to the unstarred coordinate system $OXYZ$ which is an inertial frame. A particle $\mathbf{p}$ with mass m is located by vectors $\mathbf{r}^*$ and $\mathbf{r}$ with respect to the origins of the coordinate frames $O^*X^*Y^*Z^*$ and $OXYZ$, respectively. Origin O^* is located by a vector $\mathbf{h}$ with respect to the origin O. The relation between the position vectors $\mathbf{r}$ and $\mathbf{r}^*$ is given by (Fig. 3.5)

$$\mathbf{r} = \mathbf{r}^* + \mathbf{h} \tag{3.3-15}$$

If the starred coordinate system $O^*X^*Y^*Z^*$ is moving (rotating and translating) with respect to the unstarred coordinate system $OXYZ$, then

$$\mathbf{v}(t) \triangleq \frac{d\mathbf{r}}{dt} = \frac{d\mathbf{r}^*}{dt} + \frac{d\mathbf{h}}{dt} = \mathbf{v}^* + \mathbf{v}_h \tag{3.3-16}$$

where $\mathbf{v}^*$ and $\mathbf{v}$ are the velocities of the moving particle $\mathbf{p}$ relative to the coordinate frames $O^*X^*Y^*Z^*$ and $OXYZ$, respectively, and $\mathbf{v}_h$ is the velocity of the starred coordinate system $O^*X^*Y^*Z^*$ relative to the unstarred coordinate system $OXYZ$. Using Eq. (3.3-13), Eq. (3.3-16) can be expressed as

$$\mathbf{v}(t) = \frac{d\mathbf{r}^*}{dt} + \frac{d\mathbf{h}}{dt} = \frac{d^*\mathbf{r}^*}{dt} + \omega \times \mathbf{r}^* + \frac{d\mathbf{h}}{dt} \tag{3.3-17}$$

Similarly, the acceleration of the particle $\mathbf{p}$ with respect to the unstarred coordinate system is

$$\mathbf{a}(t) = \frac{d\mathbf{v}(t)}{dt} = \frac{d^2\mathbf{r}^*}{dt^2} + \frac{d^2\mathbf{h}}{dt^2} = \mathbf{a}^* + \mathbf{a}_h \tag{3.3-18}$$

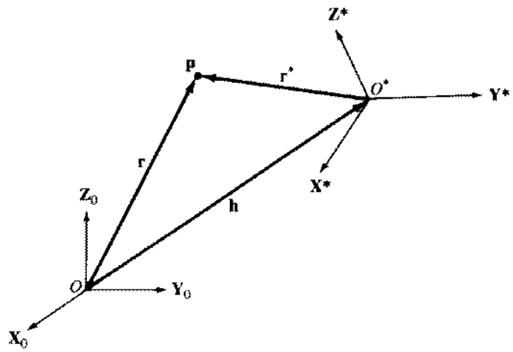


Figure 3.5 Moving coordinate system.

where $\mathbf{a}^*$ and $\mathbf{a}$ are the accelerations of the moving particle $\mathbf{p}$ relative to the coordinate frames $O^*X^*Y^*Z^*$ and $OXYZ$, respectively, and $\mathbf{a}_h$ is the acceleration of the starred coordinate system $O^*X^*Y^*Z^*$ relative to the unstarred coordinate system $OXYZ$. Using Eq. (3.3-14), Eq. (3.3-17) can be expressed as

$$\mathbf{a}(t) = \frac{d^{*2}\mathbf{r}^*}{dt^2} + 2\,\omega \times \frac{d^*\mathbf{r}^*}{dt} + \omega \times (\omega \times \mathbf{r}^*) + \frac{d\omega}{dt} \times \mathbf{r}^* + \frac{d^2\mathbf{h}}{dt^2} \tag{3.3-19}$$

With this introduction to moving coordinate systems, we would like to apply this concept to the link coordinate systems that we established for a robot arm to obtain the kinematics information of the links, and then apply the d'Alembert principle to these translating and/or rotating coordinate systems to derive the motion equations of the robot arm.

3.3.3 Kinematics of the Links

The objective of this section is to derive a set of mathematical equations that, based on the moving coordinate systems described in Sec. 3.3.2, describe the kinematic relationship of the moving-rotating links of a robot arm with respect to the base coordinate system.

With reference to Fig. 3.6, recall that an orthonormal coordinate system $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$ is established at joint i. Coordinate system $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ is then the base coordinate system while the coordinate systems $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$ and $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ are attached to link $i-1$ with origin O^* and link i with origin O', respectively. Origin O' is located by a position vector $\mathbf{p}_i$ with respect to the origin O and by a position vector $\mathbf{p}_i^*$ from the origin O^* with respect to the base coordinate system. Origin O^* is located by a position vector $\mathbf{p}_{i-1}$ from the origin O with respect to the base coordinate system.

Let $\mathbf{v}_{i-1}$ and ω_{i-1} be the linear and angular velocities of the coordinate system $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$ with respect to the base coordinate system $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$, respectively. Let ω_i, and ω_i^* be the angular velocity of O' with respect to $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ and $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$, respectively. Then, the linear velocity $\mathbf{v}_i$ and the angular velocity ω_i of the coordinate system $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ with respect to the base coordinate system are [from Eq. (3.3-17)], respectively,

$$\mathbf{v}_i = \frac{d^*\mathbf{p}_i^*}{dt} + \omega_{i-1} \times \mathbf{p}_i^* + \mathbf{v}_{i-1} \tag{3.3-20}$$

and

$$\omega_i = \omega_{i-1} + \omega_i^* \tag{3.3-21}$$

where $d^*(\ )/dt$ denotes the time derivative with respect to the moving coordinate system $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$. The linear acceleration $\dot{\mathbf{v}}_i$ and the angular acceleration $\dot{\omega}_i$ of the coordinate system $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ with respect to the base coordinate system are [from Eq. (3.3-19)], respectively,

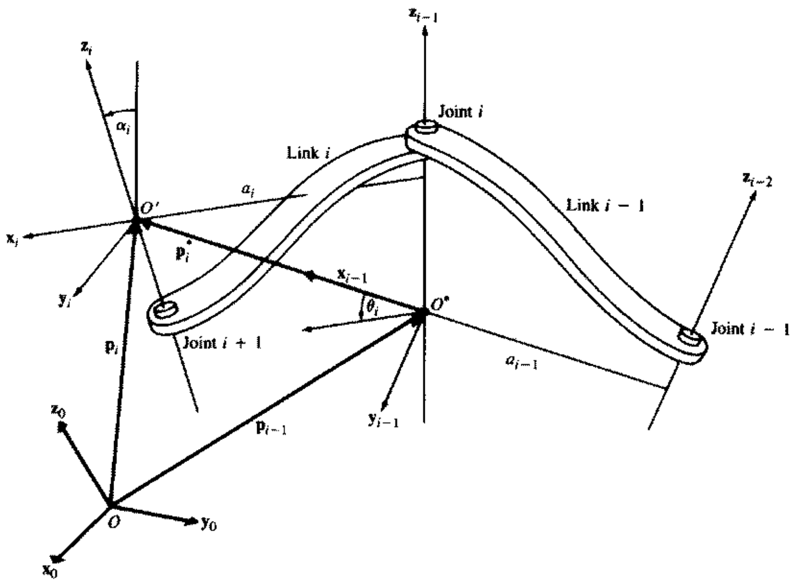


Figure 3.6 Relationship between O, O^* and O' frames.

$$\dot{\mathbf{v}}_i = \frac{d^{*2}\mathbf{p}_i^*}{dt^2} + \dot{\omega}_{i-1} \times \mathbf{p}_i^* + 2\omega_{i-1} \times \frac{d^*\mathbf{p}_i^*}{dt} \tag{3.3-22}$$
$$+ \omega_{i-1} \times (\omega_{i-1} \times \mathbf{p}_i^*) + \dot{\mathbf{v}}_{i-1}$$

and
$$\dot{\omega}_i = \dot{\omega}_{i-1} + \dot{\omega}_i^* \tag{3.3-23}$$

then, from Eq. (3.3-13), the angular acceleration of the coordinate system $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ with respect to $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$ is

$$\dot{\omega}_i^* = \frac{d^*\omega_i^*}{dt} + \omega_{i-1} \times \omega_i^* \tag{3.3-24}$$

therefore, Eq. (3.3-23) can be expressed as

$$\dot{\omega}_i = \dot{\omega}_{i-1} + \frac{d^*\omega_i^*}{dt} + \omega_{i-1} \times \omega_i^* \tag{3.3-25}$$

Recalling from the definition of link-joint parameters and the procedure for establishing link coordinate systems for a robot arm, the coordinate systems $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$ and $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ are attached to links $i-1$ and i, respectively. If link i is translational in the coordinate system $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$, it travels in the direction of $\mathbf{z}_{i-1}$ with a joint velocity $\dot{q}_i$ relative to link $i-1$. If it is rotational in coordinate system $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$, it has an angular velocity of ω_i^* and the angular motion of link i is about the $\mathbf{z}_{i-1}$ axis. Therefore,

$$\omega_i^* = \begin{cases} \mathbf{z}_{i-1}\dot{q}_i & \text{if link } i \text{ is rotational} \\ 0 & \text{if link } i \text{ is translational} \end{cases} \tag{3.3-26}$$

where $\dot{q}_i$ is the magnitude of angular velocity of link i with respect to the coordinate system $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$. Similarly,

$$\frac{d^*\omega_i^*}{dt} = \begin{cases} \mathbf{z}_{i-1}\ddot{q}_i & \text{if link } i \text{ is rotational} \\ 0 & \text{if link } i \text{ is translational} \end{cases} \tag{3.3-27}$$

Using Eqs. (3.3-26) and (3.3-27), Eqs. (3.3-21) and (3.3-25) can be expressed, respectively, as

$$\omega_i = \begin{cases} \omega_{i-1} + \mathbf{z}_{i-1}\dot{q}_i & \text{if link } i \text{ is rotational} \\ \omega_{i-1} & \text{if link } i \text{ is translational} \end{cases} \tag{3.3-28}$$

$$\dot{\omega}_i = \begin{cases} \dot{\omega}_{i-1} + \mathbf{z}_{i-1}\ddot{q}_i + \omega_{i-1} \times (\mathbf{z}_{i-1}\dot{q}_i) & \text{if link } i \text{ is rotational} \\ \dot{\omega}_{i-1} & \text{if link } i \text{ is translational} \end{cases} \tag{3.3-29}$$

Using Eq. (3.3-8), the linear velocity and acceleration of link i with respect to link $i-1$ can be obtained, respectively, as

$$\frac{d^*\mathbf{p}_i^*}{dt} = \begin{cases} \omega_i^* \times \mathbf{p}_i^* & \text{if link } i \text{ is rotational} \\ \mathbf{z}_{i-1}\dot{q}_i & \text{if link } i \text{ is translational} \end{cases} \tag{3.3-30}$$

$$\frac{d^{*2}\mathbf{p}_i^*}{dt^2} = \begin{cases} \dfrac{d^*\omega_i^*}{dt} \times \mathbf{p}_i^* + \omega_i^* \times (\omega_i^* \times \mathbf{p}_i^*) & \text{if link } i \text{ is rotational} \\ \mathbf{z}_{i-1}\ddot{q}_i & \text{if link } i \text{ is translational} \end{cases} \tag{3.3-31}$$

Therefore, using Eqs. (3.3-30) and (3.3-21), the linear velocity of link i with respect to the reference frame is [from Eq. (3.3-20)]

$$\mathbf{v}_i = \begin{cases} \omega_i \times \mathbf{p}_i^* + \mathbf{v}_{i-1} & \text{if link } i \text{ is rotational} \\ \mathbf{z}_{i-1}\dot{q}_i + \omega_i \times \mathbf{p}_i^* + \mathbf{v}_{i-1} & \text{if link } i \text{ is translational} \end{cases} \tag{3.3-32}$$

Using the following vector cross-product identities,

$$(\mathbf{a} \times \mathbf{b}) \times \mathbf{c} = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{a}(\mathbf{b} \cdot \mathbf{c}) \tag{3.3-33}$$

$$\mathbf{a} \times (\mathbf{b} \times \mathbf{c}) = \mathbf{b}(\mathbf{a} \cdot \mathbf{c}) - \mathbf{c}(\mathbf{a} \cdot \mathbf{b}) \tag{3.3-34}$$

and Eqs. (3.3-26) to (3.3-31), the acceleration of link i with respect to the reference system is [from Eq. (3.3-22)]

$$\dot{\mathbf{v}}_i = \begin{cases} \dot{\omega}_i \times \mathbf{p}_i^* + \omega_i \times (\omega_i \times \mathbf{p}_i^*) + \dot{\mathbf{v}}_{i-1} & \text{if link } i \text{ is rotational} \\ \mathbf{z}_{i-1}\ddot{q}_i + \dot{\omega}_i \times \mathbf{p}_i^* + 2\omega_i \times (\mathbf{z}_{i-1}\dot{q}_i) & \text{if link } i \text{ is} \\ \quad + \omega_i \times (\omega_i \times \mathbf{p}_i^*) + \dot{\mathbf{v}}_{i-1} & \text{translational} \end{cases} \tag{3.3-35}$$

Note that $\omega_i = \omega_{i-1}$ if link i is translational in Eq. (3.3-32). Equations (3.3-28), (3.3-29), (3.3-32), and (3.3-35) describe the kinematics information of link i that are useful in deriving the motion equations of a robot arm.

3.3.4 Recursive Equations of Motion for Manipulators

From the above kinematic information of each link, we would like to describe the motion of the robot arm links by applying d'Alembert's principle to each link. d'Alembert's principle applies the conditions of static equilibrium to problems in dynamics by considering both the externally applied driving forces and the reaction forces of mechanical elements which resist motion. d'Alembert's principle applies for all instants of time. It is actually a slightly modified form of Newton's second law of motion, and can be stated as:

For any body, the algebraic sum of externally applied forces and the forces resisting motion in any given direction is zero.

Consider a link i as shown in Fig. 3.7, and let the origin O' be situated at its center of mass. Then, by corresponding the variables defined in Fig. 3.6 with variables defined in Fig. 3.7, the remaining undefined variables, expressed with respect to the base reference system $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$, are:

m_i = total mass of link i

$\bar{\mathbf{r}}_i$ = position of the center of mass of link i from the origin of the base reference frame

$\bar{\mathbf{s}}_i$ = position of the center of mass of link i from the origin of the coordinate system $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$

$\mathbf{p}_i^*$ = the origin of the ith coordinate frame with respect to the $(i-1)$th coordinate system

$\bar{\mathbf{v}}_i = \dfrac{d\bar{\mathbf{r}}_i}{dt}$, linear velocity of the center of mass of link i

$\bar{\mathbf{a}}_i = \dfrac{d\bar{\mathbf{v}}_i}{dt}$, linear acceleration of the center of mass of link i

$\mathbf{F}_i$ = total external force exerted on link i at the center of mass

$\mathbf{N}_i$ = total external moment exerted on link i at the center of mass
$\mathbf{I}_i$ = inertia matrix of link i about its center of mass with reference to the coordinate system $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$
$\mathbf{f}_i$ = force exerted on link i by link $i-1$ at the coordinate frame $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$ to support link i and the links above it
$\mathbf{n}_i$ = moment exerted on link i by link $i-1$ at the coordinate frame $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$

Then, omitting the viscous damping effects of all the joints, and applying the d'Alembert principle to each link, we have

$$\mathbf{F}_i = \frac{d(m_i \bar{\mathbf{v}}_i)}{dt} = m_i \bar{\mathbf{a}}_i \tag{3.3-36}$$

and

$$\mathbf{N}_i = \frac{d(\mathbf{I}_i \omega_i)}{dt} = \mathbf{I}_i \dot{\omega}_i + \omega_i \times (\mathbf{I}_i \omega_i) \tag{3.3-37}$$

where, using Eqs. (3.3-32) and (3.3-35), the linear velocity and acceleration of the center of mass of link i are, respectively,†

$$\bar{\mathbf{v}}_i = \omega_i \times \bar{\mathbf{s}}_i + \mathbf{v}_i \tag{3.3-38}$$

and

$$\bar{\mathbf{a}}_i = \dot{\omega}_i \times \bar{\mathbf{s}}_i + \omega_i \times (\omega_i \times \bar{\mathbf{s}}_i) + \dot{\mathbf{v}}_i \tag{3.3-39}$$

Then, from Fig. 3.7, and looking at all the forces and moments acting on link i, the total external force $\mathbf{F}_i$ and moment $\mathbf{N}_i$ are those exerted on link i by gravity and neighboring links, link $i-1$ and link $i+1$. That is,

$$\mathbf{F}_i = \mathbf{f}_i - \mathbf{f}_{i+1} \tag{3.3-40}$$

and

$$\mathbf{N}_i = \mathbf{n}_i - \mathbf{n}_{i+1} + (\mathbf{p}_{i-1} - \bar{\mathbf{r}}_i) \times \mathbf{f}_i - (\mathbf{p}_i - \bar{\mathbf{r}}_i) \times \mathbf{f}_{i+1} \tag{3.3-41}$$

$$= \mathbf{n}_i - \mathbf{n}_{i+1} + (\mathbf{p}_{i-1} - \bar{\mathbf{r}}_i) \times \mathbf{F}_i - \mathbf{p}_i^* \times \mathbf{f}_{i+1} \tag{3.3-42}$$

Then, the above equations can be rewritten into recursive equations using the fact that $\bar{\mathbf{r}}_i - \mathbf{p}_{i-1} = \mathbf{p}_i^* + \bar{\mathbf{s}}_i$

$$\mathbf{f}_i = \mathbf{F}_i + \mathbf{f}_{i+1} = m_i \bar{\mathbf{a}}_i + \mathbf{f}_{i+1} \tag{3.3-43}$$

and

$$\mathbf{n}_i = \mathbf{n}_{i+1} + \mathbf{p}_i^* \times \mathbf{f}_{i+1} + (\mathbf{p}_i^* + \bar{\mathbf{s}}_i) \times \mathbf{F}_i + \mathbf{N}_i \tag{3.3-44}$$

The above equations are recursive and can be used to derive the forces and moments $(\mathbf{f}_i, \mathbf{n}_i)$ at the links for $i = 1, 2, \ldots, n$ for an n-link manipulator, noting that $\mathbf{f}_{n+1}$ and $\mathbf{n}_{n+1}$ are, respectively, the forces and moments exerted by the manipulator hand upon an external object.

