
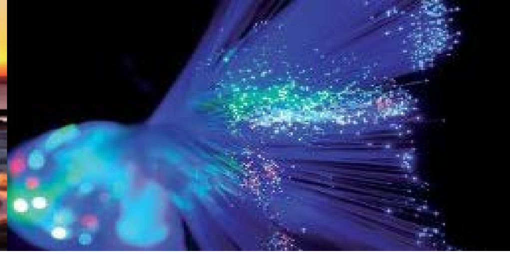
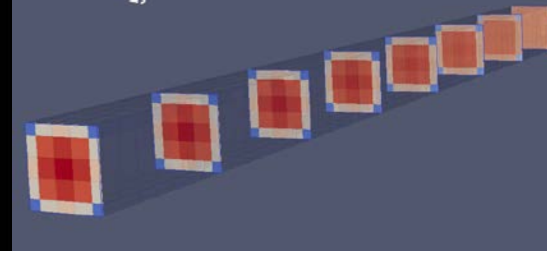
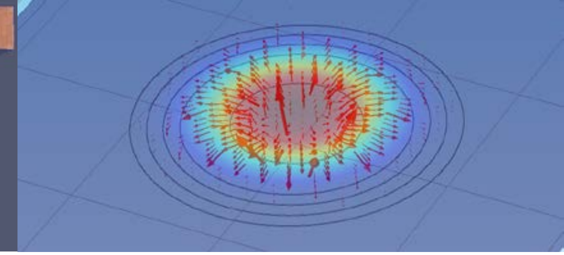

INTRODUCTION TO NUMERICAL ANALYSIS

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2. MATHEMATICAL BACKGROUND

2.2 Calculus
2.3 Vectors
2.4 Matrices and linear algebra
2.5 Ordinary differential equations
2.6 Partial differentiation
2.7 Taylor series
2.8 Inner product and orthogonality

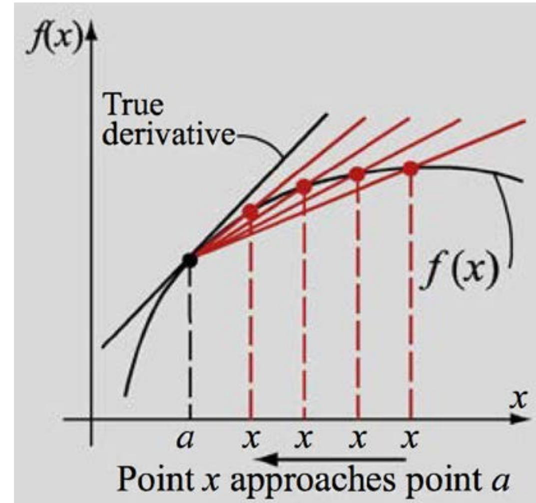
Objectives of this chapter

- To present some background useful for analysis of numerical methods
- To review some fundamental concepts and terms from calculus
  - Useful in the derivation of the numerical methods themselves

2.2 Concepts from pre-calculus and calculus

❖ Derivatives of a function

$$\left.\frac{dy}{dx}\right|_{x=a} = f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$



$$\frac{d^2y}{dx^2} = \frac{d\left(\frac{dy}{dx}\right)}{dx} \qquad \frac{d^3y}{dx^3} = \frac{d}{dx}\left(\frac{d^2y}{dx^2}\right)$$

- Chain rule for ordinary differentiation
  - Useful for differentiating functions whose arguments themselves are functions

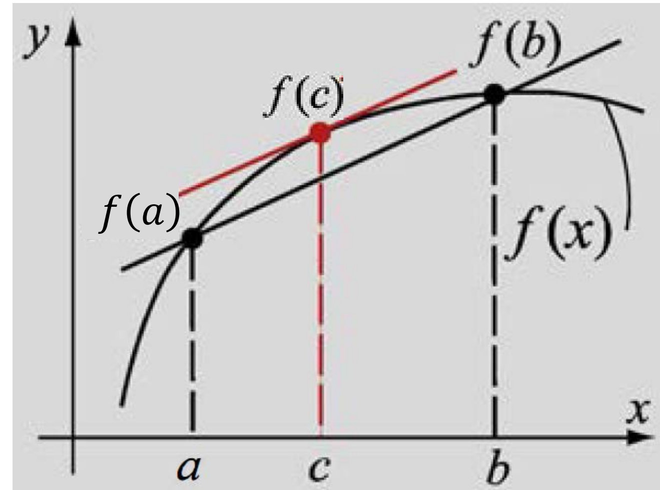
$$y = f(u) \qquad u = g(x) \qquad y = f(g(x))$$

