Topics in Ship Structural Design
(Hull Buckling and Ultimate Strength)

Lecture 3 Plate Bending

Reference : Ship Structural Design Ch.09

NAOE

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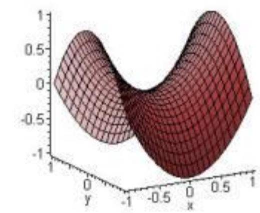
"Long" Plates(Cylindrical Bending)

- An equation relating the deflections to the loading can be developed as for the beam.
- Cylindrical Bending:
  - ✓ A plate which is bent about one axis only(a>>b)
  - ✓ Transverse deformation does not occur, since such a deformation would required a saddle shape deformation.

Saddle shape

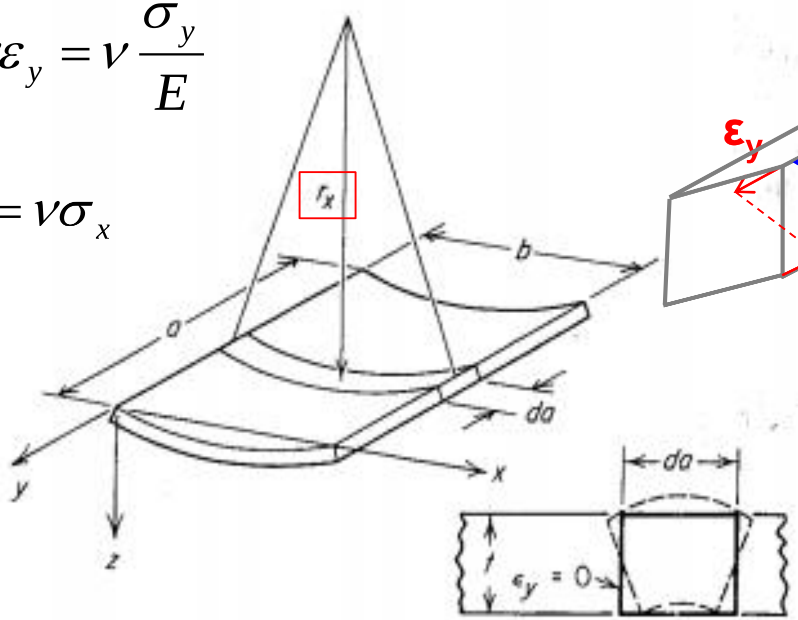


$$\varepsilon_x = \frac{\sigma_x}{E} - \nu \frac{\sigma_y}{E} \Leftarrow \Delta\varepsilon_x = \nu\varepsilon_y = \nu\frac{\sigma_y}{E}$$

$$\varepsilon_y = \frac{\sigma_y}{E} - \nu \frac{\sigma_x}{E} = 0 \Rightarrow \sigma_y = \nu\sigma_x$$

$$\varepsilon_x = \frac{\sigma_x(1-\nu^2)}{E}$$

$$\sigma_x = \frac{E}{1-\nu^2}\varepsilon_x = E'\varepsilon_x$$



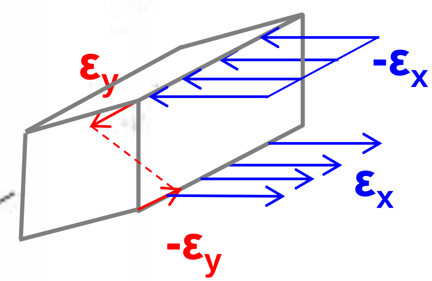
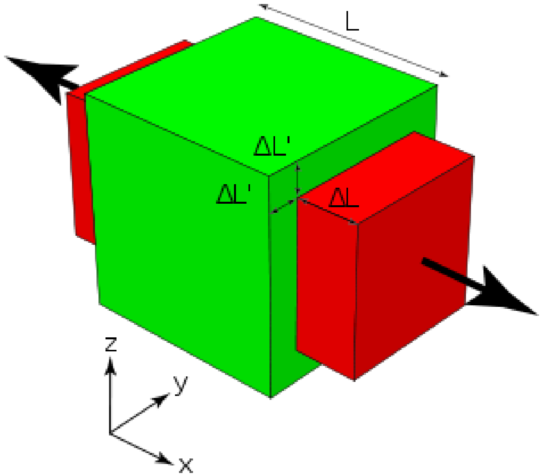


Plate in cylindrical bending

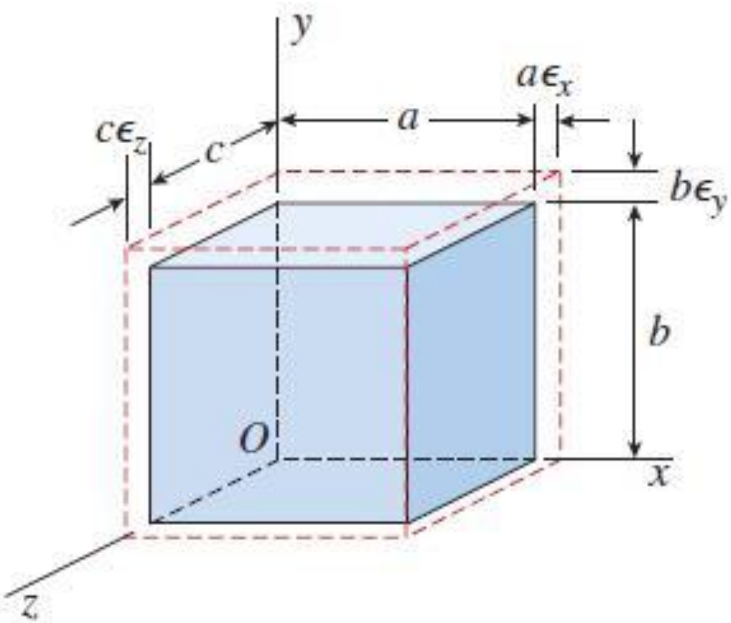
Poisson Ratio



$$\nu = -\frac{\varepsilon_{trans}}{\varepsilon_{axial}}$$

Strain $\Delta\varepsilon_y$ induced by ε_y

$$\Delta\varepsilon_x = \nu\varepsilon_y = \nu\frac{\sigma_y}{E}$$



$$\varepsilon_x = \frac{1}{E}(\sigma_x - \nu\sigma_y - \nu\sigma_z)$$

$$\varepsilon_y = \frac{1}{E}(\sigma_y - \nu\sigma_z - \nu\sigma_x)$$

$$\varepsilon_z = \frac{1}{E}(\sigma_z - \nu\sigma_x - \nu\sigma_y)$$

Formulas between a beam and a long plate

- E´ : effective modulus of elastictiy, always > E.

$$E' = \frac{E}{(1-\nu^2)}$$

→ A plate is always "stiffer" than a row of beams

- For a long prismatically loaded plate, the extra stiffness may be fully taken into account for by using E´ in place of E.

- For simply supported plates:

$$\omega_{\max} = \frac{5pb^4}{384E'I} = \frac{5pb^4(1\text{-}\nu^2)}{32Et^3}$$

- For clamped plates:

$$\omega_{\max} = \frac{pb^4}{384E'I} = \frac{pb^4(1\text{-}\nu^2)}{32Et^3}$$

Formulas between a beam and a long plate

- Moment curvature relation may be obtained from

  External bending moment = moment of stress force

$$M = \int_{-t/2}^{t/2} \sigma_x z dz = \int_{-t/2}^{t/2} \frac{E}{1-\nu^2} \left(\frac{z^2}{r_x}\right) dz = \frac{Et^3}{12(1-\nu^2)} \left(\frac{1}{r_x}\right)$$

where $\boxed{\sigma_x = \frac{E}{(1-\nu^2)} \varepsilon_x}$ $\boxed{\varepsilon_x = z/r_x}$

$$M = \frac{Et^3}{12(1-\nu^2)} \left(\frac{1}{r_x}\right) = \frac{D}{r_x} \qquad D = \frac{Et^3}{12(1-\nu^2)}$$