From Chap. 2, the kinematic relationship between the neighboring links and the establishment of coordinate systems show that if joint i is rotational, then it

† Here $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ is the moving-rotating coordinate frame.

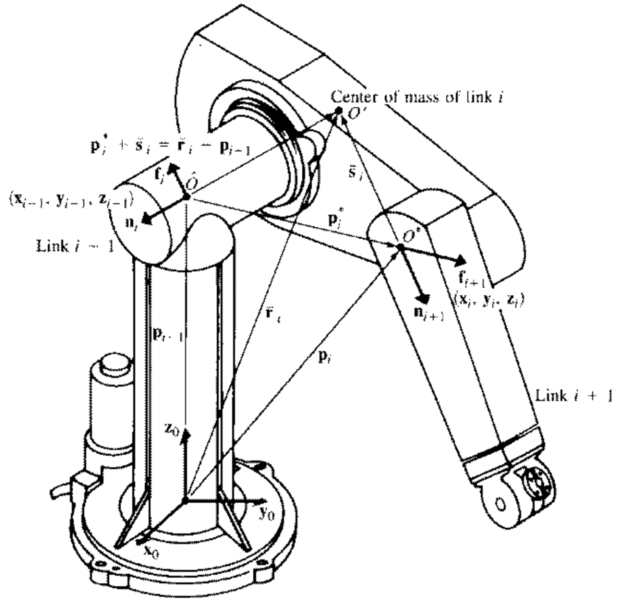


Figure 3.7 Forces and moments on link i.

actually rotates q_i radians in the coordinate system $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$ about the $\mathbf{z}_{i-1}$ axis. Thus, the input torque at joint i is the sum of the projection of $\mathbf{n}_i$ onto the $\mathbf{z}_{i-1}$ axis and the viscous damping moment in that coordinate system. However, if joint i is translational, then it translates q_i unit relative to the coordinate system $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$ along the $\mathbf{z}_{i-1}$ axis. Then, the input force τ_i at that joint is the sum of the projection of $\mathbf{f}_i$ onto the $\mathbf{z}_{i-1}$ axis and the viscous damping force in that coordinate system. Hence, the input torque/force for joint i is

$$\tau_i = \begin{cases} \mathbf{n}_i^T \mathbf{z}_{i-1} + b_i \dot{q}_i & \text{if link } i \text{ is rotational} \\ \mathbf{f}_i^T \mathbf{z}_{i-1} + b_i \dot{q}_i & \text{if link } i \text{ is translational} \end{cases} \tag{3.3-45}$$

where b_i is the viscous damping coefficient for joint i in the above equations.

If the supporting base is bolted on the platform and link 0 is stationary, then $\boldsymbol{\omega}_0 = \dot{\boldsymbol{\omega}}_0 = 0$ and $\mathbf{v}_0 = 0$ and $\dot{\mathbf{v}}_0$ (to include gravity) is

$$\dot{\mathbf{v}}_0 = \mathbf{g} = \begin{bmatrix} g_x \\ g_y \\ g_z \end{bmatrix} \quad \text{where} |\mathbf{g}| = 9.8062 \text{ m/s}^2 \tag{3.3-46}$$

In summary, the Newton-Euler equations of motion consist of a set of forward and backward recursive equations. They are Eqs. (3.3-28), (3.3-29), (3.3-35), (3.3-39), and (3.3-43) to (3.3-45) and are listed in Table 3.2. For the forward recursive equations, linear velocity and acceleration, angular velocity and acceleration of each individual link, are propagated from the base reference system to the end-effector. For the backward recursive equations, the torques and forces exerted on each link are computed recursively from the end-effector to the base reference system. Hence, the forward equations propagate kinematics information of each link from the base reference frame to the hand, while the backward equations compute the necessary torques/forces for each joint from the hand to the base reference system.

Table 3.2 Recursive Newton-Euler equations of motion

Forward equations: $i = 1, 2, \ldots, n$

$$\omega_i = \begin{cases} \omega_{i-1} + \mathbf{z}_{i-1}\dot{q}_i & \text{if link } i \text{ is rotational} \\ \omega_{i-1} & \text{if link } i \text{ is translational} \end{cases}$$

$$\dot{\omega}_i = \begin{cases} \dot{\omega}_{i-1} + \mathbf{z}_{i-1}\ddot{q}_i + \omega_{i-1} \times (\mathbf{z}_{i-1}\dot{q}_i) & \text{if link } i \text{ is rotational} \\ \dot{\omega}_{i-1} & \text{if link } i \text{ is translational} \end{cases}$$

$$\dot{\mathbf{v}}_i = \begin{cases} \dot{\omega}_i \times \mathbf{p}_i^* + \omega_i \times (\omega_i \times \mathbf{p}_i^*) + \dot{\mathbf{v}}_{i-1} & \text{if link } i \text{ is rotational} \\ \mathbf{z}_{i-1}\ddot{q}_i + \dot{\omega}_i \times \mathbf{p}_i^* + 2\omega_i \times (\mathbf{z}_{i-1}\dot{q}_i) & \\ \quad + \omega_i \times (\omega_i \times \mathbf{p}_i^*) + \dot{\mathbf{v}}_{i-1} & \text{if link } i \text{ is translational} \end{cases}$$

$$\bar{\mathbf{a}}_i = \dot{\omega}_i \times \bar{\mathbf{s}}_i + \omega_i \times (\omega_i \times \bar{\mathbf{s}}_i) + \dot{\mathbf{v}}_i$$

Backward equations: $i = n, n-1, \ldots, 1$

$$\mathbf{F}_i = m_i \bar{\mathbf{a}}_i$$

$$\mathbf{N}_i = \mathbf{I}_i \dot{\omega}_i + \omega_i \times (\mathbf{I}_i \omega_i)$$

$$\mathbf{f}_i = \mathbf{F}_i + \mathbf{f}_{i+1}$$

$$\mathbf{n}_i = \mathbf{n}_{i+1} + \mathbf{p}_i^* \times \mathbf{f}_{i+1} + (\mathbf{p}_i^* + \bar{\mathbf{s}}_i) \times \mathbf{F}_i + \mathbf{N}_i$$

$$\tau_i = \begin{cases} \mathbf{n}_i^T \mathbf{z}_{i-1} + b_i \dot{q}_i & \text{if link } i \text{ is rotational} \\ \mathbf{f}_i^T \mathbf{z}_{i-1} + b_i \dot{q}_i & \text{if link } i \text{ is translational} \end{cases}$$

where b_i is the viscous damping coefficient for joint i.

The "usual" initial conditions are $\omega_0 = \dot{\omega}_0 = \mathbf{v}_0 = \mathbf{0}$ and $\dot{\mathbf{v}}_0 = (g_x, g_y, g_z)^T$ (to include gravity), where $|\mathbf{g}| = 9.8062 \text{ m/s}^2$.

3.3.5 Recursive Equations of Motion of a Link About Its Own Coordinate Frame

The above equations of motion of a robot arm indicate that the resulting N-E dynamic equations, excluding gear friction, are a set of compact forward and backward recursive equations. This set of recursive equations can be applied to the robot links sequentially. The forward recursion propagates kinematics information such as angular velocities, angular accelerations, and linear accelerations from the base reference frame (inertial frame) to the end-effector. The backward recursion propagates the forces exerted on each link from the end-effector of the manipulator to the base reference frame, and the applied joint torques are computed from these forces.

One obvious drawback of the above recursive equations of motion is that all the inertial matrices $\mathbf{I}_i$ and the physical geometric parameters ($\bar{\mathbf{r}}_i$, $\bar{\mathbf{s}}_i$, $\mathbf{p}_{i-1}$, $\mathbf{p}_i^*$) are referenced to the base coordinate system. As a result, they change as the robot arm is moving. Luh et al. [1980*a*] improved the above N-E equations of motion by referencing all velocities, accelerations, inertial matrices, location of center of mass of each link, and forces/moments to their own link coordinate systems. Because of the nature of the formulation and the method of systematically computing the joint torques, computations are much simpler. The most important consequence of this modification is that the computation time of the applied torques is found linearly proportional to the number of joints of the robot arm and independent of the robot arm configuration. This enables the implementation of a simple real-time control algorithm for a robot arm in the joint-variable space.

Let ${}^{i-1}\mathbf{R}_i$ be a 3×3 rotation matrix which transforms any vector with reference to coordinate frame $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ to the coordinate system $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$. This is the upper left 3×3 submatrix of ${}^{i-1}\mathbf{A}_i$.

It has been shown before that

$$({}^{i-1}\mathbf{R}_i)^{-1} = {}^{i}\mathbf{R}_{i-1} = ({}^{i-1}\mathbf{R}_i)^T \tag{3.3-47}$$

where

$${}^{i-1}\mathbf{R}_i = \begin{bmatrix} \cos\theta_i & -\cos\alpha_i\sin\theta_i & \sin\alpha_i\sin\theta_i \\ \sin\theta_i & \cos\alpha_i\cos\theta_i & -\sin\alpha_i\cos\theta_i \\ 0 & \sin\alpha_i & \cos\alpha_i \end{bmatrix} \tag{3.3-48}$$

and

$$[{}^{i-1}\mathbf{R}_i]^{-1} = \begin{bmatrix} \cos\theta_i & \sin\theta_i & 0 \\ -\cos\alpha_i\sin\theta_i & \cos\alpha_i\cos\theta_i & \sin\alpha_i \\ \sin\alpha_i\sin\theta_i & -\sin\alpha_i\cos\theta_i & \cos\alpha_i \end{bmatrix} \tag{3.3-49}$$

Instead of computing ω_i, $\dot{\omega}_i$, $\dot{\mathbf{v}}_i$, $\bar{\mathbf{a}}_i$, $\mathbf{p}_i^*$, $\bar{\mathbf{s}}_i$, $\mathbf{F}_i$, $\mathbf{N}_i$, $\mathbf{f}_i$, $\mathbf{n}_i$, and τ_i which are referenced to the base coordinate system, we compute ${}^{i}\mathbf{R}_0\omega_i$, ${}^{i}\mathbf{R}_0\dot{\omega}_i$, ${}^{i}\mathbf{R}_0\dot{\mathbf{v}}_i$, ${}^{i}\mathbf{R}_0\dot{\mathbf{a}}_i$, ${}^{i}\mathbf{R}_0\mathbf{F}_i$, ${}^{i}\mathbf{R}_0\mathbf{N}_i$, ${}^{i}\mathbf{R}_0\mathbf{f}_i$, ${}^{i}\mathbf{R}_0\mathbf{n}_i$, and ${}^{i}\mathbf{R}_0\tau_i$ which are referenced to its own link coordinate system $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$. Hence, Eqs. (3.3-28), (3.3-29), (3.3-35), (3.3-39), (3.3-36), (3.3-37), (3.3-43), (3.3-44), and (3.3-45), respectively, become:

$$^i\mathbf{R}_0\omega_i = \begin{cases} ^i\mathbf{R}_{i-1}(^{i-1}\mathbf{R}_0\omega_{i-1} + \mathbf{z}_0\dot{q}_i) & \text{if link } i \text{ is rotational} \\ ^i\mathbf{R}_{i-1}(^{i-1}\mathbf{R}_0\omega_{i-1}) & \text{if link } i \text{ is translational} \end{cases} \tag{3.3-50}$$

$$^i\mathbf{R}_0\dot{\omega}_i = \begin{cases} ^i\mathbf{R}_{i-1}[^{i-1}\mathbf{R}_0\dot{\omega}_{i-1} + \mathbf{z}_0\ddot{q}_i + (^{i-1}\mathbf{R}_0\omega_{i-1}) \times \mathbf{z}_0\dot{q}_i] & \text{if link } i \text{ is rotational} \\ ^i\mathbf{R}_{i-1}(^{i-1}\mathbf{R}_0\dot{\omega}_{i-1}) & \text{if link } i \text{ is translational} \end{cases} \tag{3.3-51}$$

$$^i\mathbf{R}_0\dot{\mathbf{v}}_i = \begin{cases} (^i\mathbf{R}_0\dot{\omega}_i) \times (^i\mathbf{R}_0\mathbf{p}_i^*) + (^i\mathbf{R}_0\omega_i) \\ \quad \times [(^i\mathbf{R}_0\omega_i) \times (^i\mathbf{R}_0\mathbf{p}_i^*)] + {}^i\mathbf{R}_{i-1}(^{i-1}\mathbf{R}_0\dot{\mathbf{v}}_{i-1}) & \text{if link } i \text{ is rotational} \\ ^i\mathbf{R}_{i-1}(\mathbf{z}_0\ddot{q}_i + {}^{i-1}\mathbf{R}_0\dot{\mathbf{v}}_{i-1}) + (^i\mathbf{R}_0\dot{\omega}_i) \times (^i\mathbf{R}_0\mathbf{p}_i^*) \\ \quad + 2(^i\mathbf{R}_0\omega_i) \times (^i\mathbf{R}_{i-1}\mathbf{z}_0\dot{q}_i) \\ \quad + (^i\mathbf{R}_0\omega_i) \times [(^i\mathbf{R}_0\omega_i) \times (^i\mathbf{R}_0\mathbf{p}_i^*)] & \text{if link } i \text{ is translational} \end{cases} \tag{3.3-52}$$

$$^i\mathbf{R}_0\bar{\mathbf{a}}_i = (^i\mathbf{R}_0\dot{\omega}_i) \times (^i\mathbf{R}_0\bar{\mathbf{s}}_i) + (^i\mathbf{R}_0\omega_i) \times [(^i\mathbf{R}_0\omega_i) \times (^i\mathbf{R}_0\bar{\mathbf{s}}_i)] + {}^i\mathbf{R}_0\dot{\mathbf{v}}_i \tag{3.3-53}$$

$$^i\mathbf{R}_0\mathbf{F}_i = m_i\,{}^i\mathbf{R}_0\bar{\mathbf{a}}_i \tag{3.3-54}$$

$$^i\mathbf{R}_0\mathbf{N}_i = (^i\mathbf{R}_0\mathbf{I}_i{}^0\mathbf{R}_i)(^i\mathbf{R}_0\dot{\omega}_i) + (^i\mathbf{R}_0\omega_i) \times [(^i\mathbf{R}_0\mathbf{I}_i{}^0\mathbf{R}_i)(^i\mathbf{R}_0\omega_i)] \tag{3.3-55}$$

$$^i\mathbf{R}_0\mathbf{f}_i = {}^i\mathbf{R}_{i+1}(^{i+1}\mathbf{R}_0\mathbf{f}_{i+1}) + {}^i\mathbf{R}_0\mathbf{F}_i \tag{3.3-56}$$

$$^i\mathbf{R}_0\mathbf{n}_i = {}^i\mathbf{R}_{i+1}[^{i+1}\mathbf{R}_0\mathbf{n}_{i+1} + (^{i+1}\mathbf{R}_0\mathbf{p}_i^*) \times (^{i+1}\mathbf{R}_0\mathbf{f}_{i+1})] + (^i\mathbf{R}_0\mathbf{p}_i^* + {}^i\mathbf{R}_0\bar{\mathbf{s}}_i) \times (^i\mathbf{R}_0\mathbf{F}_i) + {}^i\mathbf{R}_0\mathbf{N}_i \tag{3.3-57}$$

and

$$\tau_i = \begin{cases} (^i\mathbf{R}_0\mathbf{n}_i)^T(^i\mathbf{R}_{i-1}\mathbf{z}_0) + b_i\dot{q}_i & \text{if link } i \text{ is rotational} \\ (^i\mathbf{R}_0\mathbf{f}_i)^T(^i\mathbf{R}_{i-1}\mathbf{z}_0) + b_i\dot{q}_i & \text{if link } i \text{ is translational} \end{cases} \tag{3.3-58}$$

where $\mathbf{z}_0 = (0, 0, 1)^T$, $^i\mathbf{R}_0\bar{\mathbf{s}}_i$ is the center of mass of link i referred to its own link coordinate system $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$, and $^i\mathbf{R}_0\mathbf{p}_i^*$ is the location of $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$ from the origin of $(\mathbf{x}_{i-1}, \mathbf{y}_{i-1}, \mathbf{z}_{i-1})$ with respect to the ith coordinate frame and is found to be

$$ {}^i\mathbf{R}_0\,\mathbf{p}_i^* = \begin{bmatrix} a_i \\ d_i \sin\alpha_i \\ d_i \cos\alpha_i \end{bmatrix} \qquad (3.3\text{-}59) $$

and $({}^i\mathbf{R}_0\mathbf{I}_i{}^0\mathbf{R}_i)$ is the inertia matrix of link i about its center of mass referred to its own link coordinate system $(\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i)$.

Hence, in summary, efficient Newton-Euler equations of motion are a set of forward and backward recursive equations with the dynamics and kinematics of each link referenced to its own coordinate system. A list of the recursive equations are found in Table 3.3.

3.3.6 Computational Algorithm

The Newton-Euler equations of motion represent the most efficient set of computational equations running on a uniprocessor computer at the present time. The computational complexity of the Newton-Euler equations of motion has been tabulated in Table 3.4. The total mathematical operations (multiplications and additions) are proportional to n, the number of degrees of freedom of the robot arm.

Since the equations of motion obtained are recursive in nature, it is advisable to state an algorithmic approach for computing the input joint torque/force for each joint actuator. Such an algorithm is given below.

Algorithm 3.1: Newton-Euler approach. Given an n-link manipulator, this computational procedure generates the nominal joint torque/force for all the joint actuators. Computations are based on the equations in Table 3.3.