$$\frac{dy}{dx} = \left(\frac{dy}{du}\right)\left(\frac{du}{dx}\right)$$

Derivatives of a function

- Mean value theorem for derivatives
  - Within the interval there exists a point c such that the value of the derivative of $f(x)$ is exactly equal to the slope of the secant line joining the endpoints $(a, f(a))$ and $(b, f(b))$.

$$f'(c) = \left.\frac{dy}{dx}\right|_{x=c} = \frac{f(b)-f(a)}{b-a}$$

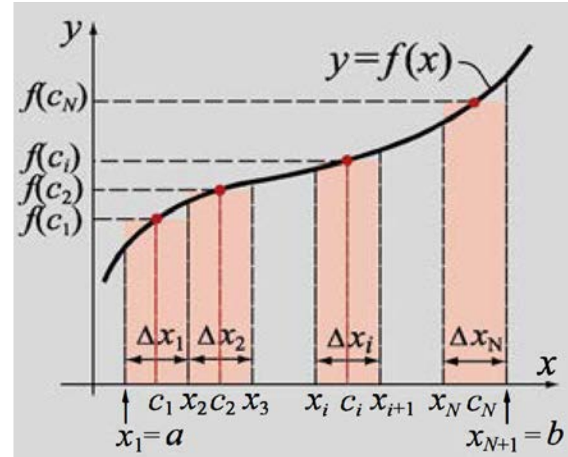


  - Useful in numerical analysis when finding bounds for the order of magnitude of numerical error

Integral of a function

- Riemann sum

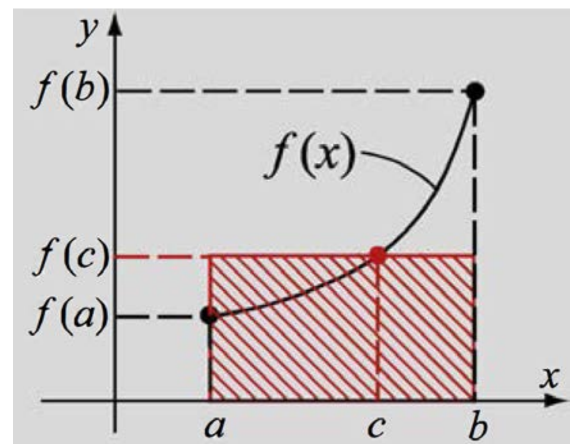
$$\sum_{i=1}^{n} f(c_i)\Delta x_i$$



- Definite integral

$$I = \int_a^b f(x)dx = \lim_{\Delta x_i \to 0} \sum_{i=1}^{n} f(c_i)\Delta x$$

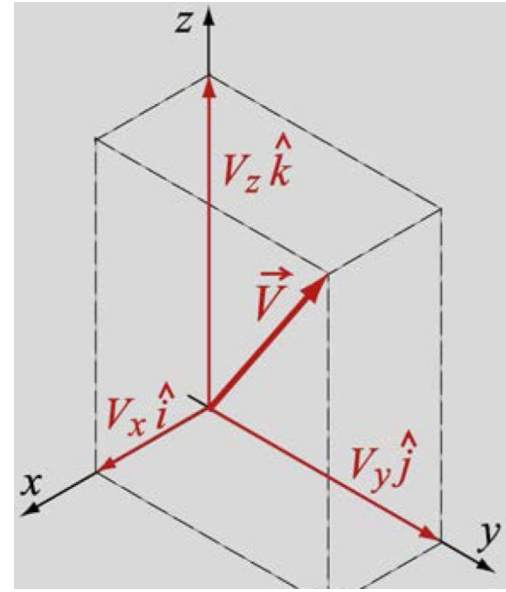
- Mean value theorem for integrals
  - Somewhere between these two rectangles there exists a rectangle whose area is exactly equal to the area under the curve.