- D : the flexural rigidity of the plate

  the constant of proportionality between moment and curavature

  analogous with EI in beam theory $\boxed{M = \frac{EI}{\rho}}$

Formulas between a beam and a long plate

- The radius of curvature of the plate can be expressed in terms of the deflection w of the plate

$$-\frac{\partial^2 w}{\partial x^2} = \frac{1}{r_x} \qquad -D\frac{\partial^2 w}{\partial x^2} = M \qquad \boxed{-v'' = \frac{M}{EI}}$$

- Also for a unit strip in a long plate the max. stress is $\sigma_{\max} = M_{\max} c/I$
  where the section modulus

$$I/c = (t^3/12)/(t/2) = t^2/6$$

- For a uniform pressure ***p***, the maximum bending moment is proportional to pb^2. The stress is expressed in

$$\sigma_{\max} = kp\left(\frac{b}{t}\right)^2$$

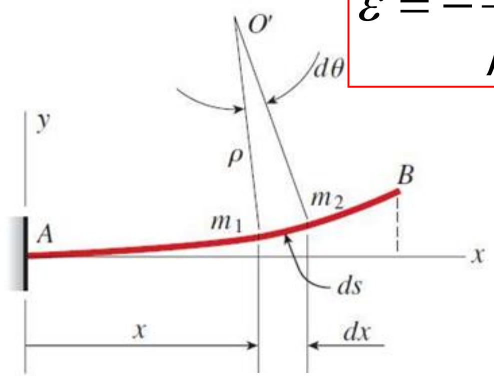
- The coefficient k depends on the boundary conditions:
  k=3/4 for simply supported edges
  k=1/2 for clamped edges

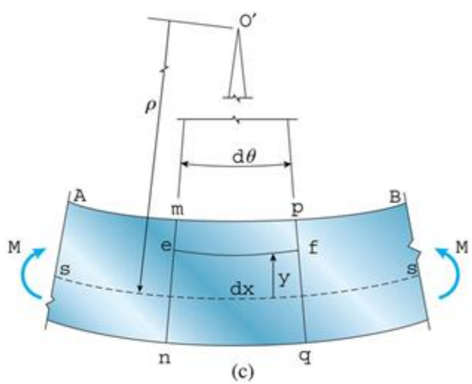
Beam Theory

stress – strain - curvature - Moment

1) strain-curvature Geometric relation

$$\varepsilon = -\frac{y}{\rho}$$





1

2)strain-stress relation

$$\sigma = E\varepsilon = -\frac{Ey}{\rho}$$

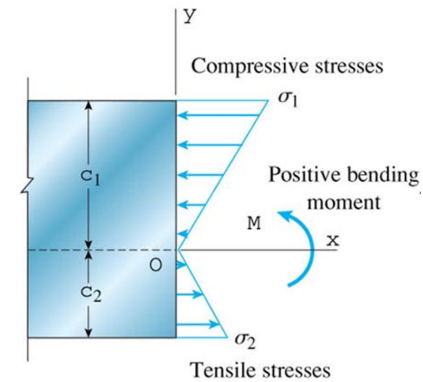
2

$$\frac{1}{\rho} = -\frac{\sigma}{Ey}$$

3

3) moment-curvature relation

$$M = -\int_A \sigma_x y dA$$



$$\therefore \kappa = \frac{1}{\rho} = \frac{M}{EI}$$

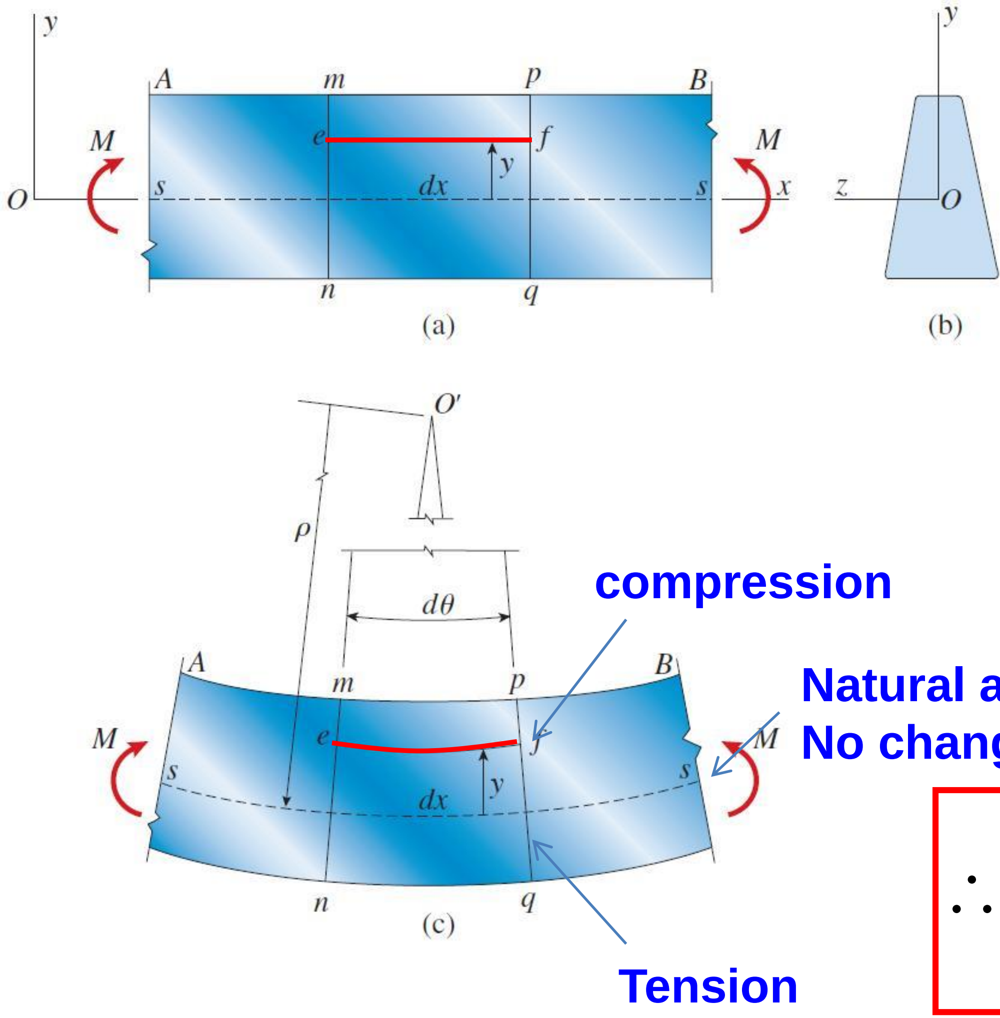
4

4) Stress-moment relation

$$\therefore \sigma_x = -\frac{My}{I}$$

Beam theory

1) Strain-Curvature Relation



Initial length of Line e-f = dx

Length after bending = L_1

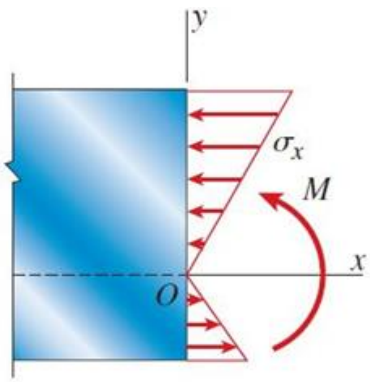
$$L_1 = (\rho - y)d\theta = dx - \frac{y}{\rho}dx$$

$$d\theta = \frac{dx}{\rho}$$

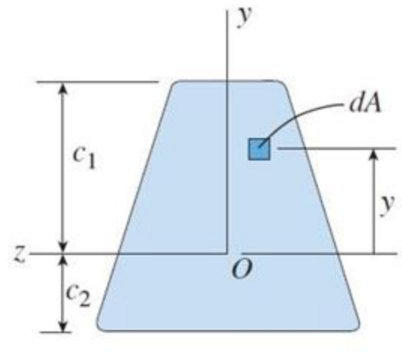
$$\therefore \varepsilon_x = \frac{L_1 - dx}{dx} = -\frac{y}{\rho} = -\kappa y$$

Beam theory

3) moment-curvature relation



(a)