Initial conditions:

$n =$ number of links (n joints)

$\omega_0 = \dot{\omega}_0 = \mathbf{v}_0 = \mathbf{0} \qquad \dot{\mathbf{v}}_0 = \mathbf{g} = (g_x, g_y, g_z)^T$ where $|\mathbf{g}| = 9.8062\ \text{m/s}^2$

Joint variables are q_i, $\dot{q}_i$, $\ddot{q}_i$ for $i = 1, 2, \ldots, n$

Link variables are i, $\mathbf{F}_i$, $\mathbf{f}_i$, $\mathbf{n}_i$, τ_i

Forward iterations:

N1. [*Set counter for iteration*] Set $i \leftarrow 1$.

N2. [*Forward iteration for kinematics information*] Compute ${}^i\mathbf{R}_0\,\omega_i$, ${}^i\mathbf{R}_0\dot{\omega}_i$, ${}^i\mathbf{R}_0\dot{\mathbf{v}}_i$, and ${}^i\mathbf{R}_0\bar{\mathbf{a}}_i$ using equations in Table 3.3.

N3. [*Check $i = n$?*] If $i = n$, go to step N4; otherwise set $i \leftarrow i + 1$ and return to step N2.

Table 3.3 Efficient recursive Newton-Euler equations of motion

Forward equations: $i = 1, 2, \ldots, n$

$$
{}^i\mathbf{R}_0\omega_i = \begin{cases} {}^i\mathbf{R}_{i-1}({}^{i-1}\mathbf{R}_0\omega_{i-1} + \mathbf{z}_0\dot{q}_i) & \text{if link } i \text{ is rotational} \\ {}^i\mathbf{R}_{i-1}({}^{i-1}\mathbf{R}_0\omega_{i-1}) & \text{if link } i \text{ is translational} \end{cases}
$$

$$
{}^i\mathbf{R}_0\dot{\omega}_i = \begin{cases} {}^i\mathbf{R}_{i-1}[{}^{i-1}\mathbf{R}_0\dot{\omega}_{i-1} + \mathbf{z}_0\ddot{q}_i + ({}^{i-1}\mathbf{R}_0\omega_{i-1}) \times \mathbf{z}_0\dot{q}_i] & \text{if link } i \text{ is rotational} \\ {}^i\mathbf{R}_{i-1}({}^{i-1}\mathbf{R}_0\dot{\omega}_{i-1}) & \text{if link } i \text{ is translational} \end{cases}
$$

$$
{}^i\mathbf{R}_0\dot{\mathbf{v}}_i = \begin{cases} ({}^i\mathbf{R}_0\dot{\omega}_i) \times ({}^i\mathbf{R}_0\mathbf{p}_i^*) + ({}^i\mathbf{R}_0\omega_i) \times [\,({}^i\mathbf{R}_0\omega_i) \times ({}^i\mathbf{R}_0\mathbf{p}_i^*)\,] \\ \quad + {}^i\mathbf{R}_{i-1}({}^{i-1}\mathbf{R}_0\dot{\mathbf{v}}_{i-1}) \quad \text{if link } i \text{ is rotational} \\ {}^i\mathbf{R}_{i-1}(\mathbf{z}_0\ddot{q}_i + {}^{i-1}\mathbf{R}_0\dot{\mathbf{v}}_{i-1}) + ({}^i\mathbf{R}_0\dot{\omega}_i) \times ({}^i\mathbf{R}_0\mathbf{p}_i^*) \\ \quad + 2({}^i\mathbf{R}_0\omega_i) \times ({}^i\mathbf{R}_{i-1}\mathbf{z}_0\dot{q}_i) \\ \quad + ({}^i\mathbf{R}_0\omega_i) \times [\,({}^i\mathbf{R}_0\omega_i) \times ({}^i\mathbf{R}_0\mathbf{p}_i^*)\,] \quad \text{if link } i \text{ is translational} \end{cases}
$$

$$
{}^i\mathbf{R}_0\bar{\mathbf{a}}_i = ({}^i\mathbf{R}_0\dot{\omega}_i) \times ({}^i\mathbf{R}_0\bar{\mathbf{s}}_i) + ({}^i\mathbf{R}_0\omega_i) \times [({}^i\mathbf{R}_0\omega_i) \times ({}^i\mathbf{R}_0\bar{\mathbf{s}}_i)] + {}^i\mathbf{R}_0\dot{\mathbf{v}}_i
$$

Backward equations: $i = n, n-1, \ldots, 1$

$$
{}^i\mathbf{R}_0\mathbf{f}_i = {}^i\mathbf{R}_{i+1}({}^{i+1}\mathbf{R}_0\mathbf{f}_{i+1}) + m_i\,{}^i\mathbf{R}_0\bar{\mathbf{a}}_i
$$

$$
\begin{aligned} {}^i\mathbf{R}_0\mathbf{n}_i = {} & {}^i\mathbf{R}_{i+1}[{}^{i+1}\mathbf{R}_0\mathbf{n}_{i+1} + ({}^{i+1}\mathbf{R}_0\mathbf{p}_i^*) \times ({}^{i+1}\mathbf{R}_0\mathbf{f}_{i+1})] + ({}^i\mathbf{R}_0\mathbf{p}_i^* + {}^i\mathbf{R}_0\bar{\mathbf{s}}_i) \times ({}^i\mathbf{R}_0\mathbf{F}_i) \\ & + ({}^i\mathbf{R}_0\mathbf{I}_i\,{}^0\mathbf{R}_i)({}^i\mathbf{R}_0\dot{\omega}_i) + ({}^i\mathbf{R}_0\omega_i) \times [({}^i\mathbf{R}_0\mathbf{I}_i\,{}^0\mathbf{R}_i)({}^i\mathbf{R}_0\omega_i)] \end{aligned}
$$

$$
\tau_i = \begin{cases} ({}^i\mathbf{R}_0\mathbf{n}_i)^T({}^i\mathbf{R}_{i-1}\mathbf{z}_0) + b_i\dot{q}_i & \text{if link } i \text{ is rotational} \\ ({}^i\mathbf{R}_0\mathbf{f}_i)^T({}^i\mathbf{R}_{i-1}\mathbf{z}_0) + b_i\dot{q}_i & \text{if link } i \text{ is translational} \end{cases}
$$

where $\mathbf{z}_0 = (0, 0, 1)^T$ and b_i is the viscous damping coefficient for joint i. The usual initial conditions are $\omega_0 = \dot{\omega}_0 = \mathbf{v}_0 = \mathbf{0}$ and $\dot{\mathbf{v}}_0 = (g_x, g_y, g_z)^T$ (to include gravity), where $|\mathbf{g}| = 9.8062$ m/s^2.

Backward iterations:

N4. [*Set* $\mathbf{f}_{n+1}$ *and* $\mathbf{n}_{n+1}$] Set $\mathbf{f}_{n+1}$ and $\mathbf{n}_{n+1}$ to the required force and moment, respectively, to carry the load. If no load, they are set to zero.

Table 3.4 Breakdown of mathematical operations of the Newton-Euler equations of motion for a PUMA robot arm

Newton-Euler equations of motion	Multiplications	Additions
${}^i\mathbf{R}_0\omega_i$	$9n$†	$7n$
${}^i\mathbf{R}_0\dot{\omega}_i$	$9n$	$9n$
${}^i\mathbf{R}_0\dot{\mathbf{v}}_i$	$27n$	$22n$
${}^i\mathbf{R}_0\mathbf{a}_i$	$15n$	$14n$
${}^i\mathbf{R}_0\mathbf{F}_i$	$3n$	0
${}^i\mathbf{R}_0\mathbf{f}_i$	$9(n-1)$	$9n-6$
${}^i\mathbf{R}_0\mathbf{N}_i$	$24n$	$18n$
${}^i\mathbf{R}_0\mathbf{n}_i$	$21n-15$	$24n-15$
Total mathematical operations	$117n-24$	$103n-21$

† n = number of degrees of freedom of the robot arm.

N5. [*Compute joint force/torque*] Compute ${}^i\mathbf{R}_0\mathbf{F}_i$, ${}^i\mathbf{R}_0\mathbf{N}_i$, ${}^i\mathbf{R}_0\mathbf{f}_i$, ${}^i\mathbf{R}_0\mathbf{n}_i$, and τ_i with $\mathbf{f}_{n+1}$ and $\mathbf{n}_{n+1}$ given.

N6. [*Backward iteration*] If $i = 1$, then stop; otherwise set $i \leftarrow i - 1$ and go to step N5.

3.3.7 A Two-Link Manipulator Example

In order to illustrate the use of the N-E equations of motion, the same two-link manipulator with revolute joints as shown in Fig. 3.2 is worked out in this section. All the rotation axes at the joints are along the **z** axis perpendicular to the paper surface. The physical dimensions, center of mass, and mass of each link and coordinate systems are given in Sec. 3.2.6.

First, we obtain the rotation matrices from Fig. 3.2 using Eqs. (3.3-48) and (3.3-49):

$$
{}^0\mathbf{R}_1 = \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad {}^1\mathbf{R}_2 = \begin{bmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad {}^0\mathbf{R}_2 = \begin{bmatrix} C_{12} & -S_{12} & 0 \\ S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

$$
{}^1\mathbf{R}_0 = \begin{bmatrix} C_1 & S_1 & 0 \\ -S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad {}^2\mathbf{R}_1 = \begin{bmatrix} C_2 & S_2 & 0 \\ -S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad {}^2\mathbf{R}_0 = \begin{bmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}
$$

From the equations in Table 3.3 we assume the following initial conditions:

$$\omega_0 = \dot{\omega}_0 = \mathbf{v}_0 = \mathbf{0} \quad \text{and} \quad \dot{\mathbf{v}}_0 = (0,\ g,\ 0)^T \quad \text{with } g = 9.8062 \text{ m/s}^2$$

Forward Equations for $i = 1, 2$. Using Eq. (3.3-50), compute the angular velocity for revolute joint for $i = 1, 2$. So for $i = 1$, with $\omega_0 = \mathbf{0}$, we have:

$$^1\mathbf{R}_0\omega_1 = {}^1\mathbf{R}_0(\omega_0 + \mathbf{z}_0\dot{\theta}_1)$$

$$= \begin{bmatrix} C_1 & S_1 & 0 \\ -S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1$$

For $i = 2$, we have:

$$^2\mathbf{R}_0\omega_2 = {}^2\mathbf{R}_1({}^1\mathbf{R}_0\omega_1 + \mathbf{z}_0\dot{\theta}_2)$$

$$= \begin{bmatrix} C_2 & S_2 & 0 \\ -S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1 + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_2 \right] = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} (\dot{\theta}_1 + \dot{\theta}_2)$$

Using Eq. (3.3-51), compute the angular acceleration for revolute joints for $i = 1, 2$:

For $i = 1$, with $\dot{\omega}_0 = \omega_0 = \mathbf{0}$, we have:

$$^1\mathbf{R}_0\dot{\omega}_1 = {}^1\mathbf{R}_0(\dot{\omega}_0 + \mathbf{z}_0\ddot{\theta}_1 + \omega_0 \times \mathbf{z}_0\dot{\theta}_1) = (0,\ 0,\ 1)^T\, \ddot{\theta}_1$$

For $i = 2$, we have:

$$^2\mathbf{R}_0\dot{\omega}_2 = {}^2\mathbf{R}_1[{}^1\mathbf{R}_0\dot{\omega}_1 + \mathbf{z}_0\ddot{\theta}_2 + ({}^1\mathbf{R}_0\omega_1) \times \mathbf{z}_0\dot{\theta}_2] = (0,\ 0,\ 1)^T\, (\ddot{\theta}_1 + \ddot{\theta}_2)$$

Using Eq. (3.3-52), compute the linear acceleration for revolute joints for $i = 1, 2$:

For $i = 1$, with $\dot{\mathbf{v}}_0 = (0,\ g,\ 0)^T$, we have:

$$^1\mathbf{R}_0\dot{\mathbf{v}}_1 = ({}^1\mathbf{R}_0\dot{\omega}_1) \times ({}^1\mathbf{R}_0\mathbf{p}_1^*) + ({}^1\mathbf{R}_0\omega_1) \times [({}^1\mathbf{R}_0\omega_1) \times ({}^1\mathbf{R}_0\mathbf{p}_1^*)] + {}^1\mathbf{R}_0\dot{\mathbf{v}}_0$$

$$= \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ddot{\theta}_1 \times \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1 \times \left\{ \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \dot{\theta}_1 \times \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix} \right\} + \begin{bmatrix} gS_1 \\ gC_1 \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} -l\dot{\theta}_1^2 + gS_1 \\ l\ddot{\theta}_1 + gC_1 \\ 0 \end{bmatrix}$$

For $i = 2$, we have:

$$ {}^2\mathbf{R}_0\dot{\mathbf{v}}_2 = ({}^2\mathbf{R}_0\dot{\omega}_2) \times ({}^2\mathbf{R}_0\mathbf{p}_2^*) + ({}^2\mathbf{R}_0\omega_2)\times[({}^2\mathbf{R}_0\omega_2) \times ({}^2\mathbf{R}_0\mathbf{p}_2^*)] + {}^2\mathbf{R}_1({}^1\mathbf{R}_0\dot{\mathbf{v}}_1) $$

$$ = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \left\{ \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix} \right\} $$

$$ + \begin{bmatrix} C_2 & S_2 & 0 \\ -S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -l\dot{\theta}_1^2 + gS_1 \\ l\ddot{\theta}_1 + gC_1 \\ 0 \end{bmatrix} $$

$$ = \begin{bmatrix} l(S_2\ddot{\theta}_1 - C_2\dot{\theta}_1^2 - \dot{\theta}_1^2 - \dot{\theta}_2^2 - 2\dot{\theta}_1\dot{\theta}_2) + gS_{12} \\ l(\ddot{\theta}_1 + \ddot{\theta}_2 + C_2\ddot{\theta}_1 + S_2\dot{\theta}_1^2) + gC_{12} \\ 0 \end{bmatrix} $$

Using Eq. (3.3-53), compute the linear acceleration at the center of mass for links 1 and 2:

For $i = 1$, we have:

$$ {}^1\mathbf{R}_0\bar{\mathbf{a}}_1 = ({}^1\mathbf{R}_0\dot{\omega}_1) \times ({}^1\mathbf{R}_0\bar{\mathbf{s}}_1) + ({}^1\mathbf{R}_0\omega_1) \times [({}^1\mathbf{R}_0\omega_1) \times ({}^1\mathbf{R}_0\bar{\mathbf{s}}_1)] + {}^1\mathbf{R}_0\dot{\mathbf{v}}_1 $$

where

$$ \bar{\mathbf{s}}_1 = \begin{bmatrix} -\frac{l}{2}C_1 \\ -\frac{l}{2}S_1 \\ 0 \end{bmatrix} \qquad {}^1\mathbf{R}_0\bar{\mathbf{s}}_1 = \begin{bmatrix} C_1 & S_1 & 0 \\ -S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{l}{2}C_1 \\ -\frac{l}{2}S_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \end{bmatrix} $$

Thus,

$$ {}^1\mathbf{R}_0\bar{\mathbf{a}}_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \ddot{\theta}_1 \times \begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \left\{ \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 \end{bmatrix} \times \begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \end{bmatrix} \right\} $$

$$ + \begin{bmatrix} -l\dot{\theta}_1^2 + gS_1 \\ l\ddot{\theta}_1 + gC_1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{l}{2}\dot{\theta}_1^2 + gS_1 \\ \frac{l}{2}\ddot{\theta}_1 + gC_1 \\ 0 \end{bmatrix} $$

For $i = 2$, we have:

$$^2\mathbf{R}_0\bar{\mathbf{a}}_2 = (^2\mathbf{R}_0\dot{\omega}_2) \times (^2\mathbf{R}_0\bar{\mathbf{s}}_2) + (^2\mathbf{R}_0\omega_2) \times [(^2\mathbf{R}_0\omega_2) \times (^2\mathbf{R}_0\bar{\mathbf{s}}_2)] + {}^2\mathbf{R}_0\dot{\mathbf{v}}_2$$

where

$$\bar{\mathbf{s}}_2 = \begin{bmatrix} -\frac{l}{2}C_{12} \\ -\frac{l}{2}S_{12} \\ 0 \end{bmatrix} \qquad {}^2\mathbf{R}_0\bar{\mathbf{s}}_2 = \begin{bmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -\frac{l}{2}C_{12} \\ -\frac{l}{2}S_{12} \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$^2\mathbf{R}_0\bar{\mathbf{a}}_2 = \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \left[\begin{bmatrix} 0 \\ 0 \\ \dot{\theta}_1 + \dot{\theta}_2 \end{bmatrix} \times \begin{bmatrix} -\frac{l}{2} \\ 0 \\ 0 \end{bmatrix} \right]$$

$$+ \begin{bmatrix} l(S_2\ddot{\theta}_1 - C_2\dot{\theta}_1^2 - \dot{\theta}_1^2 - \dot{\theta}_2^2 - 2\dot{\theta}_1\dot{\theta}_2) + gS_{12} \\ l(\ddot{\theta}_1 + \ddot{\theta}_2 + C_2\ddot{\theta}_1 + S_2\dot{\theta}_1^2) + gC_{12} \\ 0 \end{bmatrix}$$

$$= \begin{bmatrix} l(S_2\ddot{\theta}_1 - C_2\dot{\theta}_1^2 - \tfrac{1}{2}\dot{\theta}_1^2 - \tfrac{1}{2}\dot{\theta}_2^2 - \dot{\theta}_1\dot{\theta}_2) + gS_{12} \\ l(C_2\ddot{\theta}_1 + S_2\dot{\theta}_1^2 + \tfrac{1}{2}\ddot{\theta}_1 + \tfrac{1}{2}\ddot{\theta}_2) + gC_{12} \\ 0 \end{bmatrix}$$

Backward Equations for $i = 2, 1$. Assuming no-load conditions, $\mathbf{f}_3 = \mathbf{n}_3 = \mathbf{0}$. We use Eq. (3.3-56) to compute the force exerted on link i for $i = 2, 1$:

For $i = 2$, with $\mathbf{f}_3 = \mathbf{0}$, we have:

$$^2\mathbf{R}_0\mathbf{f}_2 = {}^2\mathbf{R}_3(^3\mathbf{R}_0\mathbf{f}_3) + {}^2\mathbf{R}_0\mathbf{F}_2 = {}^2\mathbf{R}_0\mathbf{F}_2 = m_2\,{}^2\mathbf{R}_0\bar{\mathbf{a}}_2$$

$$= \begin{bmatrix} m_2 l(S_2\ddot{\theta}_1 - C_2\dot{\theta}_1^2 - \tfrac{1}{2}\dot{\theta}_1^2 - \tfrac{1}{2}\dot{\theta}_2^2 - \dot{\theta}_1\dot{\theta}_2) + gm_2S_{12}) \\ m_2 l(C_2\ddot{\theta}_1 + S_2\dot{\theta}_1^2 + \tfrac{1}{2}\ddot{\theta}_1 + \tfrac{1}{2}\ddot{\theta}_2) + gm_2C_{12} \\ 0 \end{bmatrix}$$

For $i = 1$, we have:

$$^1\mathbf{R}_0\mathbf{f}_1 = {}^1\mathbf{R}_2(^2\mathbf{R}_0\mathbf{f}_2) + {}^1\mathbf{R}_0\mathbf{F}_1$$

$$= \begin{bmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} m_2 l(S_2\ddot{\theta}_1 - C_2\dot{\theta}_2 - \tfrac{1}{2}\dot{\theta}_1^2 - \tfrac{1}{2}\dot{\theta}_2^2 - \dot{\theta}_1\dot{\theta}_2) + gm_2S_{12} \\ m_2 l(C_2\ddot{\theta}_1 + S_2\dot{\theta}_1^2 + \tfrac{1}{2}\ddot{\theta}_1 + \tfrac{1}{2}\ddot{\theta}_2) + gm_2C_{12} \\ 0 \end{bmatrix} + m_1\,{}^1\mathbf{R}_0\bar{\mathbf{a}}_1$$

$$= \begin{bmatrix} m_2 l[-\dot{\theta}_1^2 - \frac{1}{2}C_2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - C_2\dot{\theta}_1\dot{\theta}_2 - \frac{1}{2}S_2(\ddot{\theta}_1 + \ddot{\theta}_2)] - m_2 g(C_{12}S_2 - C_2 S_{12}) - \frac{1}{2}m_1 l\dot{\theta}_1^2 + m_1 g S_1 \\ m_2 l[\ddot{\theta}_1 - \frac{1}{2}S_2(\dot{\theta}_1^2 + \dot{\theta}_2^2) - S_2\dot{\theta}_1\dot{\theta}_2 + \frac{1}{2}C_2(\ddot{\theta}_1 + \ddot{\theta}_2)] + m_2 g C_1 + \frac{1}{2}m_1 l\ddot{\theta}_1 + g m_1 C_1 \\ 0 \end{bmatrix}$$

Using Eq. (3.3-57), compute the moment exerted on link i for $i = 2, 1$: For $i = 2$, with $\mathbf{n}_3 = \mathbf{0}$, we have:

$$^2\mathbf{R}_0\mathbf{n}_2 = (^2\mathbf{R}_0\mathbf{p}_2^* + {}^2\mathbf{R}_0\bar{\mathbf{s}}_2) \times (^2\mathbf{R}_0\mathbf{F}_2) + {}^2\mathbf{R}_0\mathbf{N}_2$$

where

$$\mathbf{p}_2^* = \begin{bmatrix} lC_{12} \\ lS_{12} \\ 0 \end{bmatrix} \qquad {}^2\mathbf{R}_0\,\mathbf{p}_2^* = \begin{bmatrix} C_{12} & S_{12} & 0 \\ -S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} lC_{12} \\ lS_{12} \\ 0 \end{bmatrix} = \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$^2\mathbf{R}_0\mathbf{n}_2 = \begin{bmatrix} \frac{l}{2} \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} m_2 l(S_2\ddot{\theta}_1 - C_2\dot{\theta}_1^2 - \frac{1}{2}\dot{\theta}_1^2 - \frac{1}{2}\dot{\theta}_2^2 - \dot{\theta}_1\dot{\theta}_2) + g m_2 S_{12} \\ m_2 l(C_2\ddot{\theta}_1 + S_2\dot{\theta}_1^2 + \frac{1}{2}\ddot{\theta}_1 + \frac{1}{2}\ddot{\theta}_2) + g m_2 C_{12} \\ 0 \end{bmatrix}$$

$$+ \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12}m_2 l^2 & 0 \\ 0 & 0 & \frac{1}{12}m_2 l^2 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ \ddot{\theta}_1 + \ddot{\theta}_2 \end{bmatrix}$$

$$= \begin{bmatrix} 0 \\ 0 \\ \frac{1}{3}m_2 l^2\ddot{\theta}_1 + \frac{1}{3}m_2 l^2\ddot{\theta}_2 + \frac{1}{2}m_2 l^2(C_2\ddot{\theta}_1 + S_2\dot{\theta}_1^2) + \frac{1}{2}m_2 g l C_{12} \end{bmatrix}$$

For $i = 1$, we have:

$$^1\mathbf{R}_0\mathbf{n}_1 = {}^1\mathbf{R}_2[^2\mathbf{R}_0\mathbf{n}_2 + (^2\mathbf{R}_0\mathbf{p}_1^*) \times (^2\mathbf{R}_0\mathbf{f}_2)] + (^1\mathbf{R}_0\mathbf{p}_1^* + {}^1\mathbf{R}_0\bar{\mathbf{s}}_1) \times (^1\mathbf{R}_0\mathbf{F}_1) + {}^1\mathbf{R}_0\mathbf{N}_1$$

where

$$\mathbf{p}_1^* = \begin{bmatrix} lC_1 \\ lS_1 \\ 0 \end{bmatrix} \qquad {}^2\mathbf{R}_0\,\mathbf{p}_1^* = \begin{bmatrix} lC_2 \\ -lS_2 \\ 0 \end{bmatrix} \qquad {}^1\mathbf{R}_0\,\mathbf{p}_1^* = \begin{bmatrix} l \\ 0 \\ 0 \end{bmatrix}$$

Thus,

$$^1\mathbf{R}_0\mathbf{n}_1 = {}^1\mathbf{R}_2({}^2\mathbf{R}_0\mathbf{n}_2) + {}^1\mathbf{R}_2[({}^2\mathbf{R}_0\mathbf{p}_1^*) \times ({}^2\mathbf{R}_0\mathbf{f}_2)] + \left(\frac{l}{2}, 0, 0\right)^T \times {}^1\mathbf{R}_0\mathbf{F}_1 + {}^1\mathbf{R}_0\mathbf{N}_1$$

Finally, we obtain the joint torques applied to each of the joint actuators for both links, using Eq. (3.3-58):

For $i = 2$, with $b_2 = 0$, we have:

$$\tau_2 = ({}^2\mathbf{R}_0\mathbf{n}_2)^T({}^2\mathbf{R}_1\mathbf{z}_0)$$
$$= \tfrac{1}{3}m_2 l^2\ddot{\theta}_1 + \tfrac{1}{3}m_2 l^2\ddot{\theta}_2 + \tfrac{1}{2}m_2 l^2 C_2\ddot{\theta}_1 + \tfrac{1}{2}m_2 glC_{12} + \tfrac{1}{2}m_2 l^2 S_2\dot{\theta}_1^2$$

For $i = 1$, with $b_1 = 0$, we have:

$$\tau_1 = ({}^1\mathbf{R}_0\mathbf{n}_1)^T({}^1\mathbf{R}_0\mathbf{z}_0)$$
$$= \tfrac{1}{3}m_1 l^2\ddot{\theta}_1 + \tfrac{4}{3}m_2 l^2\ddot{\theta}_1 + \tfrac{1}{3}m_2 l^2\ddot{\theta}_2 + m_2 C_2 l^2\ddot{\theta}_1$$
$$+ \tfrac{1}{2}m_2 l^2 C_2\ddot{\theta}_2 - m_2 S_2 l^2\dot{\theta}_1\dot{\theta}_2 - \tfrac{1}{2}m_2 S_2 l^2\dot{\theta}_2^2 + \tfrac{1}{2}m_1 glC_1$$
$$+ \tfrac{1}{2}m_2 glC_{12} + m_2 glC_1$$

The above equations of motion agree with those obtained from the Lagrange-Euler formulation in Sec. 3.2.6.

3.4 GENERALIZED d'ALEMBERT EQUATIONS OF MOTION

Computationally, the L-E equations of motion are inefficient due to the 4×4 homogeneous matrix manipulations, while the efficiency of the N-E formulation can be seen from the vector formulation and its recursive nature. In order to obtain an efficient set of closed-form equations of motion, one can utilize the relative position vector and rotation matrix representation to describe the kinematic information of each link, obtain the kinetic and potential energies of the robot arm to form the lagrangian function, and apply the Lagrange-Euler formulation to obtain the equations of motion. In this section, we derive a Lagrange form of d'Alembert equations of motion or generalized d'Alembert equations of motion (G-D). We shall only focus on robot arms with rotary joints.

Assuming that the links of the robot arm are rigid bodies, the angular velocity ω_s of link s with respect to the base coordinate frame can be expressed as a sum of the relative angular velocities from the lower joints (see Fig. 3.8),

$$\omega_s = \sum_{j=1}^{s} \dot{\theta}_j \mathbf{z}_{j-1} \tag{3.4-1}$$

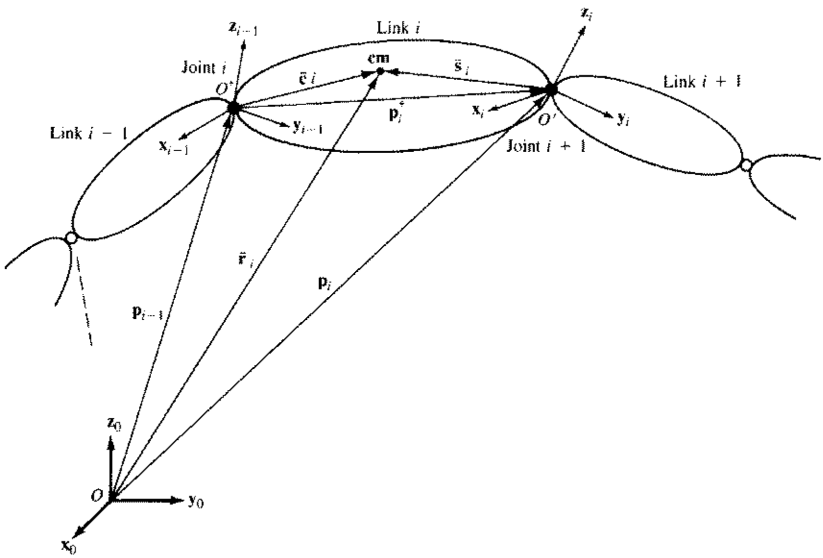


Figure 3.8 Vector definition in the generalized d'Alembert equations.

where $\mathbf{z}_{j-1}$ is the axis of rotation of joint j with reference to the base coordinate frame. Premultiplying the above angular velocity by the rotation matrix ${}^s\mathbf{R}_0$ changes its reference to the coordinate frame of link s; that is,

$$ {}^s\mathbf{R}_0\omega_s = \sum_{j=1}^{s} \dot{\theta}_j \, {}^s\mathbf{R}_0\mathbf{z}_{j-1} \tag{3.4-2} $$

In Fig. 3.8, let $\bar{\mathbf{r}}_s$ be the position vector to the center of mass of link s from the base coordinate frame. This position vector can be expressed as

$$ \bar{\mathbf{r}}_s = \sum_{j=1}^{s-1} \mathbf{p}_j^* + \bar{\mathbf{c}}_s \tag{3.4-3} $$

where $\bar{\mathbf{c}}_s$ is the position vector of the center of mass of link s from the $(s-1)$th coordinate frame with reference to the base coordinate frame.

Using Eqs. (3.4-1) to (3.4-3), the linear velocity of link s, $\mathbf{v}_s$, with respect to the base coordinate frame can be computed as a sum of the linear velocities from the lower links,

$$ \mathbf{v}_s = \sum_{k=1}^{s-1} \left[\left(\sum_{j=1}^{k} \dot{\theta}_j \mathbf{z}_{j-1} \right) \times \mathbf{p}_k^* \right] + \left(\sum_{j=1}^{s} \dot{\theta}_j \mathbf{z}_{j-1} \right) \times \bar{\mathbf{c}}_s \tag{3.4-4} $$

The kinetic energy of link s ($1 \leqslant s \leqslant n$) with mass m_s can be expressed as the summation of the kinetic energies due to the translational and rotational effects at its center of mass:

$$K_s = (K_s)_{\text{tran}} + (K_s)_{\text{rot}} = \tfrac{1}{2} m_s (\mathbf{v}_s \cdot \mathbf{v}_s) + \tfrac{1}{2}({}^s\mathbf{R}_0 \omega_s)^T \, \mathbf{I}_s ({}^s\mathbf{R}_0 \omega_s) \qquad (3.4\text{-}5)$$

where $\mathbf{I}_s$ is the inertia tensor of link s about its center of mass expressed in the sth coordinate system.

For ease of discussion and derivation, the equations of motion due to the translational, rotational, and gravitational effects of the links will be considered and treated separately. Applying the Lagrange-Euler formulation to the above translational kinetic energy of link s with respect to the generalized coordinate θ_i ($s \geqslant i$), we have

$$\frac{d}{dt}\left[\frac{\partial (K_s)_{\text{tran}}}{\partial \dot{\theta}_i}\right] - \frac{\partial (K_s)_{\text{tran}}}{\partial \theta_i}$$

$$= \frac{d}{dt}\left(m_s \, \mathbf{v}_s \cdot \frac{\partial \mathbf{v}_s}{\partial \dot{\theta}_i}\right) - m_s \, \mathbf{v}_s \cdot \frac{\partial \mathbf{v}_s}{\partial \theta_i}$$

$$= m_s \, \dot{\mathbf{v}}_s \cdot \frac{\partial \mathbf{v}_s}{\partial \dot{\theta}_i} + m_s \mathbf{v}_s \cdot \frac{d}{dt}\left(\frac{\partial \mathbf{v}_s}{\partial \dot{\theta}_i}\right) - m_s \, \mathbf{v}_s \cdot \frac{\partial \mathbf{v}_s}{\partial \theta_i} \qquad (3.4\text{-}6)$$

where

$$\frac{\partial \mathbf{v}_s}{\partial \dot{\theta}_i} = \mathbf{z}_{i-1} \times (\mathbf{p}_i^* + \mathbf{p}_{i+1}^* + \cdots + \mathbf{p}_{s-1}^* + \bar{\mathbf{c}}_s) \qquad (3.4\text{-}7)$$

$$= \mathbf{z}_{i-1} \times (\bar{\mathbf{r}}_s - \mathbf{p}_{i-1}) \qquad s \geqslant i$$

Using the identities

$$\frac{d}{dt}\left(\frac{\partial \mathbf{v}_s}{\partial \dot{\theta}_i}\right) = \frac{\partial \dot{\mathbf{v}}_s}{\partial \dot{\theta}_i} \qquad \text{and} \qquad \frac{\partial \dot{\mathbf{v}}_s}{\partial \dot{\theta}_i} = \frac{\partial \mathbf{v}_s}{\partial \theta_i} \qquad (3.4\text{-}8)$$

Eq. (3.4-6) becomes

$$\frac{d}{dt}\left[\frac{\partial (K_s)_{\text{tran}}}{\partial \dot{\theta}_i}\right] - \frac{\partial (K_s)_{\text{tran}}}{\partial \theta_i} = m_s \dot{\mathbf{v}}_s \cdot [\mathbf{z}_{i-1} \times (\bar{\mathbf{r}}_s - \mathbf{p}_{i-1})] \qquad (3.4\text{-}9)$$

Summing all the links from i to n gives the reaction torques due to the translational effect of all the links,

$$\frac{d}{dt}\left[\frac{\partial (\text{K.E.})_{\text{tran}}}{\partial \dot{\theta}_i}\right] - \frac{\partial (\text{K.E.})_{\text{tran}}}{\partial \theta_i} = \sum_{s=i}^{n}\left\{\frac{d}{dt}\left[\frac{\partial (K_s)_{\text{tran}}}{\partial \dot{\theta}_i}\right] - \frac{\partial (K_s)_{\text{tran}}}{\partial \theta_i}\right\}$$

$$= \sum_{s=i}^{n} m_s \, \dot{\mathbf{v}}_s \cdot [\mathbf{z}_{i-1} \times (\bar{\mathbf{r}}_s - \mathbf{p}_{i-1})] \qquad (3.4\text{-}10)$$

where, using Eqs. (3.3-8) and (3.3-12), the acceleration of link s is given by

$$\dot{\mathbf{v}}_s = \frac{d}{dt}\left\{ \sum_{k=1}^{s-1} \left[\left(\sum_{j=1}^{k} \dot{\theta}_j \mathbf{z}_{j-1} \right) \times \mathbf{p}_k^* \right] + \left(\sum_{j=1}^{s} \dot{\theta}_j \mathbf{z}_{j-1} \right) \times \bar{\mathbf{c}}_s \right\}$$

$$= \sum_{k=1}^{s-1} \left[\left(\sum_{j=1}^{k} \ddot{\theta}_j \mathbf{z}_{j-1} \right) \times \mathbf{p}_k^* + \left\{ \left(\sum_{j=1}^{k} \dot{\theta}_j \mathbf{z}_{j-1} \right) \times \left[\left(\sum_{j=1}^{k} \dot{\theta}_j \mathbf{z}_{j-1} \right) \times \mathbf{p}_k^* \right] \right\} \right]$$

$$+ \left[\left(\sum_{j=1}^{s} \ddot{\theta}_j \mathbf{z}_{j-1} \right) \times \bar{\mathbf{c}}_s \right] + \left\{ \left(\sum_{j=1}^{s} \dot{\theta}_j \mathbf{z}_{j-1} \right) \times \left[\left(\sum_{j=1}^{s} \dot{\theta}_j \mathbf{z}_{j-1} \right) \times \bar{\mathbf{c}}_s \right] \right\}$$

$$+ \sum_{k=2}^{s-1} \left\{ \sum_{p=2}^{k} \left[\left(\sum_{q=1}^{p-1} \dot{\theta}_q \mathbf{z}_{q-1} \right) \times \dot{\theta}_p \mathbf{z}_{p-1} \right] \times \mathbf{p}_k^* \right\}$$

$$+ \left\{ \sum_{p=2}^{s} \left[\left(\sum_{q=1}^{p-1} \dot{\theta}_q \mathbf{z}_{q-1} \right) \times \dot{\theta}_p \mathbf{z}_{p-1} \right] \times \bar{\mathbf{c}}_s \right\} \tag{3.4-11}$$