$$\int_a^b f(x)dx = f(c)(b-a) \quad \langle f \rangle = \frac{1}{(b-a)}\int_a^b f(x)dx$$

- **Components of the vector**

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k} \qquad \vec{V} = \begin{bmatrix} V_x & V_y & V_z \end{bmatrix} \text{ or } \vec{V} = \begin{bmatrix} V_x \\ V_y \\ V_z \end{bmatrix}$$



- **Magnitude of a vector**

$$\left|\vec{V}\right| = \sqrt{V_x^2 + V_y^2 + V_z^2}$$

- **Unit vector**

$$\hat{V} = \frac{\vec{V}}{\left|\vec{V}\right|} = \frac{V_x\hat{i} + V_y\hat{j} + V_z\hat{k}}{\sqrt{V_x^2 + V_y^2 + V_z^2}} = l\hat{i} + m\hat{j} + n\hat{k}$$

$$l = \frac{V_x}{\sqrt{V_x^2 + V_y^2 + V_z^2}}, \quad m = \frac{V_y}{\sqrt{V_x^2 + V_y^2 + V_z^2}}, \quad \text{and} \quad n = \frac{V_z}{\sqrt{V_x^2 + V_y^2 + V_z^2}}$$

- Direction cosines: equal to the cosine of the angle between the vectors and coordinates

Operations with vectors

- Additions and subtraction of two vectors
- Multiplication of a vector by a scalar
- Transpose of a vector
- Multiplication of two vectors
  - Dot product $\Rightarrow$ scalar quantity
  - Cross product $\Rightarrow$ vector quantity

$$\vec{V} = [V_1, V_2, \ldots, V_n] \qquad \vec{V}^T = \begin{bmatrix} V_1 \\ V_2 \\ \cdots \\ V_n \end{bmatrix}$$

Dot product:

$$\vec{V} \bullet \vec{U} = [V_i][U_i] = V_1U_1 + V_2U_2 + \ldots V_nU_n$$

$$\vec{V} \bullet \vec{U} = V_iU_i = \sum_{i=1}^{n} V_iU_i$$

$$\vec{V} \bullet \vec{U} = |\vec{V}||\vec{U}|\cos\theta$$

Cross product:

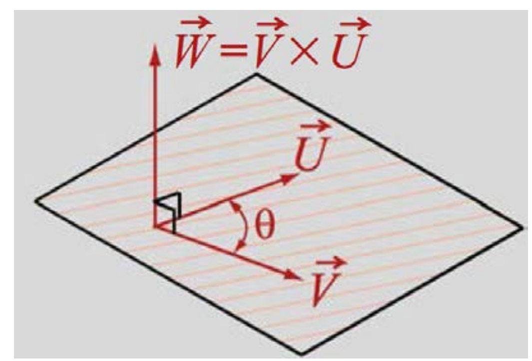
$$\vec{U} = U_x\hat{i} + U_y\hat{j} + U_z\hat{k}$$

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k}$$

$$\vec{W} = \vec{V} \otimes \vec{U}$$

$$= (V_yU_z - V_zU_y)\hat{i} + (V_zU_x - V_xU_z)\hat{j} + (V_xU_y - V_yU_x)\hat{k}$$

$$|\vec{W}| = |\vec{V}||\vec{U}|\sin\theta$$



Operations with vectors

- Linear dependence and linear independence of a set of vectors

$$\alpha_1 \vec{V}_1 + \alpha_2 \vec{V}_2 + \ldots + \alpha_n \vec{V}_n = 0$$

linear combination of vectors

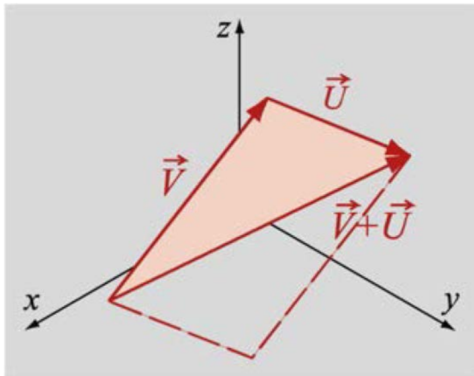
  - **Linearly independent** if and only if

$$\alpha_1 = \alpha_2 = \ldots = \alpha_n = 0$$

  - Otherwise, **linearly dependent**

- Triangle inequality

$$\left|\vec{V} + \vec{U}\right| \leq \left|\vec{V}\right| + \left|\vec{U}\right|$$