(b)

$$dM = -\sigma_x y dA$$

$$M = -\int_A \sigma_x y dA$$

$$M = \int_A \kappa E y^2 dA = \kappa E \int_A y^2 dA = \kappa E I = \frac{1}{\rho} E I$$

$$\Leftarrow \quad \boxed{\sigma = -\kappa E y}$$

$$\boxed{\therefore \kappa = \frac{1}{\rho} = \frac{M}{EI}}$$

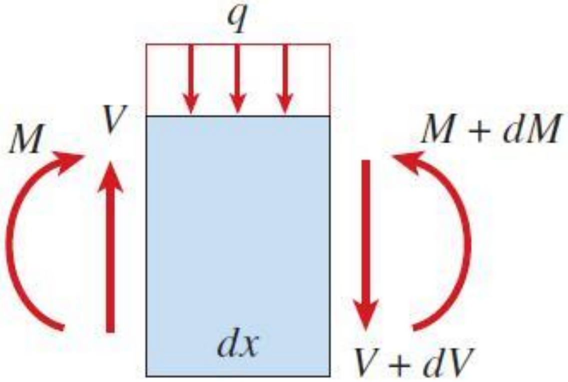
: EI = bending rigidity

$$I \equiv \int_A y^2 dA$$

: Moment of Inertia

Reference Homework #1 Derive in detail

Beam Theory : Relation between load, shear force, Moment



(a)

▪ Force equilibrium

→ Relation between shear force (V) and load (q)

$$\sum F_{vert} = 0 : V - qdx - (V + dV) = 0 \quad \Rightarrow \frac{dV}{dx} = -q$$

▪ Moment equilibrium

→ Relation between Moment (M) and shear force (V)

$$\sum M = 0 : -M - qdx\left(\frac{dx}{2}\right) - (V + dV)dx + (M + dM) = 0 \quad \Rightarrow \quad \frac{dM}{dx} = V$$

▪ Combining two relations

$$\frac{d^2M}{dx^2} = -q$$

▪ Relation between deflection and distributed load

$$\therefore \frac{d^2v}{dx^2} = \frac{M}{EI}$$

$$\boxed{\frac{d^4v}{dx^4} = -\frac{q}{EI}}$$

9.1 Small Deflection Theory

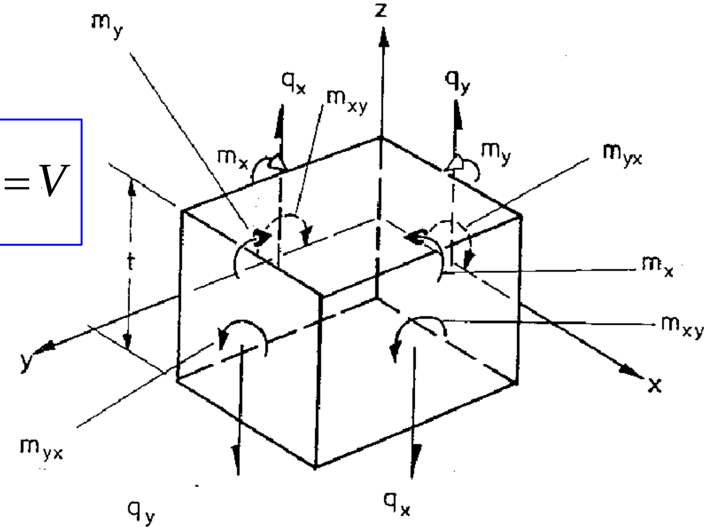
Forces and moments in a plate element

- **Force equilibrium**

→ Relation between shear force (V) and load (q)

$$\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+p=0 \qquad \boxed{\frac{dV}{dx}=-q}$$

q :shear force per unit length
m : moment per unit length



- **Moment equilibrium**

$$\frac{\partial m_{xy}}{\partial x}-\frac{\partial m_y}{\partial y}+q_y=0 \qquad \frac{\partial m_{yx}}{\partial y}+\frac{\partial m_x}{\partial x}-q_x=0 \qquad \boxed{\frac{dM}{dx}=V}$$

- **Combining two relations**

$$\frac{\partial^2 m_x}{\partial x^2}-2\frac{\partial^2 \boxed{m_{xy}}}{\partial x \partial y}+\frac{\partial^2 m_y}{\partial y^2}+p=0 \qquad \boxed{\frac{d^2M}{d^2x}=-q}$$

- **Relation between deflection and distributed load**

$$m_x=-D\left(\frac{\partial^2 w}{\partial x^2}+v\frac{\partial^2 w}{\partial y^2}\right) \qquad m_y=-D\left(\frac{\partial^2 w}{\partial y^2}+v\frac{\partial^2 w}{\partial x^2}\right)$$

$$m_{xy}=D(1-v)\frac{\partial^2 w}{\partial x \partial y}$$

$$\boxed{\frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2 \partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{p}{D}} \qquad \boxed{\frac{d^4 v}{dx^4}=-\frac{q}{EI}}$$

Derivation of the Plate Bending Equation

❖ The previous theory is only applicable if :

- Plane cross sections remain plane.
- The deflections of the plate are small(w_{max} not exceeding 3/4 t)
- The maximum stress nowhere exceeds the plate yield stress(i.e., the material remains elastic)

❖ A panel of plating will have curvature in two directions

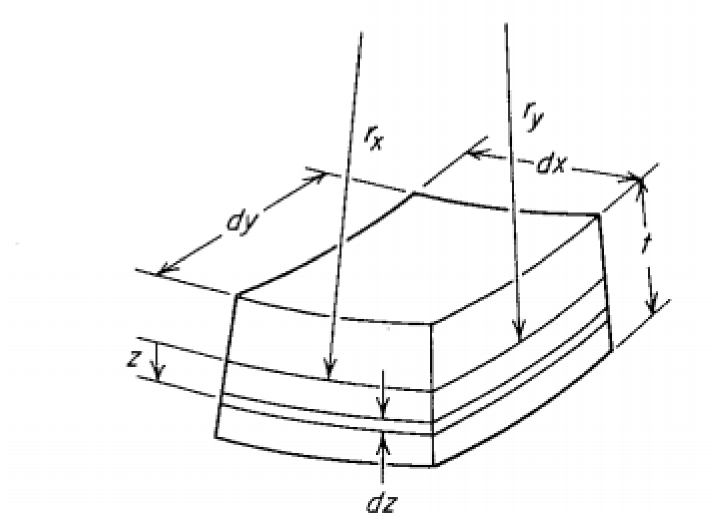
$$\varepsilon_x = \frac{\sigma_x}{E} - \frac{\nu\sigma_y}{E} \qquad \varepsilon_y = \frac{\sigma_y}{E} - \frac{\nu\sigma_x}{E}$$

$$+$$

$$\nu\varepsilon_y = \frac{\nu\sigma_y}{E} - \frac{\nu^2\sigma_x}{E}$$

$$\parallel$$

$$\frac{\sigma_x}{E}(1-\nu^2) = \varepsilon_x + \nu\varepsilon_y$$



Differential element of plating

Derivation of the Plate Bending Equation

- Assumptions stated previously give rise to the strain-curvature relations

$$\varepsilon_x = (-z)\left(\frac{\partial^2 w}{\partial x^2}\right) \quad \varepsilon_y = (-z)\left(\frac{\partial^2 w}{\partial y^2}\right) \Rightarrow \frac{\sigma_x}{E}(1\text{-}\nu^2)=\varepsilon_x+\nu\varepsilon_y$$

$$\sigma_x = \frac{E}{1-\nu^2}(-z)\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)$$