Next, the kinetic energy due to the rotational effect of link s is:

$$(K_s)_{\text{rot}} = \tfrac{1}{2}({}^s\mathbf{R}_0 \omega_s)^T \mathbf{I}_s ({}^s\mathbf{R}_0 \omega_s)$$

$$= \frac{1}{2} \left(\sum_{j=1}^{s} \dot{\theta}_j \, {}^s\mathbf{R}_0 \mathbf{z}_{j-1} \right)^T \mathbf{I}_s \left(\sum_{j=1}^{s} \dot{\theta}_j \, {}^s\mathbf{R}_0 \mathbf{z}_{j-1} \right) \tag{3.4-12}$$

Since

$$\frac{\partial (K_s)_{\text{rot}}}{\partial \dot{\theta}_i} = ({}^s\mathbf{R}_0 \mathbf{z}_{i-1})^T \mathbf{I}_s \left(\sum_{j=1}^{s} \dot{\theta}_j \, {}^s\mathbf{R}_0 \mathbf{z}_{j-1} \right) \qquad s \geqslant i \tag{3.4-13}$$

$$\frac{\partial}{\partial \theta_i} ({}^s\mathbf{R}_0 \mathbf{z}_{j-1}) = {}^s\mathbf{R}_0 \mathbf{z}_{j-1} \times {}^s\mathbf{R}_0 \mathbf{z}_{i-1} \qquad i \geqslant j \tag{3.4-14}$$

and

$$\frac{d}{dt} ({}^s\mathbf{R}_0 \mathbf{z}_{i-1}) = \sum_{j=i}^{s} \left(\frac{\partial}{\partial \theta_j} \, {}^s\mathbf{R}_0 \mathbf{z}_{i-1} \right) \frac{d\theta_j}{dt}$$

$$= {}^s\mathbf{R}_0 \mathbf{z}_{i-1} \times \left(\sum_{j=i}^{s} \dot{\theta}_j \, {}^s\mathbf{R}_0 \mathbf{z}_{j-1} \right) \tag{3.4-15}$$

then the time derivative of Eq. (3.4-13) is

$$\frac{d}{dt}\left[\frac{\partial (K_s)_{\text{rot}}}{\partial \dot{\theta}_i}\right] = \left[\frac{d}{dt}\,{}^s\mathbf{R}_0\mathbf{z}_{i-1}\right]^T \mathbf{I}_s \left(\sum_{j=1}^{s}\dot{\theta}_j\,{}^s\mathbf{R}_0\mathbf{z}_{j-1}\right)$$

$$+ ({}^s\mathbf{R}_0\mathbf{z}_{i-1})^T\mathbf{I}_s\left(\sum_{j=1}^{s}\ddot{\theta}_j\,{}^s\mathbf{R}_0\mathbf{z}_{j-1}\right) + ({}^s\mathbf{R}_0\mathbf{z}_{i-1})^T\mathbf{I}_s\left[\sum_{j=1}^{s}\dot{\theta}_j\left(\frac{d}{dt}\,{}^s\mathbf{R}_0\mathbf{z}_{j-1}\right)\right]$$

$$= \left({}^s\mathbf{R}_0\mathbf{z}_{i-1} \times \sum_{j=i}^{s}\dot{\theta}_j\,{}^s\mathbf{R}_0\mathbf{z}_{j-1}\right)^T \mathbf{I}_s\left(\sum_{j=1}^{s}\dot{\theta}_j\,{}^s\mathbf{R}_0\mathbf{z}_{j-1}\right)$$

$$+ ({}^s\mathbf{R}_0\mathbf{z}_{i-1})^T\mathbf{I}_s\left(\sum_{j=1}^{s}\ddot{\theta}_j\,{}^s\mathbf{R}_0\mathbf{z}_{j-1}\right)$$

$$+ ({}^s\mathbf{R}_0\mathbf{z}_{i-1})^T\mathbf{I}_s\left[\sum_{j=1}^{s}\left(\dot{\theta}_j\,{}^s\mathbf{R}_0\mathbf{z}_{j-1} \times \sum_{k=j+1}^{s}\dot{\theta}_k\,{}^s\mathbf{R}_0\mathbf{z}_{k-1}\right)\right] \qquad (3.4\text{-}16)$$

Next, using Eq. (3.4-14), we can find the partial derivative of $(K_s)_{\text{rot}}$ with respect to the generalized coordinate θ_i $(s \geqslant i)$; that is,

$$\frac{\partial (K_s)_{\text{rot}}}{\partial \theta_i} = \left[\left(\sum_{j=1}^{i}\dot{\theta}_j\,{}^s\mathbf{R}_0\mathbf{z}_{j-1}\right) \times {}^s\mathbf{R}_0\mathbf{z}_{i-1}\right]^T \mathbf{I}_s\left(\sum_{j=1}^{s}\dot{\theta}_j\,{}^s\mathbf{R}_0\mathbf{z}_{j-1}\right) \qquad (3.4\text{-}17)$$

Subtracting Eq. (3.4-17) from Eq. (3.4-16) and summing all the links from i to n gives us the reaction torques due to the rotational effects of all the links,

$$\frac{d}{dt}\left[\frac{\partial (\text{K.E.})_{\text{rot}}}{\partial \dot{\theta}_i}\right] - \frac{\partial (\text{K.E.})_{\text{rot}}}{\partial \theta_i} = \sum_{s=i}^{n}\left\{\frac{d}{dt}\left[\frac{\partial (K_s)_{\text{rot}}}{\partial \dot{\theta}_i}\right] - \frac{\partial (K_s)_{\text{rot}}}{\partial \theta_i}\right\}$$

$$= \sum_{s=i}^{n}\left[({}^s\mathbf{R}_0\mathbf{z}_{i-1})^T\mathbf{I}_s\left(\sum_{j=1}^{s}\ddot{\theta}_j\,{}^s\mathbf{R}_0\mathbf{z}_{j-1}\right)\right.$$

$$+ ({}^s\mathbf{R}_0\mathbf{z}_{i-1})^T\mathbf{I}_s\left\{\sum_{j=1}^{s}\left[\dot{\theta}_j\,{}^s\mathbf{R}_0\mathbf{z}_{j-1} \times \left(\sum_{k=j+1}^{s}\dot{\theta}_k\,{}^s\mathbf{R}_0\mathbf{z}_{k-1}\right)\right]\right\}$$

$$\left. + \left[{}^s\mathbf{R}_0\mathbf{z}_{i-1} \times \left(\sum_{k=1}^{s}\dot{\theta}_k\,{}^s\mathbf{R}_0\mathbf{z}_{k-1}\right)\right]^T \mathbf{I}_s\left(\sum_{j=1}^{s}\dot{\theta}_j\,{}^s\mathbf{R}_0\mathbf{z}_{j-1}\right)\right]$$

$$i = 1, 2, \ldots, n \qquad (3.4\text{-}18)$$

The potential energy of the robot arm equals to the sum of the potential energies of each link,

$$\text{P.E.} = \sum_{s=1}^{n} P_s \tag{3.4-19}$$

where P_s is the potential energy of link s given by

$$P_s = -\mathbf{g} \cdot m_s \bar{\mathbf{r}}_s = -\mathbf{g} \cdot m_s(\mathbf{p}_{i-1} + \mathbf{p}_i^* + \cdots + \bar{\mathbf{c}}_s) \tag{3.4-20}$$

where $\mathbf{g} = (g_x, g_y, g_z)^T$ and $|\mathbf{g}| = 9.8062 \text{ m/s}^2$. Applying the Lagrange-Euler formulation to the potential energy of link s with respect to the generalized coordinate θ_i ($s \geqslant i$), we have

$$\frac{d}{dt}\left[\frac{\partial(P_s)}{\partial\dot{\theta}_i}\right] - \frac{\partial(P_s)}{\partial\theta_i} = -\frac{\partial(P_s)}{\partial\theta_i} = \mathbf{g} \cdot m_s \frac{\partial(\mathbf{p}_{i-1} + \mathbf{p}_i^* + \cdots + \bar{\mathbf{c}}_s)}{\partial\theta_i}$$

$$= \mathbf{g} \cdot m_s \frac{\partial(\bar{\mathbf{r}}_s - \mathbf{p}_{i-1})}{\partial\theta_i} = \mathbf{g} \cdot m_s [\mathbf{z}_{i-1} \times (\bar{\mathbf{r}}_s - \mathbf{p}_{i-1})] \tag{3.4-21}$$

where $\mathbf{p}_{i-1}$ is not a function of θ_i. Summing all the links from i to n gives the reaction torques due to the gravity effects of all the links,

$$\frac{d}{dt}\left[\frac{\partial(\text{P.E.})}{\partial\dot{\theta}_i}\right] - \frac{\partial(\text{P.E.})}{\partial\theta_i} = -\sum_{s=i}^{n}\frac{\partial(P_s)}{\partial\theta_i}$$

$$= \sum_{s=i}^{n} \mathbf{g} \cdot m_s [\mathbf{z}_{i-1} \times (\bar{\mathbf{r}}_s - \mathbf{p}_{i-1})] \tag{3.4-22}$$

The summation of Eqs. (3.4-10), (3.4-18), and (3.4-22) is equal to the generalized applied torque exerted at joint i to drive link i,

$$\tau_i = \left\{\frac{d}{dt}\left[\frac{\partial(\text{K.E.})_{\text{tran}}}{\partial\dot{\theta}_i}\right] - \frac{\partial(\text{K.E.})_{\text{tran}}}{\partial\theta_i}\right\} + \left[\frac{d}{dt}\left(\frac{\partial(\text{K.E.})_{\text{rot}}}{\partial\dot{\theta}_i}\right) - \frac{\partial(\text{K.E.})_{\text{rot}}}{\partial\theta_i}\right] + \frac{\partial(\text{P.E.})}{\partial\theta_i}$$

$$= \sum_{s=i}^{n}\left(m_s\left\{\left[\sum_{k=1}^{s-1}\left(\sum_{j=1}^{k}\ddot{\theta}_j\mathbf{z}_{j-1}\right)\times\mathbf{p}_k^*\right] + \left[\left(\sum_{j=1}^{s}\ddot{\theta}_j\mathbf{z}_{j-1}\right)\times\bar{\mathbf{c}}_s\right]\right\}\cdot[\mathbf{z}_{i-1}\times(\bar{\mathbf{r}}_s - \mathbf{p}_{i-1})]\right)$$

$$+ \sum_{s=i}^{n}\left[({}^s\mathbf{R}_0\mathbf{z}_{i-1})^T\mathbf{I}_s\left(\sum_{j=1}^{s}\ddot{\theta}_j\,{}^s\mathbf{R}_0\mathbf{z}_{j-1}\right)\right]$$

$$+ \sum_{s=i}^{n}\left(m_s\left[\sum_{k=1}^{s-1}\left(\left\{\left(\sum_{p=1}^{k}\dot{\theta}_p\mathbf{z}_{p-1}\right)\times\left[\left(\sum_{q=1}^{k}\dot{\theta}_q\mathbf{z}_{q-1}\right)\times\mathbf{p}_k^*\right]\right\}\right.\right.\right.$$

$$\left.\left.\left. + \left\{\sum_{p=2}^{k}\left[\left(\sum_{q=1}^{p-1}\dot{\theta}_q\mathbf{z}_{q-1}\right)\times\dot{\theta}_p\mathbf{z}_{p-1}\right]\times\mathbf{p}_k^*\right\}\right)\right]\cdot[\mathbf{z}_{i-1}\times(\bar{\mathbf{r}}_s - \mathbf{p}_{i-1})]\right)$$

$$+\sum_{s=i}^{n}\left\{m_s\left[\left\{\left(\sum_{p=1}^{s}\dot{\theta}_p\mathbf{z}_{p-1}\right)\times\left[\left(\sum_{q=1}^{s}\dot{\theta}_q\mathbf{z}_{q-1}\right)\times\bar{\mathbf{c}}_s\right]\right\}\right.\right.$$

$$\left.\left.+\left\{\sum_{p=2}^{s}\left[\left(\sum_{q=1}^{p-1}\dot{\theta}_q\mathbf{z}_{q-1}\right)\times\dot{\theta}_p\mathbf{z}_{p-1}\right]\times\bar{\mathbf{c}}_s\right\}\right]\cdot[\mathbf{z}_{i-1}\times(\bar{\mathbf{r}}_s-\mathbf{p}_{i-1})]\right]$$

$$+\sum_{s=i}^{n}\left[({}^s\mathbf{R}_0\mathbf{z}_{i-1})^T\mathbf{I}_s\left\{\sum_{j=1}^{s}\left[\dot{\theta}_j{}^s\mathbf{R}_0\mathbf{z}_{j-1}\times\left(\sum_{k=j+1}^{s}\dot{\theta}_k{}^s\mathbf{R}_0\mathbf{z}_{k-1}\right)\right]\right\}\right.$$

$$\left.+\left[{}^s\mathbf{R}_0\mathbf{z}_{i-1}\times\left(\sum_{p=1}^{s}\dot{\theta}_p{}^s\mathbf{R}_0\mathbf{z}_{p-1}\right)\right]^T\mathbf{I}_s\left(\sum_{q=1}^{s}\dot{\theta}_q{}^s\mathbf{R}_0\mathbf{z}_{q-1}\right)\right]$$

$$-\mathbf{g}\cdot\left\{\mathbf{z}_{i-1}\times\left[\sum_{j=i}^{n}m_j(\bar{\mathbf{r}}_j-\mathbf{p}_{i-1})\right]\right\} \qquad (3.4\text{-}23)$$

for $i = 1, 2, \ldots, n$.

The above equation can be rewritten in a more "structured" form as (for $i = 1, 2, \ldots, n$):

$$\sum_{j=1}^{n} D_{ij}\ddot{\theta}_j(t) + h_i^{\text{tran}}(\theta,\dot{\theta}) + h_i^{\text{rot}}(\theta,\dot{\theta}) + c_i = \tau_i(t) \qquad (3.4\text{-}24)$$

where, for $i = 1, 2, \ldots, n$,

$$D_{ij} = D_{ij}^{\text{rot}} + D_{ij}^{\text{tran}} = \sum_{s=j}^{n}[({}^s\mathbf{R}_0\mathbf{z}_{i-1})^T\mathbf{I}_s\,({}^s\mathbf{R}_0\mathbf{z}_{j-1})]$$

$$+\sum_{s=j}^{n}\left\{m_s\left[\mathbf{z}_{j-1}\times\left(\sum_{k=j}^{s-1}\mathbf{p}_k^* + \bar{\mathbf{c}}_s\right)\right]\cdot[\mathbf{z}_{i-1}\times(\bar{\mathbf{r}}_s-\mathbf{p}_{i-1})]\right\} \qquad i \leqslant j$$

$$= \sum_{s=j}^{n}[({}^s\mathbf{R}_0\mathbf{z}_{i-1})^T\mathbf{I}_s\,({}^s\mathbf{R}_0\mathbf{z}_{j-1})]$$

$$+\sum_{s=j}^{n}\left\{m_s[\mathbf{z}_{j-1}\times(\bar{\mathbf{r}}_s-\mathbf{p}_{j-1})]\cdot[\mathbf{z}_{i-1}\times(\bar{\mathbf{r}}_s-\mathbf{p}_{i-1})]\right\} \qquad i \leqslant j$$

$$(3.4\text{-}25)$$

also,

$$h_i^{\text{tran}}(\theta, \dot{\theta}) = \sum_{s=i}^{n} \left\{ m_s \left[\sum_{k=1}^{s-1} \left[\left\{ \left[\sum_{p=1}^{k} \dot{\theta}_p \mathbf{z}_{p-1} \right] \times \left[\left[\sum_{q=1}^{k} \dot{\theta}_q \mathbf{z}_{q-1} \right] \times \mathbf{p}_k^* \right] \right\} \right.\right.\right.$$

$$\left.\left.\left. + \left\{ \sum_{p=2}^{k} \left[\left[\sum_{q=1}^{p-1} \dot{\theta}_q \mathbf{z}_{q-1} \right] \times \dot{\theta}_p \mathbf{z}_{p-1} \right] \times \mathbf{p}_k^* \right\} \right] \right] \cdot [\mathbf{z}_{i-1} \times (\mathbf{r}_s - \mathbf{p}_{i-1})] \right\}$$

$$+ \sum_{s=i}^{n} \left[m_s \left[\left\{ \left[\sum_{p=1}^{s} \dot{\theta}_p \mathbf{z}_{p-1} \right] \times \left[\left[\sum_{q=1}^{s} \dot{\theta}_q \mathbf{z}_{q-1} \right] \times \bar{\mathbf{c}}_s \right] \right\} \right.\right.$$

$$\left.\left. + \left\{ \sum_{p=2}^{s} \left[\left[\sum_{q=1}^{p-1} \dot{\theta}_q \mathbf{z}_{q-1} \right] \times \dot{\theta}_p \mathbf{z}_{p-1} \right] \times \bar{\mathbf{c}}_s \right\} \right] \cdot [\mathbf{z}_{i-1} \times (\mathbf{r}_s - \mathbf{p}_{i-1})] \right] \tag{3.4-26}$$

and

$$h_i^{\text{rot}}(\theta, \dot{\theta}) = \sum_{s=i}^{n} \left\{ ({}^s\mathbf{R}_0 \mathbf{z}_{i-1})^T \mathbf{I}_s \left\{ \sum_{j=1}^{s} \left[\dot{\theta}_j \, {}^s\mathbf{R}_0 \mathbf{z}_{j-1} \times \left(\sum_{k=j+1}^{s} \dot{\theta}_k \, {}^s\mathbf{R}_0 \mathbf{z}_{k-1} \right) \right] \right\} \right.$$

$$\left. + \left[{}^s\mathbf{R}_0 \mathbf{z}_{i-1} \times \left(\sum_{p=1}^{s} \dot{\theta}_p \, {}^s\mathbf{R}_0 \mathbf{z}_{p-1} \right) \right]^T \mathbf{I}_s \left(\sum_{q=1}^{s} \dot{\theta}_q \, {}^s\mathbf{R}_0 \mathbf{z}_{q-1} \right) \right\} \tag{3.4-27}$$

Finally, we have

$$c_i = -\mathbf{g} \cdot \left[\mathbf{z}_{i-1} \times \sum_{j=i}^{n} m_j (\bar{\mathbf{r}}_j - \mathbf{p}_{i-1}) \right] \tag{3.4-28}$$

The dynamic coefficients D_{ij} and c_i are functions of both the joint variables and inertial parameters of the manipulator, while the h_i^{tran} and h_i^{rot} are functions of the joint variables, the joint velocities and inertial parameters of the manipulator. These coefficients have the following physical interpretation:

1. The elements of the D_{ij} matrix are related to the inertia of the links in the manipulator. Equation (3.4-25) reveals the acceleration effects of joint j acting on joint i where the driving torque τ_i acts. The first term of Eq. (3.4-25) indicates the inertial effects of moving link j on joint i due to the *rotational* motion of link j, and vice versa. If $i = j$, it is the effective inertias felt at joint i due to the rotational motion of link i; while if $i \neq j$, it is the pseudoproducts of inertia of link j felt at joint i due to the rotational motion of link j. The second term has the same physical meaning except that it is due to the *translational* motion of link j acting on joint i.