Matrices and linear algebra

- Multiplication by a scalar
- Addition and subtraction of two matrices
- Transpose of a matrix

$$[a]^T = [a_{ij}^T] = [a_{ji}]$$

- Multiplication of matrices (I × Q) • (Q × J)

$$c_{ij} = a_{ik}b_{kj} = \sum_{k=1}^{q} a_{ik}b_{kj}$$

$$\begin{bmatrix} A_{11} & A_{12} & \cdots & A_{1n} \\ A_{21} & & & A_{2n} \\ \vdots & & & \vdots \\ A_{n1} & A_{n2} & \cdots & A_{nn} \end{bmatrix}$$

- Special matrices
  - Square matrix: same number of columns as rows
    - Diagonal elements/ off-diagonal elements
    - Super-diagonal elements or above-diagonal elements
    - Sub-diagonal element or below-diagonal elements
  - Diagonal matrix / upper triangular matrix / lower triangular matrix
  - Identity matrix / zero matrix / symmetric matrix

Inverse of a matrix

$$[a][a]^{-1} = [a]^{-1}[a] = [I]$$

Properties of matrices

$$[a][b] \neq [b][a]$$

$$([a]+[b])[c] = [a][c]+[b][c]$$

$$[a]([b]+[c]) = [a][b]+[a][c]$$

$$([a]^T)^T = [a] \qquad ([a]^{-1})^{-1} = [a]$$

$$([a][b])^T = [b]^T[a]^T \qquad ([a][b])^{-1} = [b]^{-1}[a]^{-1}$$

Determinant of a matrix

- Useful quantity that features prominently in finding the inverse of a matrix and provides useful information regarding whether or not solutions exist for a set of equations.

$$\det(A) = |A| = \sum_{j} (-1)^k a_{1,\,j_1} a_{2,\,j_2} \cdots a_{n,\,j_n}$$

Cramer's rule and solution of a system of simultaneous linear equations

- A set of linear equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + \ldots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \ldots + a_{2n}x_n &= b_2 \\ \ldots + \ldots + \ldots + \ldots &= \ldots \\ a_{n1}x_1 + a_{n2}x_2 + \ldots + a_{nn}x_n &= b_n \end{aligned} \Rightarrow \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \ldots & \ldots & \ldots & \ldots \\ a_{n1} & a_{n2} & \ldots & a_{nn} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ \ldots \\ x_n \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \ldots \\ b_n \end{bmatrix}$$

- Cramer's rule

$$x_j = \frac{\det(a'_j)}{\det(a)} \text{ for } j = 1, 2, \ldots, n$$

Solutions can exist only if $\det(a) \neq 0$

  - Two or more columns or rows are identical / or one or more columns or rows are linearly dependent on other columns or rows

Cramer's rule and solution of a system of simultaneous linear equations

- Example

Example 2-2: Solving a system of linear equations using Cramer's rule.

Find the solution of the following system of equations using Cramer's rule.

$$\begin{aligned} 2x + 3y - z &= 5 \\ 4x + 4y - 3z &= 3 \\ -2x + 3y - z &= 1 \end{aligned} \tag{2.45}$$

SOLUTION

***Step 1*:** Write the system of equations in a matrix form $[a][x] = [b]$.

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix} \tag{2.46}$$

***Step 2*:** Calculate the determinant of the matrix of coefficients.

$$\begin{aligned} \det(A) &= 2[(4\times -1)-(-3\times 3)] - 3[(4\times -1)-(-3\times -2)] - 1[(4\times 3)-(4\times -2)] \\ &= 2(5) - 3(-10) - 1(20) = 10 + 30 - 20 = 20 \end{aligned}$$

***Step 3*:** Apply Eq. (2.44) to find x, y, and z. To find x, the modified matrix a'_x is created by replacing its first column with $[b]$.

$$x = \frac{\det\left(\begin{bmatrix} 5 & 3 & -1 \\ 3 & 4 & -3 \\ 1 & 3 & -1 \end{bmatrix}\right)}{20} = \frac{(5\cdot 5)-(3\cdot 0)-(1\cdot 5)}{20} = 1$$

In the same way, to find y, the modified matrix a'_y is created by replacing its second column with $[b]$.