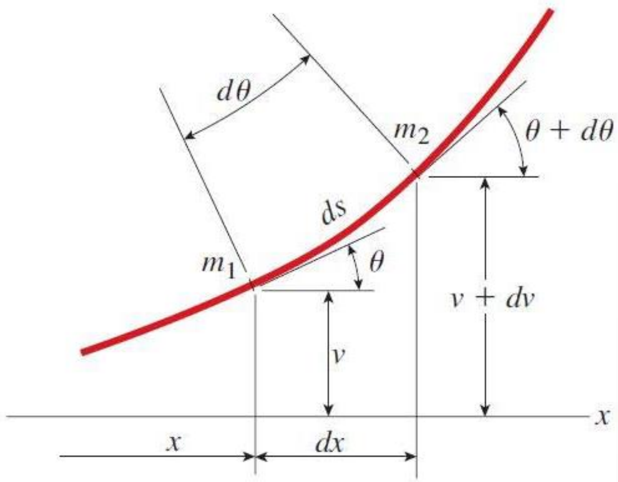
$$m_x = \int_{-t/2}^{t/2}\sigma_x z dz = -\int_{-t/2}^{t/2}\frac{E}{1-\nu^2}\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)z^2 dz$$

$$\boxed{m_x = -D\left(\frac{\partial^2 w}{\partial x^2}+\nu\frac{\partial^2 w}{\partial y^2}\right)} \qquad \boxed{D = \frac{Et^3}{12(1\text{-}\nu^2)}}$$

$$\boxed{m_y = -D\left(\frac{\partial^2 w}{\partial y^2}+\nu\frac{\partial^2 w}{\partial x^2}\right)}$$

Beam Theory : Deflection curve

▪ Relation between deflection and distributed load

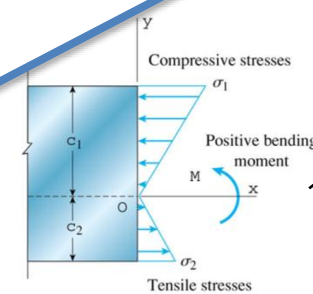


From Geometric relation

$$ds \approx dx$$

$$\left.\begin{aligned} \rho d\theta &= ds \\ \kappa &= \frac{1}{\rho} = \frac{d\theta}{ds} \\ \frac{dv}{dx} &= \tan\theta \end{aligned}\right\}$$

$$\left.\begin{aligned} \kappa &= \frac{1}{\rho} = \frac{d\theta}{dx} \\ \tan\theta &= \boxed{\frac{dv}{dx} \approx \theta} \end{aligned}\right\} \therefore \kappa = \frac{1}{\rho} = \frac{d^2v}{dx^2}$$



$$\kappa = \frac{1}{\rho} = \frac{M}{EI}$$

$$\therefore \frac{d^2v}{dx^2} = \frac{M}{EI}$$

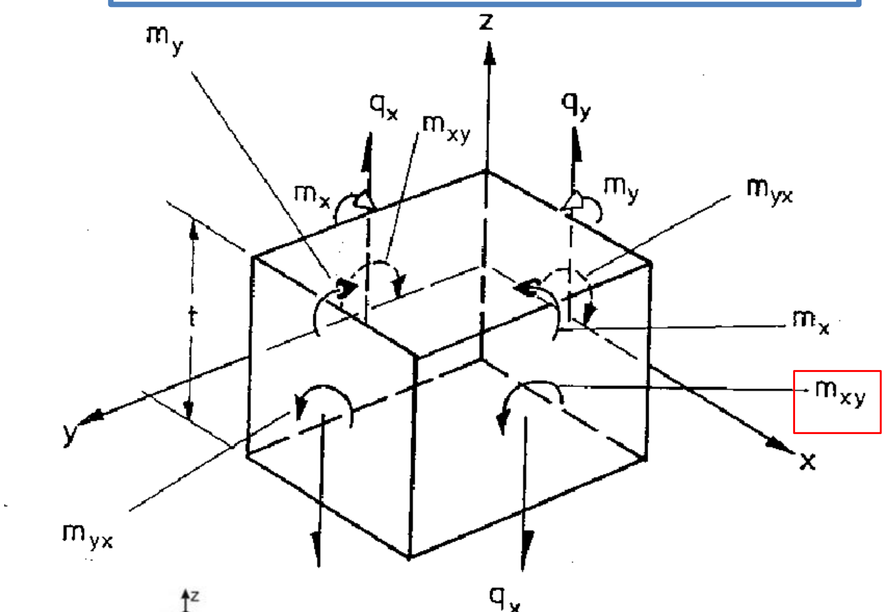
$$\boxed{\frac{d^2M}{dx^2} = -q}$$

$$\boxed{\frac{d^4v}{dx^4} = -\frac{q}{EI}}$$

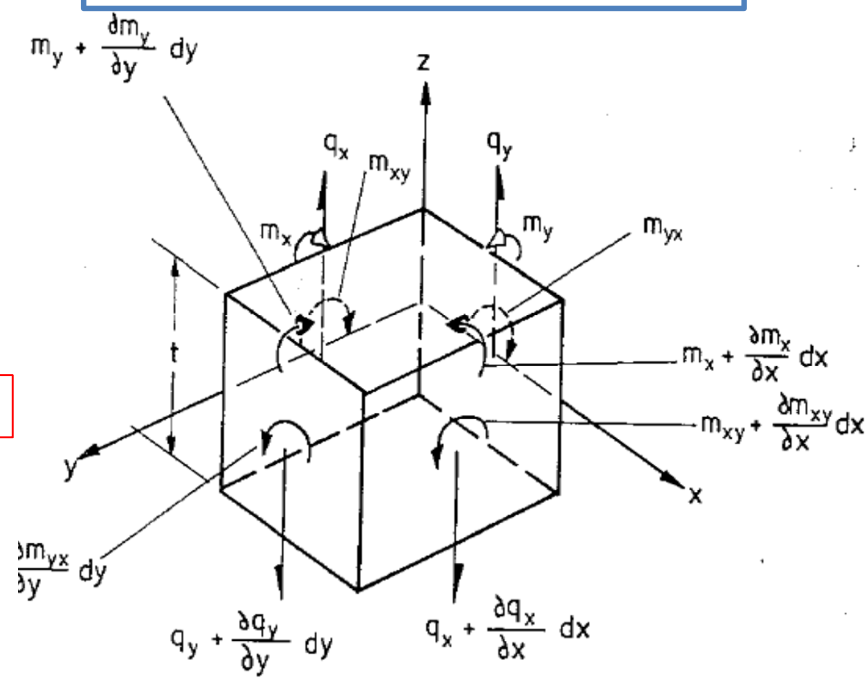
Forces and moments in a plate element

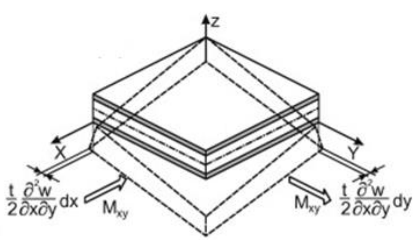
- The lateral load *pdxdy* is carried by the distributed shear forces *q* acting on the four edges of the element.
- In the general case of bending, **twisting moments** will also be generated on all the four faces