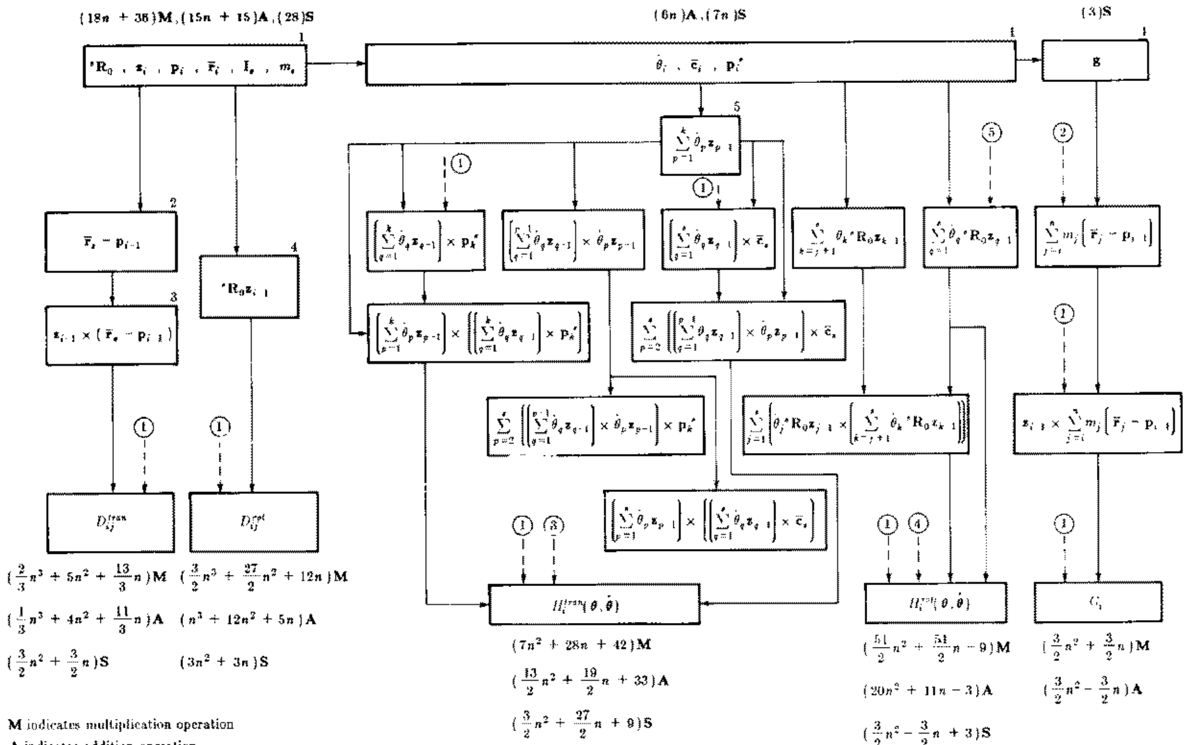
2. The $h_i^{\text{tran}}(\theta, \dot{\theta})$ term is related to the velocities of the joint variables. Equation (3.4-26) represents the combined centrifugal and Coriolis reaction torques felt at joint i due to the velocities of joints p and q resulting from the *translational* motion of links p and q. The first and third terms of Eq. (3.4-26) constitute, respectively, the centrifugal and Coriolis reaction forces from all the links below link s and link s itself, due to the translational motion of the links. If $p = q$, then it represents the centrifugal reaction forces felt at joint i. If $p \neq q$, then it indicates the Coriolis forces acting on joint i. The second and fourth terms of Eq. (3.4-26) indicate, respectively, the Coriolis reaction forces contributed from the links below link s and link s itself, due to the translational motion of the links.
3. The $h_i^{\text{rot}}(\theta, \dot{\theta})$ term is also related to the velocities of the joint variables. Similar to $h_i^{\text{tran}}(\theta, \dot{\theta})$, Eq. (3.4-27) reveals the combined centrifugal and Coriolis reaction torques felt at joint i due to the velocities of joints p and q resulting from the *rotational* motion of links p and q. The first term of Eq. (3.4-27) indicates purely the Coriolis reaction forces of joints p and q acting on joint i due to the rotational motion of the links. The second term is the combined centrifugal and Coriolis reaction forces acting on joint i. If $p = q$, then it indicates the centrifugal reaction forces felt at joint i, but if $p \neq q$, then it represents the Coriolis forces acting on joint i due to the rotational motion of the links.
4. The coefficient c_i represents the gravity effects acting on joint i from the links above joint i.

At first sight, Eqs. (3.4-25) to (3.4-28) would seem to require a large amount of computation. However, most of the cross-product terms can be computed very fast. As an indication of their computational complexities, a block diagram explicitly showing the procedure in calculating these coefficients for every set point in the trajectory in terms of multiplication and addition operations is shown in Fig. 3.9. Table 3.5 summarizes the computational complexities of the L-E, N-E, and G-D equations of motion in terms of required mathematical operations per trajectory set point.

Table 3.5 Comparison of robot arm dynamics computational complexities†

Approach	Lagrange-Euler	Newton-Euler	Generalized d'Alembert
Multiplications	$\frac{128}{3}n^4 + \frac{512}{3}n^3 + \frac{739}{3}n^2 + \frac{160}{3}n$	$132n$	$\frac{13}{6}n^3 + \frac{105}{2}n^2 + \frac{268}{3}n + 69$
Additions	$\frac{98}{3}n^4 + \frac{781}{6}n^3 + \frac{559}{3}n^2 + \frac{245}{6}n$	$111n - 4$	$\frac{4}{3}n^3 + 44n^2 + \frac{146}{3}n + 45$
Kinematics representation	4 × 4 Homogeneous matrices	Rotation matrices and position vectors	Rotation matrices and position vectors
Equations of motion	Closed-form differential equations	Recursive equations	Closed-form differential equations

† n = number of degrees of freedom of the robot arm. No effort is spent here to optimize the computation.



M indicates multiplication operation
A indicates addition operation
S indicates memory storage requirement
Ⓝ indicates output from block n

Figure 3.9 Computational procedure for D_{ij}, h_i^{tran}, h_i^{rot}, and c_i.

3.4.1 An Empirical Method for Obtaining a Simplified Dynamic Model

One of the objectives of developing the G-D equations of motion [Eqs. (3.4-24) to (3.4-28)] is to facilitate the design of a suitable controller for a manipulator in the

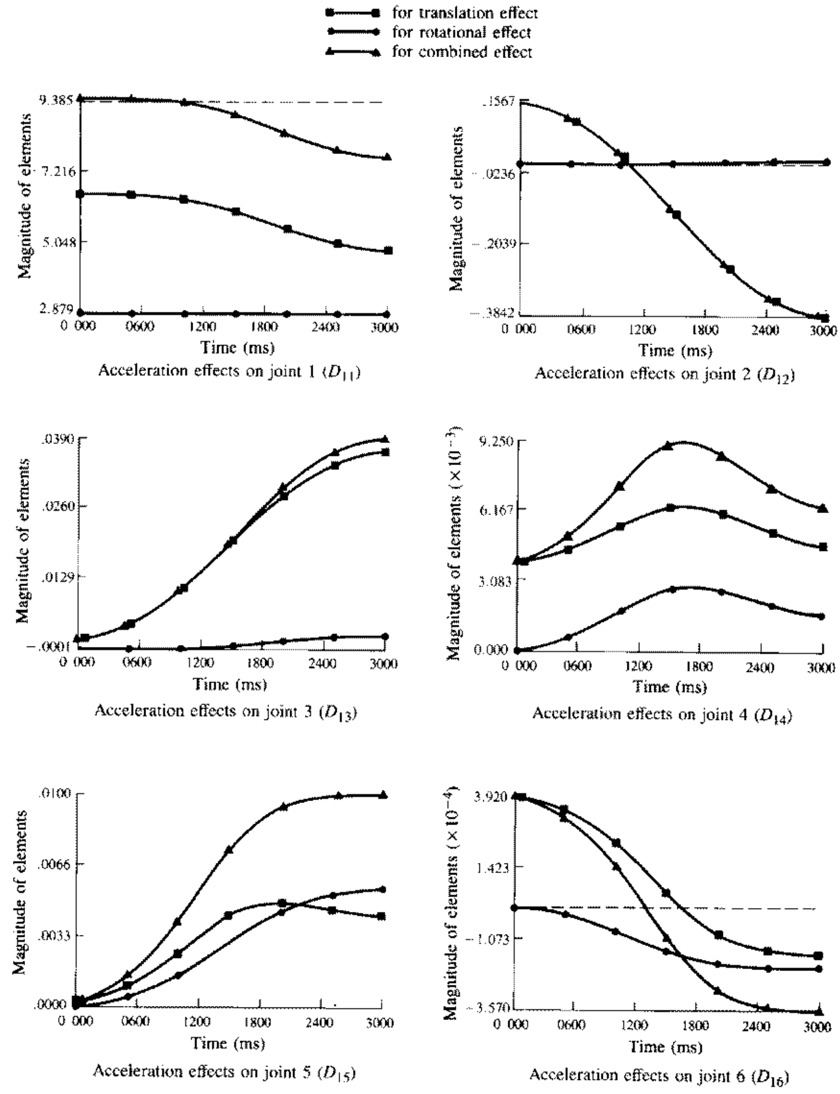


Figure 3.10 The acceleration-related D_{ij} elements.

state space or to obtain an approximate dynamic model for a manipulator. Similar to the L-E equations of motion [Eqs. (3.3-24) and (3.3-25)], the G-D equations of motion are explicitly expressed in vector-matrix form and all the interaction and coupling reaction forces can be easily identified. Furthermore, D_{ij}^{tran}, D_{ij}^{rot}, h_i^{tran}, h_i^{rot}, and c_i can be clearly identified as coming from the translational and the rotational effects of the motion. Comparing the magnitude of the translational and rotational effects for each term of the dynamic equations of motion, the extent of

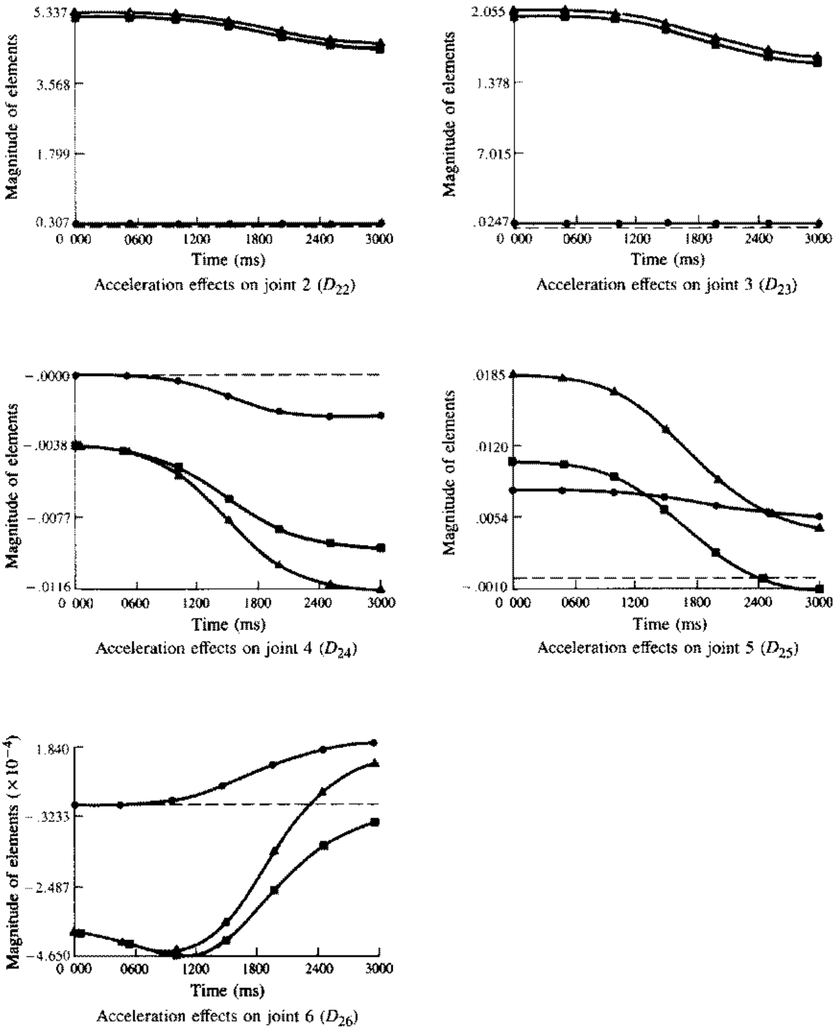


Figure 3.10 (Continued.)

dominance from the translational and rotational effects can be computed for each set point along the trajectory. The less dominant terms or elements can be neglected in calculating the dynamic equations of motion of the manipulator. This greatly aids the construction of a simplified dynamic model for control purpose.

As an example of obtaining a simplified dynamic model for a specific trajectory, we consider a PUMA 560 robot and its dynamic equations of motion along a preplanned trajectory. The D_{ij}^{tran}, D_{ij}^{rot}, h_i^{tran}, and h_i^{rot} elements along the trajectory are computed and plotted in Figs. 3.10 and 3.11. The total number of trajectory set points used is 31. Figure 3.10 shows the acceleration-related elements D_{ij}^{tran} and D_{ij}^{rot}. Figure 3.11 shows the Coriolis and centrifugal elements h_i^{tran} and h_i^{rot}. These figures show the separate and combined effects from the translational and rotational terms.

From Fig. 3.10, we can approximate the elements of the **D** matrix along the trajectory as follows: (1) The translational effect is dominant for the D_{12}, D_{22}, D_{23}, D_{33}, D_{45}, and D_{56} elements. (2) The rotational effect is dominant for the D_{44}, D_{46}, D_{55}, and D_{66} elements. (3) Both translational and rotational effects are dominant for the remaining elements of the **D** matrix. In Fig. 3.11, the elements D_{56}^{tran} and D_{45}^{tran} show a staircase shape which is due primarily to the round-off

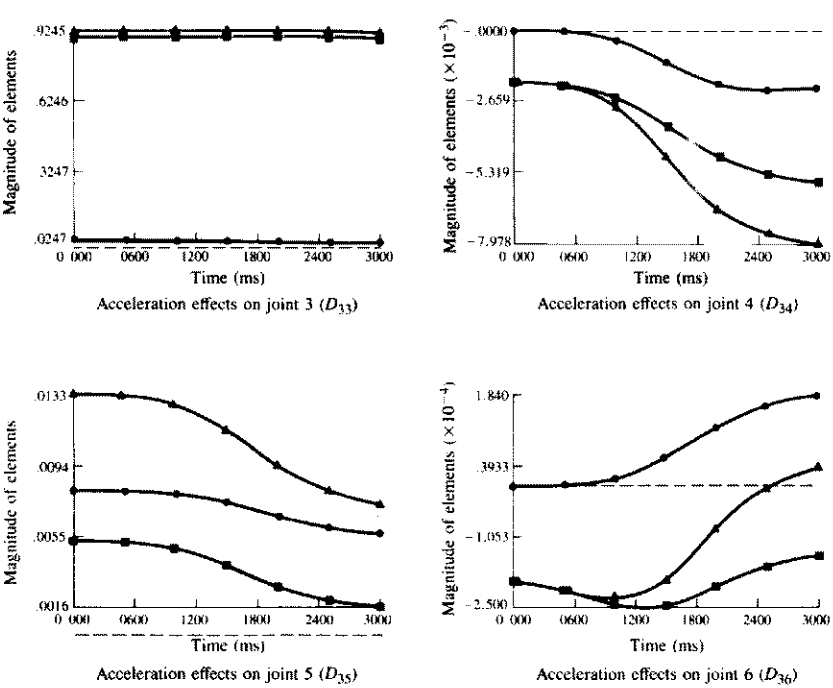


Figure 3.10 (Continued.)

error generated by the VAX-11/780 computer used in the simulation. These elements are very small in magnitude when compared with the rotational elements. Similarly, we can approximate the elements of the **h** vector as follows: (1) The translational effect is dominant for the h_1, h_2, and h_3 elements. (2) The rotational effect is dominant for the h_4 element. (3) Both translational and rotational effects are dominant for the h_5 and h_6 elements.