❖ Cramer's rule and solution of a system of simultaneous linear equations

- Example

$$y = \frac{\det\left(\begin{bmatrix} 2 & 5 & -1 \\ 4 & 3 & -3 \\ -2 & 1 & -1 \end{bmatrix}\right)}{20} = \frac{(20 \cdot 0) - (5 \cdot -10) - (1 \cdot 10)}{20} = 2$$

Finally, to determine the value of z, the modified matrix a'_z is created by replacing its third column with $[b]$.

$$z = \frac{\det\left(\begin{bmatrix} 2 & 3 & 5 \\ 4 & 4 & 3 \\ -2 & 3 & 1 \end{bmatrix}\right)}{20} = \frac{(2 \cdot -5) - (3 \cdot 10) - (5 \cdot 20)}{20} = 3$$

To check the answer, the matrix of coefficients $[a]$ is multiplied by the solution:

$$\begin{bmatrix} 2 & 3 & -1 \\ 4 & 4 & -3 \\ -2 & 3 & -1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} 2+6-3 \\ 4+8-9 \\ -2+6-3 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \\ 1 \end{bmatrix}$$

The right-hand side is equal to $[b]$, which confirms that the solution is correct.

Norms

- The magnitude of the vector

$$\left|\vec{V}\right| = \sqrt{V_x^2 + V_y^2 + V_z^2} \qquad \qquad \|[a]\|$$

- Such an equivalent measure for the "magnitude" of a matrix is also useful in comparing different matrices; it is called the **Norm**.
- There is no unique way to measure the “magnitude” or norm of a matrix
  - Will be presented later

- Properties of a norm
  - $\|[a]\| \geq 0$ and $\|[a]\| = 0$ if and only if $[a] = [0]$
  - For all numbers α, $\|\alpha[a]\| = |\alpha|\,\|[a]\|$
  - For any two matrices, $\|[a]+[b]\| \leq \|[a]\| + \|[b]\|$

2.5 Ordinary Differential Equations (ODE)

❖ Variables

- An ODE contains one dependent variable, one independent variable, and ordinary derivatives of the dependent variable.
- Independent variable: x
- Dependent variable: y

$$x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}, \ldots, \frac{d^ny}{dx^n}$$

❖ Linearity

- An ODE is linear if its dependence on y and its derivatives is linear.
- Any linear ODE can be written in the following standard or canonical form.

$$a_{n+1}(x)\frac{d^ny}{dx^n} + a_n(x)\frac{d^{n-1}y}{dx^{n-1}} + \ldots + a_3(x)\frac{d^2y}{dx^2} + a_2(x)\frac{dy}{dx} + a_1(x)y = r(x)$$

  - Coefficients are all functions of the independent variable x.
  - Ex)

$$\frac{dy}{dx} = 10x \qquad c\frac{dx}{dt} + kx = -m\frac{d^2x}{dt^2}$$

2.5 Ordinary Differential Equations (ODE)

Homogeneous / non-homogeneous ODE

$$a_{n+1}(x)\frac{d^n y}{dx^n} + a_n(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_3(x)\frac{d^2 y}{dx^2} + a_2(x)\frac{dy}{dx} + a_1(x)y = r(x)$$

- $r(x) = 0$: homogeneous ODE
- $r(x) \neq 0$: non-homogeneous ODE

Order of an ODE

- Determined by the order of the highest derivative
- The number of constants or integration constants is equal to the order of ODE.
  - Ex) second order ODE $\Rightarrow$ two undetermined constants $\Rightarrow$ two constraints must be specified.
- Constraints
  - Independent variable: position $\Rightarrow$ boundary conditions
  - Independent variable: time $\Rightarrow$ initial conditions