Without the lateral load *pdxdy*



With the lateral load *pdxdy*

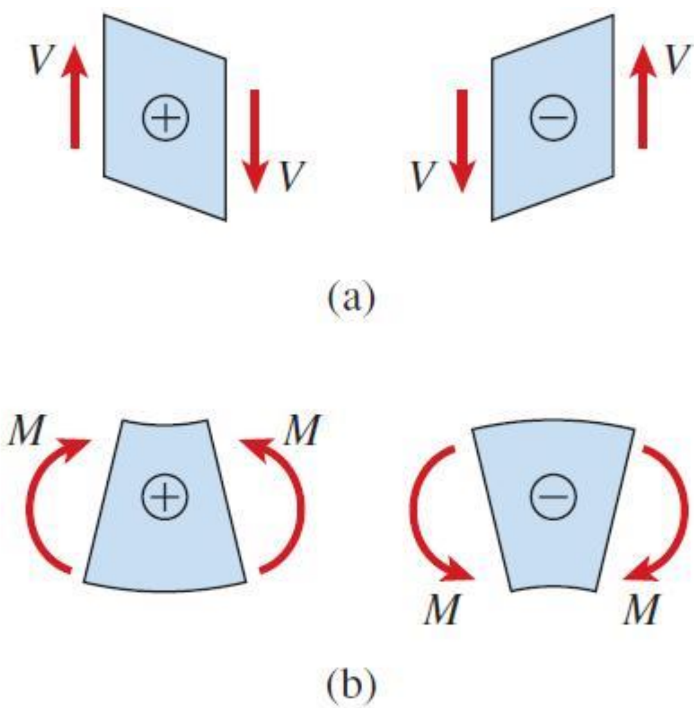




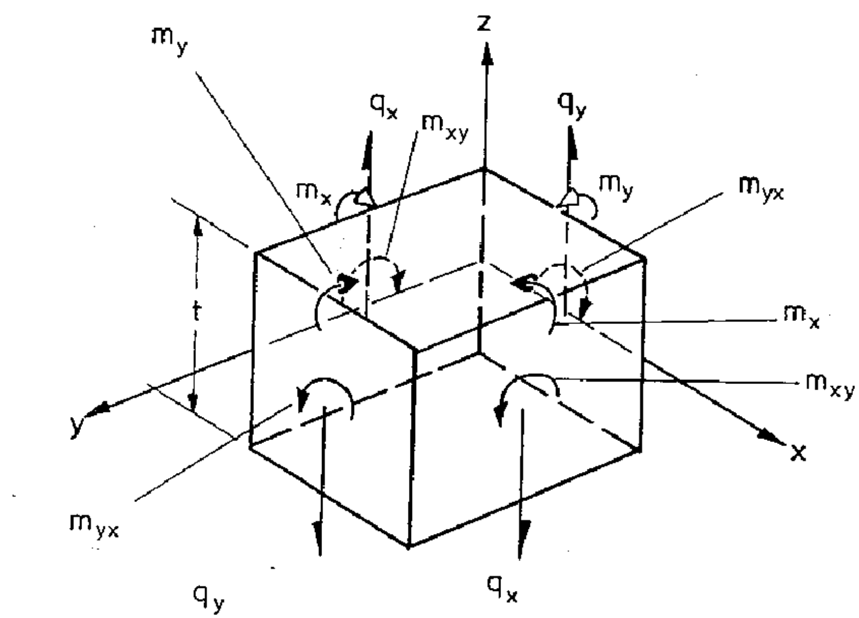
Forces and moments in a plate element

Forces and moments in a plate element

- Sine convention in beam theory
  - Determined how deform the material
  - Shear force : rotate an element clockwise (+)
  - Bending moment : shorten the upper skin (+)



Sine convention in beam theory



Sine convention in plate bending

Forces and moments in a plate element

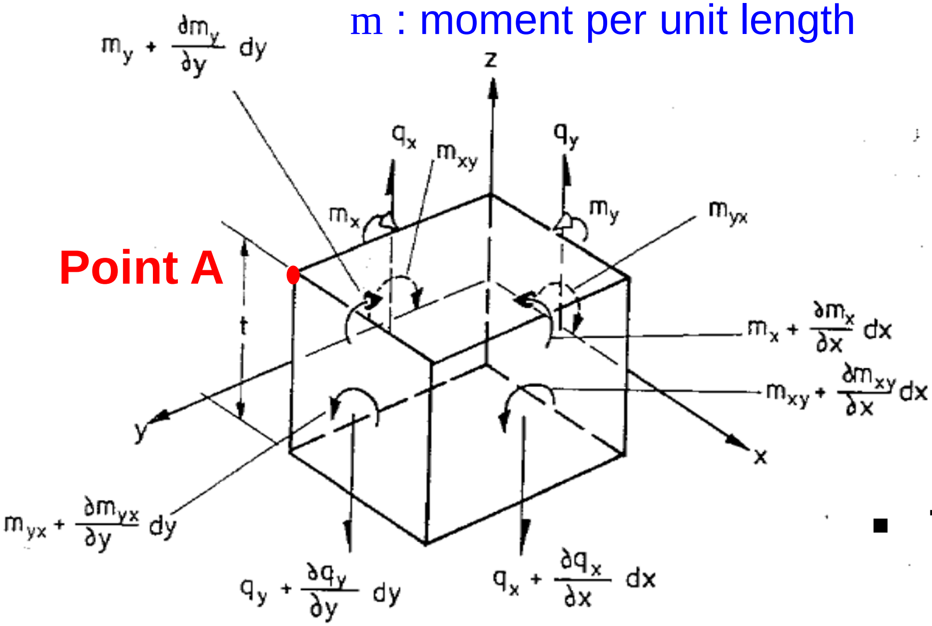
- From equilibrium of vertical forces :

$$(\cancel{q_x} + \frac{\partial q_x}{\partial x}dx)dy - \cancel{q_x} dy$$
$$+(\cancel{q_y} + \frac{\partial q_y}{\partial y}dy)dx - \cancel{q_y} dx + pdxdy = 0$$

$$\Rightarrow \quad \frac{\partial q_x}{\partial x} + \frac{\partial q_y}{\partial y} + p = 0$$

- Taking moments parallel to the x-axis, at Point A

q :shear force per unit length
m : moment per unit length



$$(m_{xy} + \frac{\partial m_{xy}}{\partial x}dx)dy - m_{xy}dy + m_y dx$$
$$-(m_y + \frac{\partial m_y}{\partial y}dy)dx + q_y dxdy = 0$$

$$\Downarrow$$

$$\frac{\partial m_{xy}}{\partial x} - \frac{\partial m_y}{\partial y} + q_y = 0$$

- **Taking moments parallel to the y-axis**

$$\frac{\partial m_{yx}}{\partial y} + \frac{\partial m_x}{\partial x} - q_x = 0$$

Forces and moments in a plate element

- Because of the principle of complementary shear stress, it follows that m_{yx}=-m_{xy}, so that

$$\frac{\partial m_{yx}}{\partial y}+\frac{\partial m_x}{\partial x}-q_x=0 \Rightarrow \boxed{-\frac{\partial m_{xy}}{\partial y}+\frac{\partial m_x}{\partial x}-q_x=0} \quad \boxed{\frac{\partial m_{xy}}{\partial x}-\frac{\partial m_y}{\partial y}+q_y=0}$$

- Substituting for q_x and q_y

$$\boxed{\frac{\partial q_x}{\partial x}+\frac{\partial q_y}{\partial y}+p=0}$$

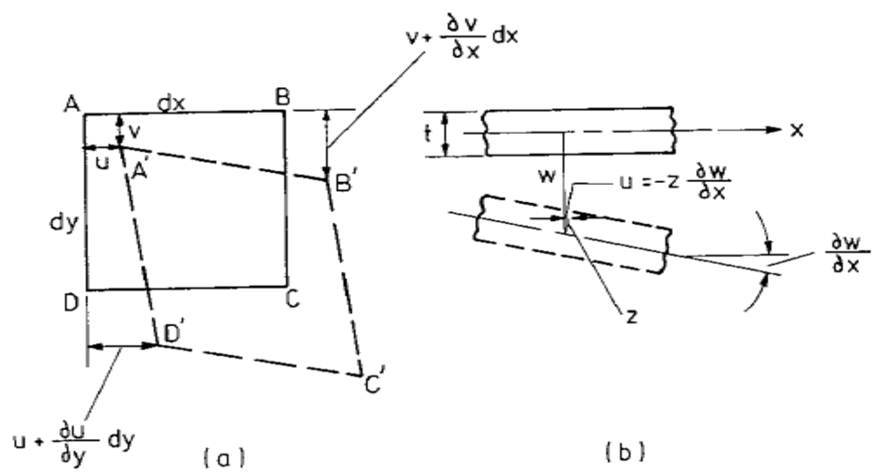
$$\frac{\partial}{\partial x}\left(-\frac{\partial m_{xy}}{\partial y}+\frac{\partial m_x}{\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{\partial m_y}{\partial y}-\frac{\partial m_{xy}}{\partial x}\right)+p=0$$

$$\frac{\partial^2 m_x}{\partial x^2}-2\frac{\partial^2 m_{xy}}{\partial x \partial y}+\frac{\partial^2 m_y}{\partial y^2}+p=0$$