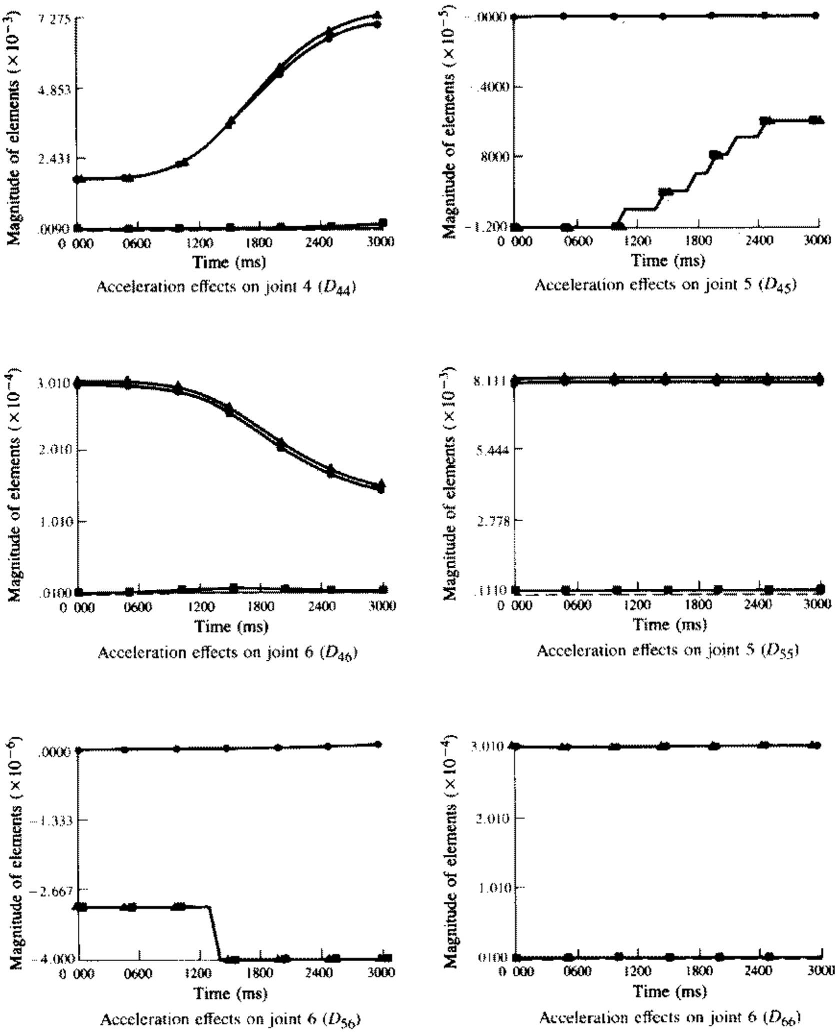


Figure 3.10 (Continued.)

The above simplification depends on the specific trajectory being considered. The resulting simplified model retains most of the major interaction and coupling reaction forces/torques at a reduced computation time, which greatly aids the design of an appropriate law for controlling the robot arm.

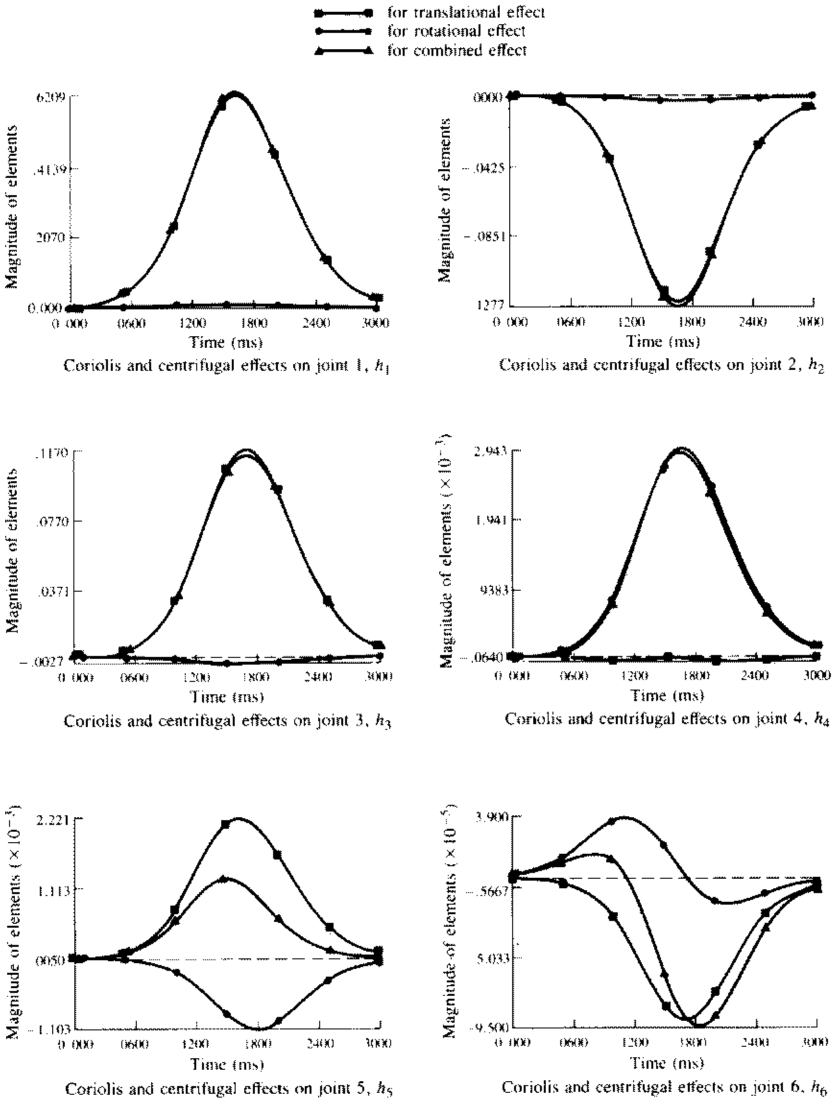


Figure 3.11 The Coriolis and centrifugal terms h_i.

3.4.2 A Two-Link Manipulator Example

Consider the two-link manipulator shown in Fig. 3.2. We would like to derive the generalized d'Alembert equations of motion for it. Letting m_1 and m_2 represent the link masses, and assuming that each link is l units long, yields the following expressions for the link inertia tensors:

$$\mathbf{I}_1 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12} m_1 l^2 & 0 \\ 0 & 0 & \frac{1}{12} m_1 l^2 \end{bmatrix} \qquad \mathbf{I}_2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \frac{1}{12} m_2 l^2 & 0 \\ 0 & 0 & \frac{1}{12} m_2 l^2 \end{bmatrix}$$

The rotation matrices are, respectively,

$${}^0\mathbf{R}_1 = \begin{bmatrix} C_1 & -S_1 & 0 \\ S_1 & C_1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \qquad {}^1\mathbf{R}_2 = \begin{bmatrix} C_2 & -S_2 & 0 \\ S_2 & C_2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$${}^0\mathbf{R}_2 = {}^0\mathbf{R}_1 {}^1\mathbf{R}_2 = \begin{bmatrix} C_{12} & -S_{12} & 0 \\ S_{12} & C_{12} & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

and

$${}^1\mathbf{R}_0 = ({}^0\mathbf{R}_1)^T \qquad {}^2\mathbf{R}_0 = ({}^0\mathbf{R}_2)^T$$

where $C_i = \cos\theta_i$, $S_i = \sin\theta_i$, $C_{ij} = \cos(\theta_i + \theta_j)$, and $S_{ij} = \sin(\theta_i + \theta_j)$.

The physical parameters of the manipulator such as $\mathbf{p}_i^*$, $\bar{\mathbf{c}}_i$, $\bar{\mathbf{r}}_i$, and $\mathbf{p}_i$ are:

$$\mathbf{p}_1^* = \mathbf{p}_1 = \begin{bmatrix} lC_1 \\ lS_1 \\ 0 \end{bmatrix} \qquad \mathbf{p}_2^* = \begin{bmatrix} lC_{12} \\ lS_{12} \\ 0 \end{bmatrix} \qquad \mathbf{p}_2 = \begin{bmatrix} l(C_1 + C_{12}) \\ l(S_1 + S_{12}) \\ 0 \end{bmatrix}$$

$$\bar{\mathbf{c}}_1 = \bar{\mathbf{r}}_1 = \begin{bmatrix} \frac{l}{2}C_1 \\ \frac{l}{2}S_1 \\ 0 \end{bmatrix} \qquad \bar{\mathbf{c}}_2 = \begin{bmatrix} \frac{l}{2}C_{12} \\ \frac{l}{2}S_{12} \\ 0 \end{bmatrix} \qquad \bar{\mathbf{r}}_2 = \begin{bmatrix} lC_1 + \frac{l}{2}C_{12} \\ lS_1 + \frac{l}{2}S_{12} \\ 0 \end{bmatrix}$$

Using Eq. (3.4-25), we obtain the elements of the **D** matrix, as follows:

$$D_{11} = ({}^1\mathbf{R}_0\mathbf{z}_0)^T\mathbf{I}_1({}^1\mathbf{R}_0\mathbf{z}_0) + ({}^2\mathbf{R}_0\mathbf{z}_0)^T\mathbf{I}_2({}^2\mathbf{R}_0\mathbf{z}_0) + m_1(\mathbf{z}_0 \times \bar{\mathbf{c}}_1) \cdot (\mathbf{z}_0 \times \bar{\mathbf{r}}_1)$$

$$+ m_2[\mathbf{z}_0 \times (\mathbf{p}_1^* + \bar{\mathbf{c}}_2)] \cdot (\mathbf{z}_0 \times \bar{\mathbf{r}}_2)$$

$$= (0,0,1)\,\mathbf{I}_1 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + (0,0,1)\,\mathbf{I}_2 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$+ m_1 \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} \frac{l}{2}C_1 \\ \frac{l}{2}S_1 \\ 0 \end{bmatrix} \right] \cdot \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} \frac{l}{2}C_1 \\ \frac{l}{2}S_1 \\ 0 \end{bmatrix} \right]$$

$$+ m_2 \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} lC_1 + \frac{l}{2}C_{12} \\ lS_1 + \frac{l}{2}S_{12} \\ 0 \end{bmatrix} \right] \cdot \left[\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \times \begin{bmatrix} lC_1 + \frac{l}{2}C_{12} \\ lS_1 + \frac{l}{2}S_{12} \\ 0 \end{bmatrix} \right]$$

$$= \tfrac{1}{3}m_1 l^2 + \tfrac{4}{3}m_2 l^2 + C_2 m_2 l^2$$

$$D_{12} = D_{21} = ({}^2\mathbf{R}_0\mathbf{z}_0)^T\mathbf{I}_2({}^2\mathbf{R}_0\mathbf{z}_1) + m_2(\mathbf{z}_1 \times \bar{\mathbf{c}}_2) \cdot (\mathbf{z}_0 \times \bar{\mathbf{r}}_2) = \tfrac{1}{3}m_2 l^2 + \tfrac{1}{2}m_2 C_2 l^2$$

$$D_{22} = ({}^2\mathbf{R}_0\mathbf{z}_1)^T\mathbf{I}_2({}^2\mathbf{R}_0\mathbf{z}_1) + m_2(\mathbf{z}_1 \times \bar{\mathbf{c}}_2) \cdot [\mathbf{z}_1 \times (\bar{\mathbf{r}}_2 - \mathbf{p}_1)]$$

$$= \tfrac{1}{12}m_2 l^2 + \tfrac{1}{4}m_2 l^2 = \tfrac{1}{3}m_2 l^2$$

Thus,

$$[D_{ij}] = \begin{bmatrix} D_{11} & D_{12} \\ D_{21} & D_{22} \end{bmatrix} = \begin{bmatrix} \tfrac{1}{3}m_1 l^2 + \tfrac{4}{3}m_2 l^2 + m_2 l^2 C_2 & \tfrac{1}{3}m_2 l^2 + \tfrac{1}{2}m_2 C_2 l^2 \\ \tfrac{1}{3}m_2 l^2 + \tfrac{1}{2}m_2 C_2 l^2 & \tfrac{1}{3}m_2 l^2 \end{bmatrix}$$

To derive the $h_i^{\text{tran}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ and $h_i^{\text{rot}}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$ components, we need to consider only the following terms in Eqs. (3.4-26) and (3.4-27) in our example because the other terms are zero.

$$h_1^{\text{tran}} = m_2[\dot\theta_1 \mathbf{z}_0 \times (\dot\theta_1 \mathbf{z}_0 \times \mathbf{p}_1^*)] \cdot (\mathbf{z}_0 \times \bar{\mathbf{r}}_2) + m_1[\dot\theta_1 \mathbf{z}_0 \times (\dot\theta_1 \mathbf{z}_0 \times \bar{\mathbf{c}}_1)]$$

$$\cdot (\mathbf{z}_0 \times \bar{\mathbf{r}}_1) + m_2[(\dot\theta_1 \mathbf{z}_0 + \dot\theta_2 \mathbf{z}_1) \times [(\dot\theta_1 \mathbf{z}_0 + \dot\theta_2 \mathbf{z}_1) \times \bar{\mathbf{c}}_2)$$

$$+ (\dot\theta_1 \mathbf{z}_0 \times \dot\theta_2 \mathbf{z}_1) \times \bar{\mathbf{c}}_2] \cdot (\mathbf{z}_0 \times \bar{\mathbf{r}}_2)$$

$$= \tfrac{1}{2} m_2 l^2 S_2 \dot\theta_1^2 - \tfrac{1}{2} m_2 l^2 S_2 \dot\theta_1^2 - \tfrac{1}{2} m_2 l^2 S_2 \dot\theta_2^2 - m_2 l^2 S_2 \dot\theta_1 \dot\theta_2$$

$$h_1^{\text{rot}} = ({}^1\mathbf{R}_0\mathbf{z}_0 \times \dot\theta_1\, {}^1\mathbf{R}_0\mathbf{z}_0)^T \mathbf{I}_1(\dot\theta_1\, {}^1\mathbf{R}_0\mathbf{z}_0) + ({}^2\mathbf{R}_0\mathbf{z}_0)^T \mathbf{I}_2(\dot\theta_1\, {}^2\mathbf{R}_0\mathbf{z}_0 \times \dot\theta_2\, {}^2\mathbf{R}_0\mathbf{z}_1)$$

$$+ [{}^2\mathbf{R}_0\mathbf{z}_0 \times (\dot\theta_1\, {}^2\mathbf{R}_0\mathbf{z}_0 + \dot\theta_2\, {}^2\mathbf{R}_0\mathbf{z}_1)]^T \mathbf{I}_2(\dot\theta_1\, {}^2\mathbf{R}_0\mathbf{z}_0 + \dot\theta_2\, {}^2\mathbf{R}_0\mathbf{z}_1)$$

$$= 0$$

Thus,

$$h_1 = h_1^{\text{tran}} + h_1^{\text{rot}} = -\tfrac{1}{2} m_2 l^2 S_2 \dot\theta_2^2 - m_2 l^2 S_2 \dot\theta_1 \dot\theta_2$$

Similarly, we can find

$$h_2^{\text{tran}} = m_2[\dot\theta_1 \mathbf{z}_0 \times (\dot\theta_1 \mathbf{z}_0 \times \mathbf{p}_1^*)] \cdot [\mathbf{z}_1 \times (\bar{\mathbf{r}}_2 - \mathbf{p}_1)] + m_2\{(\dot\theta_1 \mathbf{z}_0 + \dot\theta_2 \mathbf{z}_1)$$

$$\times [(\dot\theta_1 \mathbf{z}_0 + \dot\theta_2 \mathbf{z}_1) \times \bar{\mathbf{c}}_2] + (\dot\theta_1 \mathbf{z}_0 \times \dot\theta_2 \mathbf{z}_1) \times \bar{\mathbf{c}}_2\} \cdot [\mathbf{z}_1 \times (\bar{\mathbf{r}}_2 - \mathbf{p}_1)]$$

$$= \tfrac{1}{2} m_2 l^2 S_2 \dot\theta_1^2$$

$$h_2^{\text{rot}} = ({}^2\mathbf{R}_0\mathbf{z}_1)^T \mathbf{I}_2 (\dot\theta_1\, {}^2\mathbf{R}_0\mathbf{z}_0 \times \dot\theta_2\, {}^2\mathbf{R}_0\mathbf{z}_1)$$

$$+ [{}^2\mathbf{R}_0\mathbf{z}_1 \times (\dot\theta_1\, {}^2\mathbf{R}_0\mathbf{z}_0 + \dot\theta_2\, {}^2\mathbf{R}_0\mathbf{z}_1)]^T \mathbf{I}_2(\dot\theta_1\, {}^2\mathbf{R}_0\mathbf{z}_0 + \dot\theta_2\, {}^2\mathbf{R}_0\mathbf{z}_1)$$

$$= 0$$

We note that $h_1^{\text{rot}} = h_2^{\text{rot}} = 0$ which simplifies the design of feedback control law. Thus,

$$h_2 = h_2^{\text{tran}} + h_2^{\text{rot}} = \tfrac{1}{2} m_2 l^2 S_2 \dot\theta_1^2$$

Therefore,

$$\mathbf{h} = \begin{bmatrix} h_1 \\ h_2 \end{bmatrix} = \begin{bmatrix} -\tfrac{1}{2} m_2 S_2 l^2 \dot\theta_2^2 - m_2 S_2 l^2 \dot\theta_1 \dot\theta_2 \\ \tfrac{1}{2} m_2 S_2 l^2 \dot\theta_1^2 \end{bmatrix}$$