Nonlinear ODE

$$a_{n+1}(x)\frac{d^n y}{dx^n} + a_n(x)\frac{d^{n-1} y}{dx^{n-1}} + \ldots + a_3(x)\frac{d^2 y}{dx^2} + a_2(x)\frac{dy}{dx} + a_1(x)y = r(x)$$

- If coefficients, a, are functions y of or its derivatives
- If the linear term $a_1 y$, is replaced with a nonlinear function of y
  - Ex)

$$\frac{d^2 y}{dt^2} + \sin y = 4 \qquad y\frac{d^2 y}{dt^2} + 3y = 8 \qquad \left(\frac{dy}{dt}\right)\frac{d^2 y}{dt^2} + y = 9 \qquad \frac{d^2 y}{dt^2} + 8y = \tan y$$

Analytical solutions to some important linear ODEs

- General solution to a linear, non-homogeneous ODE
  - Homogeneous solution + particular solution
  - Boundary or initial conditions must be substituted to solve for the undetermined constants.

- General solution to a non-homogeneous linear first-order ODE

$$\frac{dy}{dx} + P(x)y = Q(x) \Rightarrow \frac{d}{dx}(y\mu) = Q(x)\mu(x) \Rightarrow y(x)\mu(x) = \int Q(x)\mu(x)dx + C_1$$

$$\mu(x) = e^{\int P(x)dx}$$

$$\boxed{y(x) = \frac{1}{\mu(x)}\int Q(x)\mu(x)dx + \frac{C_1}{\mu(x)}}$$

Analytical solutions to some important linear ODEs

- General solution to a homogeneous second-order linear ODE

$$\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0 \Rightarrow s^2 + bs + c = 0 \Rightarrow s = \frac{-b \pm \sqrt{b^2-4c}}{2}$$

$$y = e^{sx}$$

$$y(x) = e^{-bx/2}\left[C_1 e^{\frac{x}{2}\sqrt{b^2-4c}} + C_2 e^{-\frac{x}{2}\sqrt{b^2-4c}}\right]$$

- If the discriminant is negative,

$$y(x) = e^{-bx/2}\left[C_1 e^{\frac{i}{2}x\sqrt{b^2-4c}} + C_2 e^{-\frac{i}{2}x\sqrt{b^2-4c}}\right]$$

  - Euler's formula

$$e^{iz} = \cos(z) + i\sin(z)$$

$$y(x) = e^{-bx/2}\left[C_1\cos\left(x\frac{\sqrt{4c-b^2}}{2}\right) + C_1 i\sin\left(x\frac{\sqrt{4c-b^2}}{2}\right) + C_2\cos\left(x\frac{\sqrt{4c-b^2}}{2}\right) - C_2 i\sin\left(x\frac{\sqrt{4c-b^2}}{2}\right)\right]$$

Analytical solutions to some important linear ODEs

- General solution to a homogeneous second-order linear ODE

$$y(x) = e^{-bx/2}\left[(C_1+C_2)\cos\left(x\frac{\sqrt{4c-b^2}}{2}\right)+(C_1-C_2)i\sin\left(x\frac{\sqrt{4c-b^2}}{2}\right)\right]$$

$$\Downarrow$$

$$y(x) = e^{-bx/2}\left[D_1\sin\left(x\frac{\sqrt{4c-b^2}}{2}\right)+D_2\cos\left(x\frac{\sqrt{4c-b^2}}{2}\right)\right]$$

2.6 Functions of Two or More Independent Variables

Definition of partial derivative

$$\frac{\partial f}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x, y)}{\Delta x} \qquad \frac{\partial f}{\partial y} = \lim_{\Delta y \to 0} \frac{f(x, y+\Delta y) - f(x, y)}{\Delta y}$$

High order partial derivative

$$f_{xx} = \frac{\partial^2 f}{\partial x^2} \qquad f_{xy} = \frac{\partial^2 f}{\partial y \partial x} \qquad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$$