Forces and moments in a plate element

- If a point A in the plate a distance z from the neutral surface is displaced a distance v in the y direction,
- The change in slop of the line AB will be

$$\frac{v+\frac{\partial v}{\partial x}dx-v}{dx}=\frac{\partial v}{\partial x}$$



- The change in slop of the line AD

$$\frac{\partial u}{\partial x}$$

- The shear strain is

$$\gamma=\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}$$

- The shear stress

$$\tau=G\left(\frac{\partial v}{\partial x}+\frac{\partial u}{\partial y}\right)$$

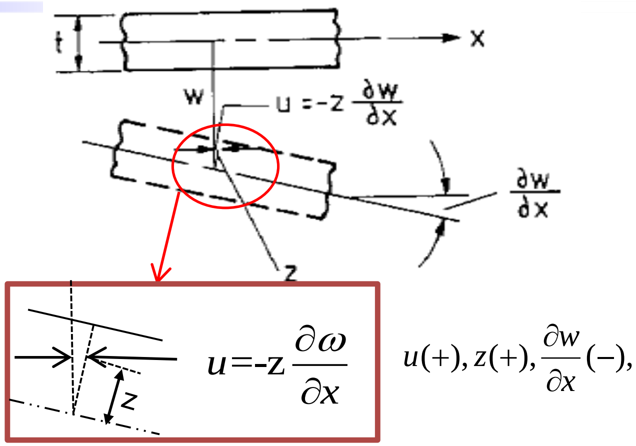
Shear strain and plate deflection

Shear strain is induced by deflection. That is, the vertical deflection generates shear strain.

Forces and moments in a plate element

- from the geometry of the slope of w in each direction (x and y):

$$u = -z\frac{\partial w}{\partial x} \qquad v = -z\frac{\partial w}{\partial y}$$

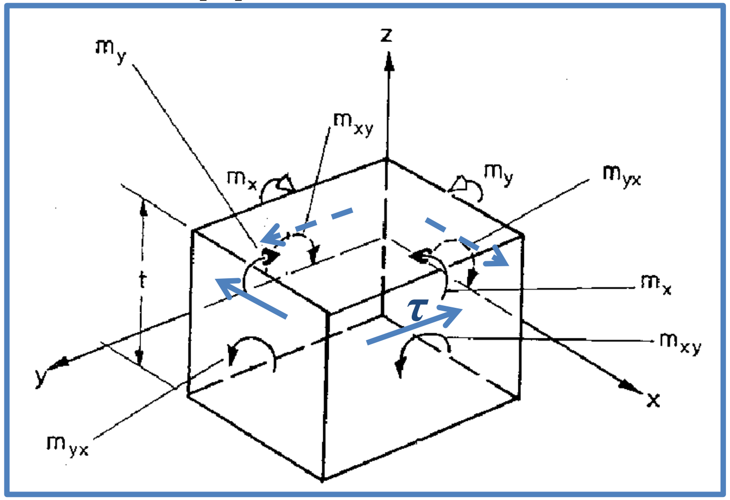


$u(+), z(+), \frac{\partial w}{\partial x}(-),$

- Making these substitutions gives

$$\tau = -2Gz\frac{\partial^2 w}{\partial x \partial y}$$

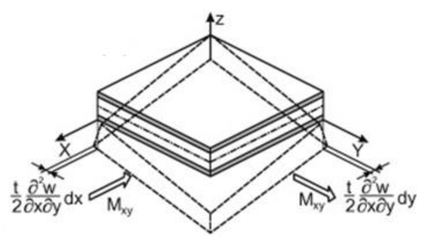
(+) convention

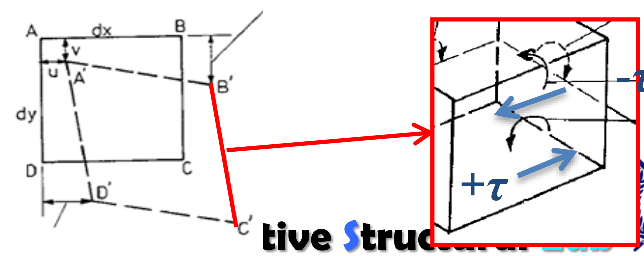


- The twisting moment

$$m_{xy} = -\int_{-t/2}^{t/2} -2G\frac{\partial^2 w}{\partial x \partial y} z^2 dz = \frac{Gt^3}{6}\frac{\partial^2 w}{\partial x \partial y} \qquad G = \frac{E}{2(1+v)}$$

$$m_{xy} = \frac{Et^3}{12(1+v)}\frac{\partial^2 w}{\partial x \partial y} = \frac{Et^3(1-v)}{12(1+v)(1-v)}\frac{\partial^2 w}{\partial x \partial y} = D(1-v)\frac{\partial^2 w}{\partial x \partial y}$$





Forces and moments in a plate element

- Substitute m_x, m_y, m_{xy}

$$\boxed{m_x = -D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)} \quad \boxed{m_y = -D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)}$$

$$\boxed{m_{xy} = D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}}$$

$$\Rightarrow \quad \frac{\partial^2 m_x}{\partial x^2} - 2\frac{\partial^2 m_{xy}}{\partial x \partial y} + \frac{\partial^2 m_y}{\partial y^2} + p = 0$$

- Finally

$$\frac{\partial^2}{\partial x^2}\left[-D\left(\frac{\partial^2 w}{\partial x^2} + \nu\frac{\partial^2 w}{\partial y^2}\right)\right] - 2\frac{\partial^2}{\partial x \partial y}\left[D(1-\nu)\frac{\partial^2 w}{\partial x \partial y}\right] + \frac{\partial^2}{\partial y^2}\left[-D\left(\frac{\partial^2 w}{\partial y^2} + \nu\frac{\partial^2 w}{\partial x^2}\right)\right] = -p$$

- Equation of equilibrium for the plate

$$\Rightarrow \quad \frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D} \qquad \boxed{\nabla^4 w = p/D}$$

- Assumption
  - small deflection, max deflection < ¾ t
  - no stretching of the middle plane (no membrane effect)

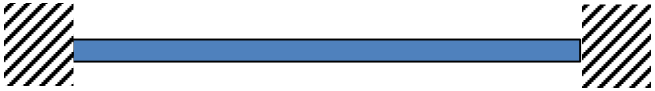
Boundary Conditions

- The types of restraint around the boundary of a plate can be idealized as follows:
  1. Simply supported: edges free to rotate and to move in the plane

  2. Pinned: edges free to rotate but not free to move in the plane

  3. Clamped but free to slide: edges not free to rotate but free to move in the plane of the plate;

  4. Rigidly clamped: edges not free to rotate or move in the plane



- In most plated frame structures (particularly in vehicles and free-standing structure) : little restraint against edge pull-in.

  → BC. 1 and 3 are generally applicable

Simply Supprted Plates

- The general expression for the load

$$p(x,y)=\sum_{m=1}^{\infty}\sum_{n=1}^{\infty}A_{mn}\sin\frac{m\pi y}{a}\sin\frac{n\pi x}{b}$$

- The coefficient A_{mn} can be obtained by Fourier analysis for any particular load condition. For a uniform pressure p_0.

$$A_{mn}=\frac{16p_0}{\pi^2 mn}$$

Homework #2 Prove it.