To derive the elements of the **c** vector, we use Eq. (3.4-28):

$$c_1 = -\mathbf{g} \cdot [\mathbf{z}_0 \times (m_1 \bar{\mathbf{r}}_1 + m_2 \bar{\mathbf{r}}_2)] = (\tfrac{1}{2} m_1 + m_2) g l C_1 + \tfrac{1}{2} m_2 g l C_{12}$$

$$c_2 = -\mathbf{g} \cdot [\mathbf{z}_1 \times m_2(\bar{\mathbf{r}}_2 - \mathbf{p}_1)] = \tfrac{1}{2} m_2 g l C_{12}$$

where $\mathbf{g} = (0, -g, 0)^T$. Thus, the gravity loading vector $\mathbf{c}$ becomes

$$\mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} (\frac{1}{2}m_1 + m_2)glC_1 + \frac{1}{2}m_2glC_{12} \\ \frac{1}{2}m_2glC_{12} \end{bmatrix}$$

where $g = 9.8062$ m/s^2. Based on the above results, it follows that the equations of motion of the two-link robot arm by the generalized d'Alembert method are:

$$\begin{bmatrix} \tau_1(t) \\ \tau_2(t) \end{bmatrix} = \begin{bmatrix} \frac{1}{3}m_1l^2 + \frac{4}{3}m_2l^2 + m_2C_2l^2 & \frac{1}{3}m_2l^2 + \frac{1}{2}m_2C_2l^2 \\ \frac{1}{3}m_2l^2 + \frac{1}{2}m_2C_2l^2 & \frac{1}{3}m_2l^2 \end{bmatrix} \begin{bmatrix} \ddot{\theta}_1(t) \\ \ddot{\theta}_2(t) \end{bmatrix}$$

$$+ \begin{bmatrix} -\frac{1}{2}m_2S_2l^2\dot{\theta}_2^2 - m_2S_2l^2\dot{\theta}_1\dot{\theta}_2 \\ \frac{1}{2}m_2S_2l^2\dot{\theta}_1^2 \end{bmatrix}$$

$$+ \begin{bmatrix} (\frac{1}{2}m_1 + m_2)glC_1 + \frac{1}{2}m_2glC_{12} \\ \frac{1}{2}m_2glC_{12} \end{bmatrix}$$

3.5 CONCLUDING REMARKS

Three different formulations for robot arm dynamics have been presented and discussed. The L-E equations of motion can be expressed in a well structured form, but they are computationally difficult to utilize for real-time control purposes unless they are simplified. The N-E formulation results in a very efficient set of recursive equations, but they are difficult to use for deriving advanced control laws. The G-D equations of motion give fairly well "structured" equations at the expense of higher computational cost. In addition to having faster computation time than the L-E equations of motion, the G-D equations of motion explicitly indicate the contributions of the translational and rotational effects of the links. Such information is useful for control analysis in obtaining an appropriate approximate model of a manipulator. Furthermore, the G-D equations of motion can be used in manipulator design. To briefly summarize the results, a user is able to choose between a formulation which is highly structured but computationally inefficient (L-E), a formulation which has efficient computations at the expense of the "structure" of the equations of motion (N-E), and a formulation which retains the "structure" of the problem with only a moderate computational penalty (G-D).

REFERENCES

Further reading on general concepts on dynamics can be found in several excellent mechanics books (Symon [1971] and Crandall et al. [1968]). The derivation of

Lagrange-Euler equations of motion using the 4 × 4 homogeneous transformation matrix was first carried out by Uicker [1965]. The report by Lewis [1974] contains a more detailed derivation of Lagrange-Euler equations of motion for a six-joint manipulator. An excellent report written by Bejczy [1974] reviews the details of the dynamics and control of an extended Stanford robot arm (the JPL arm). The report also discusses a scheme for obtaining simplified equations of motion. Exploiting the recursive nature of the lagrangian formulation, Hollerbach [1980] further improved the computation time of the generalized torques based on the lagrangian formulation.

Simplification of L-E equations of motion can be achieved via a differential transformation (Paul [1981]), a model reduction method (Bejczy and Lee [1983]), and an equivalent-composite approach (Luh and Lin [1981*b*]). The differential transformation technique converts the partial derivative of the homogeneous transformation matrices into a matrix product of the transformation and a differential matrix, thus reducing the acceleration-related matrix D_{ik} to a much simpler form. However, the Coriolis and centrifugal term, h_{ikm}, which contains the second-order partial derivative was not simplified by Paul [1981]. Bejczy and Lee [1983] developed the model reduction method which is based on the homogeneous transformation and on the lagrangian dynamics and utilized matrix numeric analysis technique to simplify the Coriolis and centrifugal term. Luh and Lin [1981*b*] utilized the N-E equations of motion and compared their terms in a computer to eliminate various terms and then rearranged the remaining terms to form the equations of motion in a symbolic form.

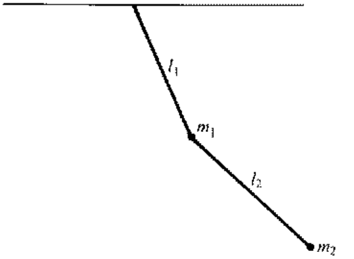
As an alternative to deriving more efficient equations of motion is to develop efficient algorithms for computing the generalized forces/torques based on the N-E equations of motion. Armstrong [1979], and Orin et al. [1979] were among the first to exploit the recursive nature of the Newton-Euler equations of motion. Luh et al. [1980*a*] improved the computations by referencing all velocities, accelerations, inertial matrices, location of the center of mass of each link, and forces/moments, to their own link coordinate frames. Walker and Orin [1982] extended the N-E formulation to computing the joint accelerations for computer simulation of robot motion.

Though the structure of the L-E and the N-E equations of motion are different, Turney et al. [1980] explicitly verified that one can obtain the L-E motion equations from the N-E equations, while Silver [1982] investigated the equivalence of the L-E and the N-E equations of motion through tensor analysis. Huston and Kelly [1982] developed an algorithmic approach for deriving the equations of motion suitable for computer implementation. Lee et al. [1983], based on the generalized d'Alembert principle, derived equations of motion which are expressed explicitly in vector-matrix form suitable for control analysis.

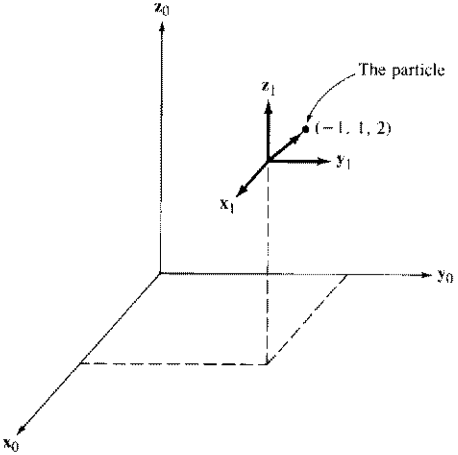
Neuman and Tourassis [1983] and Murray and Neuman [1984] developed computer software for obtaining the equations of motion of manipulators in symbolic form. Neuman and Tourassis [1985] developed a discrete dynamic model of a manipulator.

PROBLEMS

3.1 (*a*) What is the meaning of the generalized coordinates for a robot arm? (*b*) Give two *different* sets of generalized coordinates for the robot arm shown in the figure below. Draw two separate figures of the arm indicating the generalized coordinates that you chose.



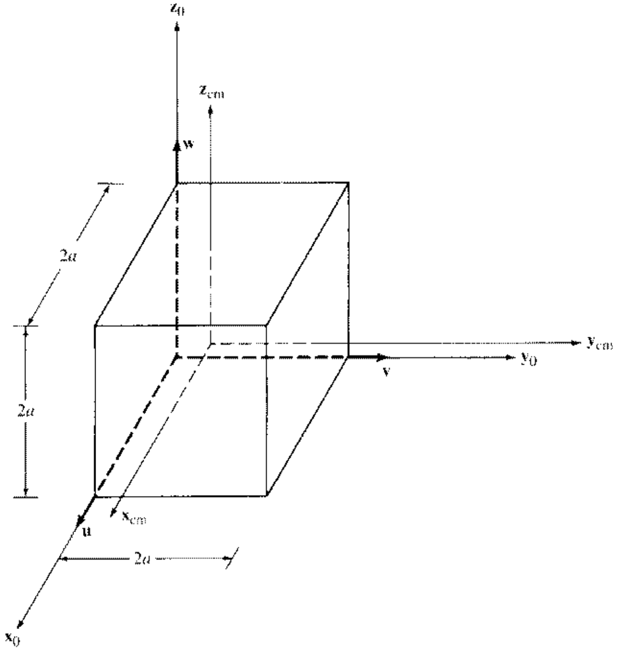
3.2 As shown in the figure below, a particle fixed in an intermediate coordinate frame $(\mathbf{x}_1, \mathbf{y}_1, \mathbf{x}_1)$ is located at $(-1, 1, 2)$ in that coordinate frame. The intermediate coordinate frame is moving translationally with a velocity of $3t\mathbf{i} + 2t\mathbf{j} + 4\mathbf{k}$ with respect to the reference frame $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{x}_0)$ where $\mathbf{i}$, $\mathbf{j}$, and $\mathbf{k}$ are unit vectors along the $\mathbf{x}_0$, $\mathbf{y}_0$, and $\mathbf{z}_0$ axes, respectively. Find the acceleration of the particle with respect to the reference frame.



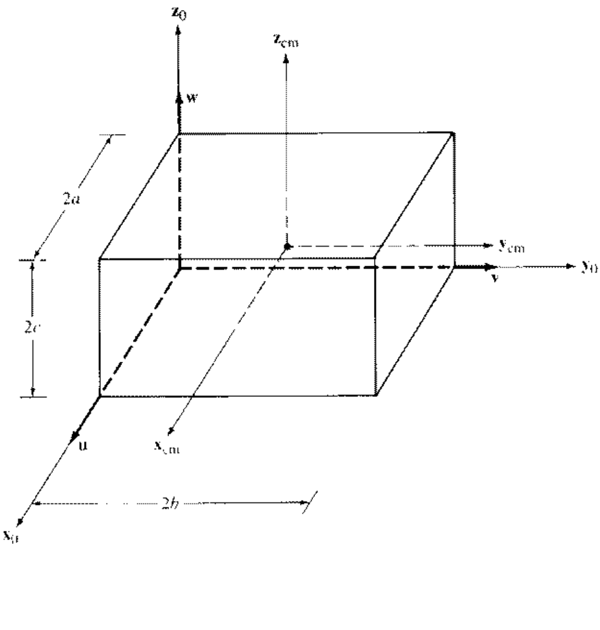
3.3 With reference to Secs. 3.3.1 and 3.3.2, a particle at rest in the starred coordinate system is located by a vector $\mathbf{r}(t) = 3t\mathbf{i} + 2t\mathbf{j} + 4\mathbf{k}$ with respect to the unstarred coordinate system (reference frame), where $(\mathbf{i}, \mathbf{j}, \mathbf{k})$ are unit vectors along the principal axes of the reference frame. If the starred coordinate frame is *only* rotating with respect to the reference frame with $\omega = (0, 0, 1)^T$, find the Coriolis and centripetal accelerations.

3.4 Discuss the differences between Eq. (3.3-13) and Eq. (3.3-17) when (*a*) $\mathbf{h} = \mathbf{0}$ and (*b*) $d\mathbf{h}/dt = \mathbf{0}$ (that is, $\mathbf{h}$ is a constant vector).

3.5 With references to the cube of mass M and side $2a$ shown in the figure below, $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ is the reference coordinate frame, $(\mathbf{u}, \mathbf{v}, \mathbf{w})$ is the body-attached coordinate frame, and $(\mathbf{x}_{cm}, \mathbf{y}_{cm}, \mathbf{z}_{cm})$ is another body-attached coordinate frame at the center of mass of the cube. (*a*) Find the inertia tensor in the ($\mathbf{x}_0$, $\mathbf{y}_0$, $\mathbf{z}_0$) coordinate system. (*b*) Find the inertia tensor at the center of mass in the ($\mathbf{x}_{cm}$, $\mathbf{y}_{cm}$, $\mathbf{z}_{cm}$) coordinate system.



3.6 Repeat Prob. 3.5 for this rectangular block of mass M and sides $2a$, $2b$, and $2c$:



3.7 Assume that the cube in Prob. 3.5 is being rotated through an angle of α about the $\mathbf{z}_0$ axis and then rotated through an angle of θ about the $\mathbf{u}$ axis. Determine the inertia tensor in the $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ coordinate system.

3.8 Repeat Prob. 3.7 for the rectangular block in Prob. 3.6.

3.9 We learned that the Newton-Euler formulation of the dynamic model of a manipulator is computationally more efficient than the Lagrange-Euler formulation. However, most researchers still use the Lagrange-Euler formulation. Why is this so? (Give two reasons.)

3.10 A robotics researcher argues that if a robot arm is always moving at a very slow speed, then its Coriolis and centrifugal forces/torques can be omitted from the equations of motion formulated by the Lagrange-Euler approach. Will these "approximate" equations of motion be computationally more efficient than the Newton-Euler equations of motion? Explain and justify your answer.

3.11 We discussed two formulations for robot arm dynamics in this chapter, namely, the Lagrange-Euler formulation and the Newton-Euler formulation. Since they describe the same physical system, their equations of motion should be "equivalent." Given a set point on a preplanned trajectory at time t_1, $(\mathbf{q}^d(t_1), \dot{\mathbf{q}}^d(t_1), \ddot{\mathbf{q}}^d(t_1))$, one should be able to find the $\mathbf{D}(\mathbf{q}^d(t_1))$, the $\mathbf{h}(\mathbf{q}^d(t_1)), \dot{\mathbf{q}}^d(t_1))$, and the $\mathbf{c}(\mathbf{q}^d(t_1))$ matrices from the L-E equations of motion. Instead of finding them from the L-E equations of motion, can you state a *procedure* indicating how you can obtain the above matrices from the N-E equations of motion using the same set point from the trajectory?

3.12 The dynamic coefficients of the equations of motion of a manipulator can be obtained from the N-E equations of motion using the technique of *probing* as discussed in Prob. 3.11. Assume that N multiplications and M additions are required to compute the torques applied to the joint motors for a particular robot. What is the *smallest* number of multiplications and additions in terms of N, M, and n needed to find all the elements in the $\mathbf{D}(\mathbf{q})$ matrix in the L-E equations of motion, where n is the number of degrees of freedom of the robot?

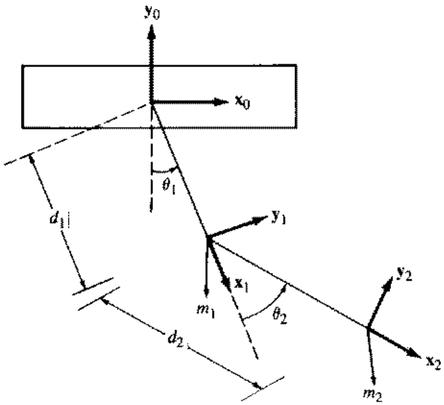
3.13 In the Lagrange-Euler derivation of equations of motion, the gravity vector $\mathbf{g}$ given in Eq. (3.3-22) is a row vector of the form $(0, 0, -|g|, 0)$, where there is a negative sign for a level system. In the Newton-Euler formulation, the gravity effect as given in Table 3.2 is $(0, 0, |g|)^T$ for a level system, and there is no negative sign. Explain the discrepancy.

3.14 In the recursive Newton-Euler equations of motion referred to its own link coordinate frame, the matrix $({}^i\mathbf{R}_0\ \mathbf{I}_i\ {}^0\mathbf{R}_i)$ is the inertial tensor of link i about the ith coordinate frame. Derive the relationship between this matrix and the pseudo-inertia matrix $\mathbf{J}_i$ of the Lagrange-Euler equations of motion.

3.15 Compare the differences between the representation of angular velocity and kinetic energy of the Lagrange-Euler and Newton-Euler equations of motion in the following table (fill in the blanks):

	Lagrange-Euler	Newton-Euler
Angular velocity		
Kinetic energy		

3.16 The two-link robot arm shown in the figure below is attached to the ceiling and under the influence of the gravitational acceleration $g = 9.8062$ m/sec^2; $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ is the reference frame; θ_1, θ_2 are the generalized coordinates; d_1, d_2 are the lengths of the links; and m_1, m_2 are the respective masses. Under the assumption of lumped equivalent masses, the mass of each link is lumped at the end of the link. (*a*) Find the link transformation matrices ${}^{i-1}\mathbf{A}_i$, $i = 1, 2$. (*b*) Find the pseudo-inertia matrix $\mathbf{J}_i$ for each link. (*c*) Derive the Lagrange-Euler equations of motion by first finding the elements in the $\mathbf{D}(\boldsymbol{\theta})$, $\mathbf{h}(\boldsymbol{\theta}, \dot{\boldsymbol{\theta}})$, and $\mathbf{c}(\boldsymbol{\theta})$ matrices.



3.17 Given the same two-link robot arm as in Prob. 3.16, do the following steps to derive the Newton-Euler equations of motion and then compare them with the Lagrange-Euler equations of motion. (*a*) What are the initial conditions for the recursive Newton-Euler equations of motion? (*b*) Find the inertia tensor ${}^i\mathbf{R}_0\mathbf{I}_i\,{}^0\mathbf{R}_i$ for each link. (*c*) Find the other constants that will be needed for the recursive Newton-Euler equations of motion, such as ${}^i\mathbf{R}_0\bar{\mathbf{s}}_i$ and ${}^i\mathbf{R}_0\,\mathbf{p}_i^*$. (*d*) Derive the Newton-Euler equations of motion for this robot arm, assuming that $\mathbf{f}_{n+1}$ and $\mathbf{n}_{n+1}$ have zero reaction force/torque.

3.18 Use the Lagrange-Euler formulation to derive the equations of motion for the two-link $\theta - d$ robot arm shown below, where $(\mathbf{x}_0, \mathbf{y}_0, \mathbf{z}_0)$ is the reference frame, θ and d are the generalized coordinates, and m_1, m_2 are the link masses. Mass m_1 of link 1 is assumed to be located at a constant distance r_1 from the axis of rotation of joint 1, and mass m_2 of link 2 is assumed to be located at the end point of link 2.