Chain rule

- Total differential of a function of two variables

$$df = \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy \qquad f(x, y)$$

- Argument $x, y \Rightarrow$ function of other variables

$$f(x, y) = f(x(t), y(t))$$
$$f(x, y) = f(x(u, v), y(u, v))$$

2.6 Functions of Two or More Independent Variables

❖ **Chain rule** $f(x, y)$

- x, y : function of single independent variable t

$$\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt}$$

- x, y : function of two independent variable u, v

$$\left.\frac{\partial f}{\partial v}\right|_u = \left.\frac{\partial f}{\partial x}\frac{\partial x}{\partial v}\right|_u + \left.\frac{\partial f}{\partial y}\frac{\partial y}{\partial v}\right|_u \qquad \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial v}$$

- y : function of x

$$\frac{df}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial y}\frac{\partial y}{\partial x}$$

- $f(x, y, z)$ and z depends x and y

$$\frac{\partial f}{dx} = \frac{\partial f}{\partial x} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial x} \qquad \frac{\partial f}{dy} = \frac{\partial f}{\partial y} + \frac{\partial f}{\partial z}\frac{\partial z}{\partial y}$$

2.6 Functions of Two or More Independent Variables

Jacobian

$$f_1(x, y) = a \text{ and } f_2(x, y) = b$$

- Jacobian matrix

$$[J] = \begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}$$

Jacobian determinant (Jacobian)

$$J(f_1, f_2) = \det\left(\begin{bmatrix} \frac{\partial f_1}{\partial x} & \frac{\partial f_1}{\partial y} \\ \frac{\partial f_2}{\partial x} & \frac{\partial f_2}{\partial y} \end{bmatrix}\right) = \left(\frac{\partial f_1}{\partial x}\right)\left(\frac{\partial f_2}{\partial y}\right) - \left(\frac{\partial f_1}{\partial y}\right)\left(\frac{\partial f_2}{\partial x}\right)$$

- Jacobian matrix of general form

$$J(f_1, f_2, \ldots, f_n) = \det\left(\begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \frac{\partial f_1}{\partial x_2} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \frac{\partial f_2}{\partial x_1} & \frac{\partial f_2}{\partial x_2} & \cdots & \frac{\partial f_2}{\partial x_n} \\ \cdots & \cdots & \cdots & \cdots \\ \frac{\partial f_n}{\partial x_1} & \frac{\partial f_n}{\partial x_2} & \cdots & \frac{\partial f_n}{\partial x_n} \end{bmatrix}\right)$$

2.7 Taylor Series Expansion of Functions

Taylor series expansion

- Function of one variable

$$f(x) = f(x_0) + (x - x_0)\frac{df}{dx}\bigg|_{x = x_0} + \frac{(x - x_0)^2}{2!}\frac{d^2 f}{dx^2}\bigg|_{x = x_0} + \frac{(x - x_0)^3}{3!}\frac{d^3 f}{dx^3}\bigg|_{x = x_0}$$

$$+ \ldots + \frac{(x - x_0)^n}{n!}\frac{d^n f}{dx^n}\bigg|_{x = x_0} + R_n(x)$$

- Function of two variables

$$f(x, y) = f(x_0, y_0) + \frac{1}{1!}\left[(x - x_0)\frac{\partial f}{\partial x}\bigg|_{x_0, y_0} + (y - y_0)\frac{\partial f}{\partial y}\bigg|_{x_0, y_0}\right] +$$

$$\frac{1}{2!}\left[(x - x_0)^2\frac{\partial^2 f}{\partial x^2}\bigg|_{x_0, y_0} + 2(x - x_0)(y - y_0)\frac{\partial^2 f}{\partial x \partial y}\bigg|_{x_0, y_0} + (y - y_0)^2\frac{\partial^2 f}{\partial y^2}\bigg|_{x_0, y_0}\right] +$$

$$+ \ldots + \frac{1}{n!}\left[\sum_{k=0}^{n}\frac{n!}{k!(n-k)!}(x - x_0)^k (y - y_0)^{n-k}\frac{\partial^n f}{\partial x^k \partial y^{n-k}}\bigg|_{x_0, y_0}\right]$$