- For the biharmonic equation, a sinusoidal load distributi on produces a sinusoidal deflection

$$\omega=\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}B_{mn}\sin\frac{m\pi y}{a}\sin\frac{n\pi x}{b} \Rightarrow \frac{\partial^4 w}{\partial x^4}+2\frac{\partial^4 w}{\partial x^2\partial y^2}+\frac{\partial^4 w}{\partial y^4}=\frac{p}{D}$$

$$B_{mn}\left(\frac{\pi^4 m^4}{a^4}+\frac{2\pi^4 m^2 n^2}{a^2 b^2}+\frac{\pi^4 n^4}{b^4}\right)=\frac{16p_0}{\pi^6 Dmn}$$

$$B_{mn}=\frac{16p_0}{\pi^6 Dmn\left(\dfrac{m^2}{a^2}+\dfrac{n^2}{b^2}\right)^2}$$

Simply Supprted Plates

- Due to the symmetry of the problem, M and n only take odd values.

$$w=\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty}\frac{16p_0}{\pi^6 Dmn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)}\sin\frac{m\pi y}{a}\sin\frac{m\pi x}{b}$$

- Effect of boundary condition

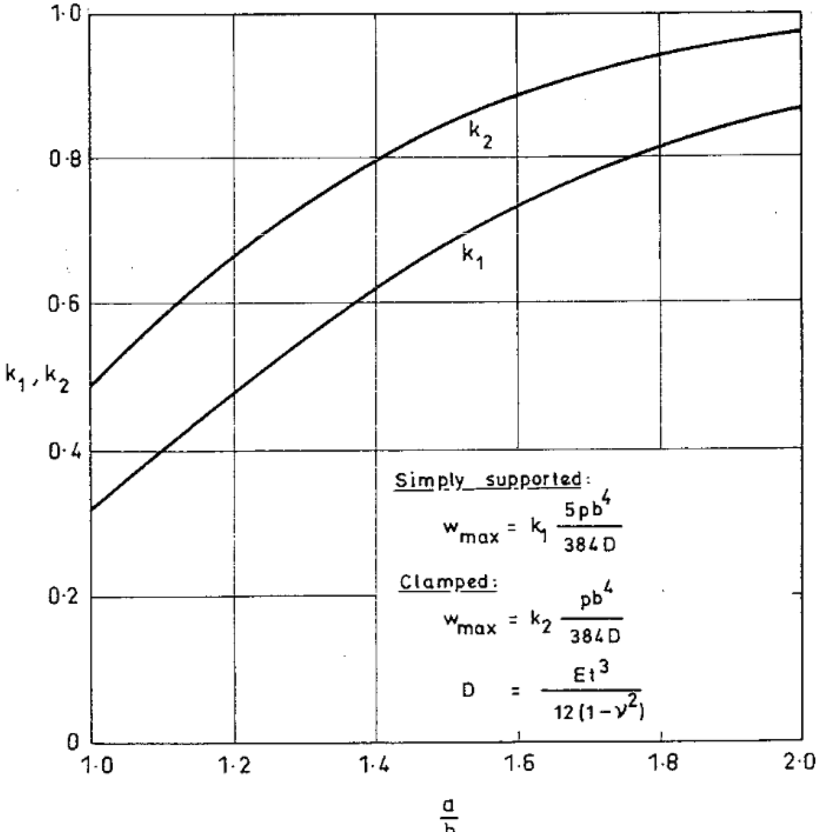
  Clamped < Simply supported

  about 4~5 times.

- Effect of aspect ratio

  when a=b, the deflection is minimum

Homework #3 Plot the curve using any kinds of software.



Maximum deflection of rectangular plates under uniform pressure

Simply Supported Plates

- To determine the bending stress in the plate, calculate the bending stress.

$$\frac{\partial^2 w}{\partial x^2} = -\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \frac{16 p_0}{\pi^6 Dmn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \frac{\pi^2 n^2}{b^2} \sin\frac{m\pi y}{a} \sin\frac{n\pi x}{b} = -\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \frac{16 p_0 n}{\pi^4 Dmb^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \sin\frac{m\pi y}{a} \sin\frac{n\pi x}{b}$$

$$\frac{\partial^2 w}{\partial y^2} = -\sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \frac{16 p_0 m}{\pi^4 Dna^2\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2} \sin\frac{m\pi y}{a} \sin\frac{n\pi x}{b}$$

$$m_x = -D\left(\frac{\partial^2 \omega}{\partial x^2} + \nu\frac{\partial^2 \omega}{\partial y^2}\right) = \sum_{m=1}^{m=\infty}\sum_{n=1}^{n=\infty} \frac{16 p_0}{\pi^4 mn\left(\frac{m^2}{a^2}+\frac{n^2}{b^2}\right)^2}\left(\frac{n^2}{b^2} + \boxed{\nu\frac{m^2}{a^2}}\right) \sin\frac{m\pi y}{a} \sin\frac{n\pi x}{b}$$

- The bending moment will always be greater across the shorter span.

  Homework #4 Prove this.

- The symbol b is used for shorter dimension, then $a/b > 1$
- m_x is the larger of the two moments and has its maximum value in the center of the plate.

Clamped Plates

- The usual methods are the energy(or Ritz) method and the method of Levy
- The energy method, while giving only approximate results
- The Levy type solution is achieved by the superposition of three loading systems applied to a simply supported plate :
  (1) uniformly distributed along the short edges : deflection w_1
  (2) moments distributed along the short edges : deflection w_2
  (3) moments distributed along the long edges : deflection w_3

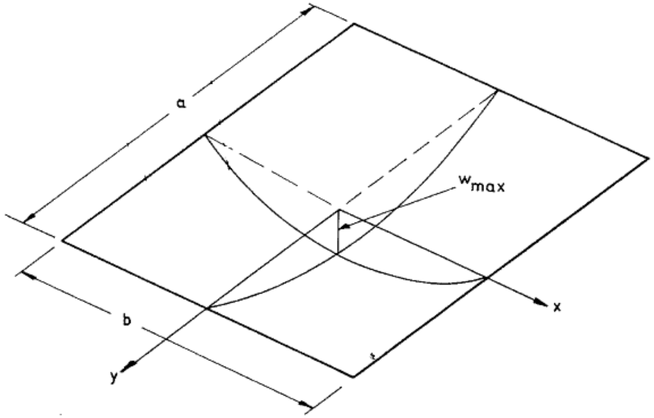
short edges :

$$\left(\frac{\partial \omega_1}{\partial y}\right)_{y=\pm a/2} + \left(\frac{\partial \omega_2}{\partial y} + \frac{\partial \omega_3}{\partial y}\right)_{y=\pm a/2} = 0$$

long edges :

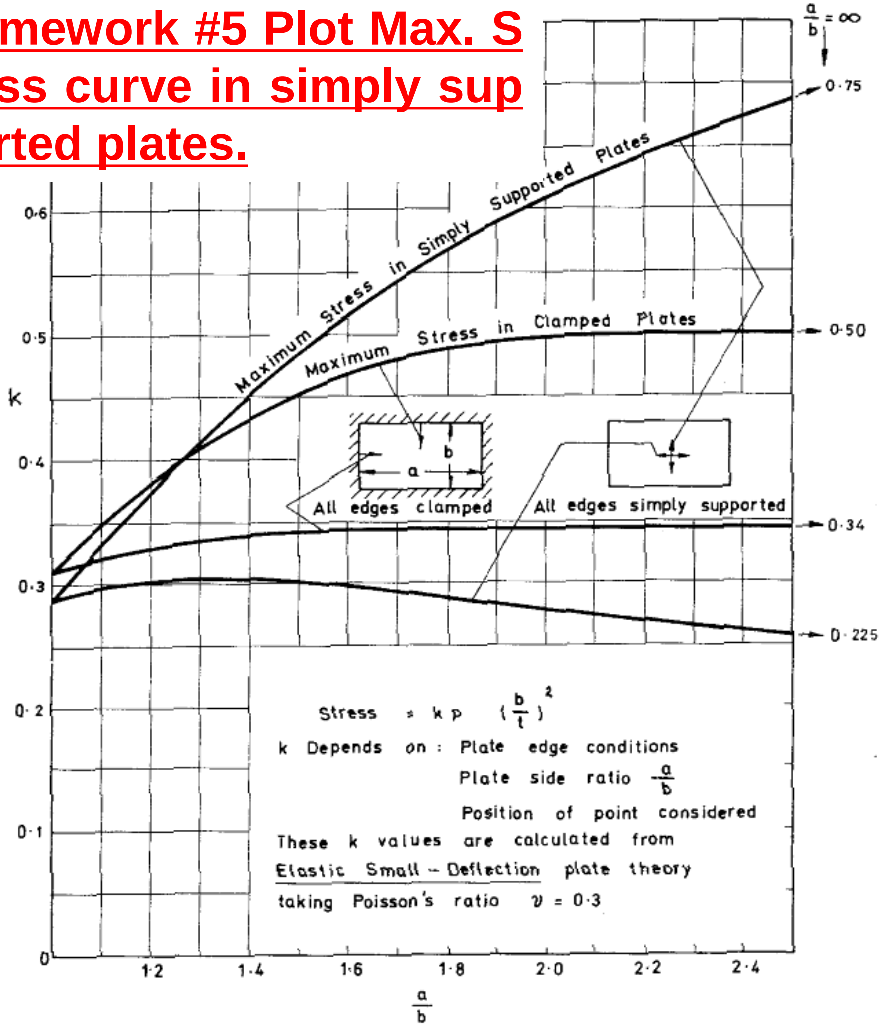
$$\left(\frac{\partial \omega_1}{\partial x}\right)_{x=\pm b/2} + \left(\frac{\partial \omega_2}{\partial x} + \frac{\partial \omega_3}{\partial x}\right)_{x=\pm b/2} = 0$$

(where x=0, y=0 is at the center of the plate)



Clamped Plates

Homework #5 Plot Max. Stress curve in simply supported plates.