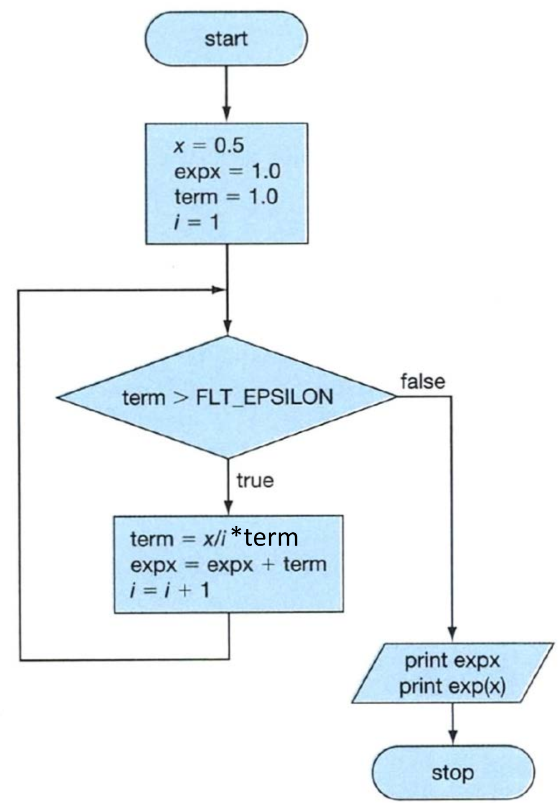
Taylor series expansion

- Example
  - MATLAB 사용
  - $\sin(x), e^x$ 등을 Taylor series 전개한 후 built-in 함수 값과 비교
  - 항의 크기가 machine epsilon 보다 작을 때까지 연산 계산 계속

$$e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \ldots$$

$$= \sum_{i=0}^{\infty} \frac{x^i}{(i)!}$$

$$e^x \approx \sum_{i=0}^{n} \frac{x^i}{(i)!}$$



2.8 Inner Product and Orthogonality

Inner product of two vectors

$$\langle \vec{V} \mid \vec{U} \rangle \qquad \text{dot product } \vec{V} \bullet \vec{U}$$

$$\vec{V} = V_x\hat{i} + V_y\hat{j} + V_z\hat{k} \qquad \vec{U} = U_x\hat{i} + U_y\hat{j} + U_z\hat{k}$$

$$\langle \vec{V} \mid \vec{U} \rangle = \vec{V} \bullet \vec{U} = |\vec{V}||\vec{U}|\cos\theta = V_xU_x + V_yU_y + V_zU_z$$

- If two vectors are orthogonal

$$\langle \vec{V} \mid \vec{U} \rangle = 0$$

❖ Inner product of sine and cosine functions

$$\langle \sin(kx)|\sin(mx)\rangle = \int_{-\pi}^{\pi} \sin(kx)\sin(mx)dx = \begin{cases} 0 & k \neq m \\ \pi & k = m \end{cases}$$

$$\langle \cos(kx)|\cos(mx)\rangle = \int_{-\pi}^{\pi} \cos(kx)\cos(mx)dx = \begin{cases} 0 & k \neq m \\ \pi & k = m \end{cases}$$

$$\langle \sin(kx)|\cos(mx)\rangle = \int_{-\pi}^{\pi} \sin(kx)\cos(mx)dx = 0 \quad \text{for both } k = m \text{ and } k \neq m$$

- For arbitrary domain [a,b]

$$\left\langle \sin\left(\frac{\pi kx}{L}\right)\middle|\sin\left(\frac{\pi mx}{L}\right)\right\rangle = \int_a^b \sin\left(\frac{\pi kx}{L}\right)\sin\left(\frac{\pi mx}{L}\right)dx = \begin{cases} 0 & k \neq m \\ L & k = m \end{cases}$$

$$\left\langle \cos\left(\frac{\pi kx}{L}\right)\middle|\cos\left(\frac{\pi mx}{L}\right)\right\rangle = \int_a^b \cos\left(\frac{\pi kx}{L}\right)\cos\left(\frac{\pi mx}{L}\right)dx = \begin{cases} 0 & k \neq m \\ L & k = m \end{cases}$$

$$\left\langle \sin\left(\frac{\pi kx}{L}\right)\middle|\cos\left(\frac{\pi mx}{L}\right)\right\rangle = \int_a^b \sin\left(\frac{\pi kx}{L}\right)\cos\left(\frac{\pi mx}{L}\right)dx = 0 \quad \text{for both } k = m \text{ and } k \neq m$$

where $L = (b-a)/2$.