Stresses in rectangular plates under uniform lateral pressure

$$\text{Stress} = kp\left(\frac{b}{t}\right)^2$$

❖ k depends on :

- plate edge conditions
- Plate side ratio a/b
- Position of point considered

The effect of aspect ratio is smaller for clamped plates, and beyond about a/b=2

such a plate behaves essentially as a clamped strip and the influence of aspect aspect ratio is negligible.

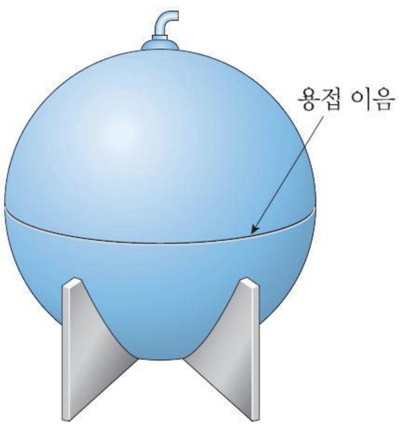
Large-Deflection Plate Theory

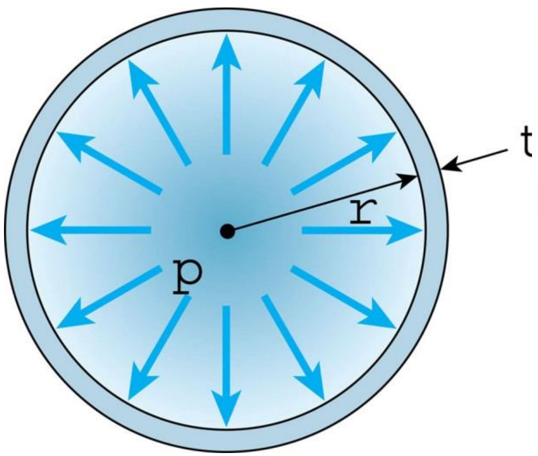
- Membrane stress arises when the deflection becomes large and/or when the edges are prevented from pulling in.
- Small-deflection theory fails to allow for membrane stresses.
- As the deflection increases, an increasing proportion of the load is carried by this membrane action.
- The lateral load is supported by both bending and membrane action.
- Amore comprehensive plate theory, usually referred to as "large-deflection" plate theory.

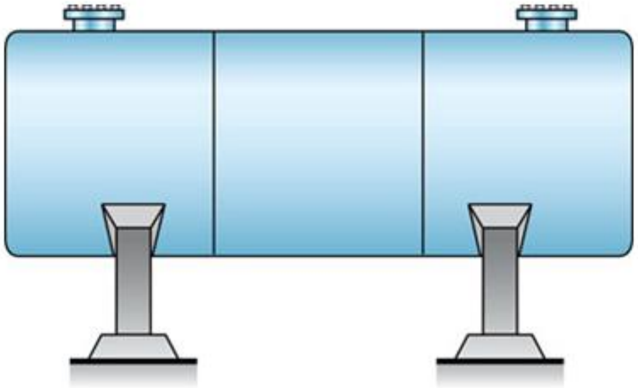
Membrane Tension(Edges Restrained Against Pull-in)

❖ What is membrane tension?

- **Membrane stress :** Stresses that act tangentially to the curved surface of a shell
- Pressure vessel : closed structures containing liquids or gases under pressure
- Gage pressure (Internal – external pressure) is resisted by membrane tension





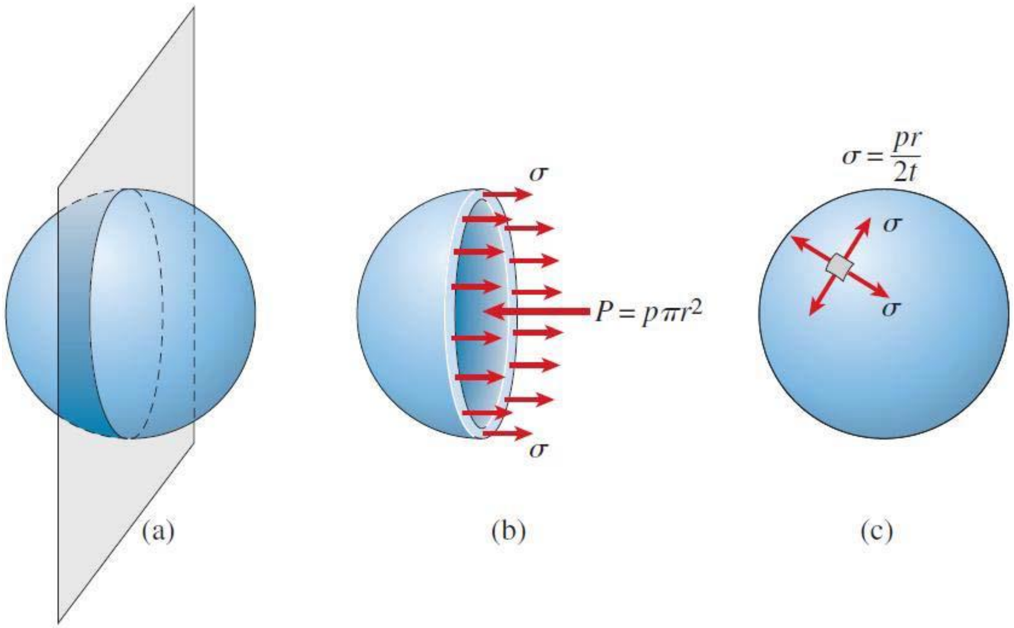


Spherical Pressure vessel

Cylindrical Pressure vessel

Membrane stress

❖ Spherical Pressure vessel



$$\sum F_{horiz} = 0$$

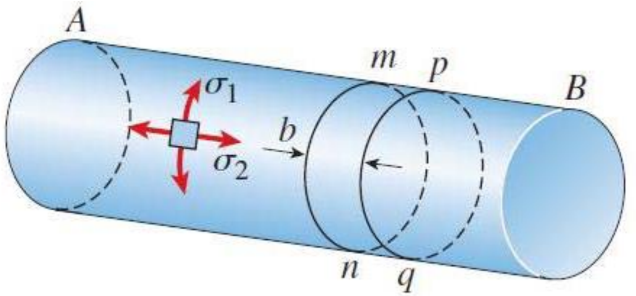
$$\sigma(2\pi r_m t) = p(\pi r^2)$$

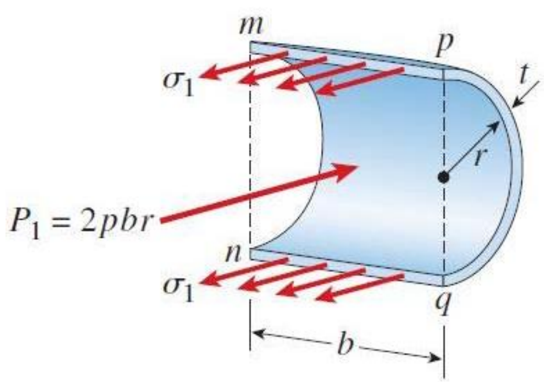
$$r_m = r + \frac{t}{2}$$

$$\therefore \sigma = \frac{pr^2}{2r_m t} \approx \frac{pr}{2t}$$

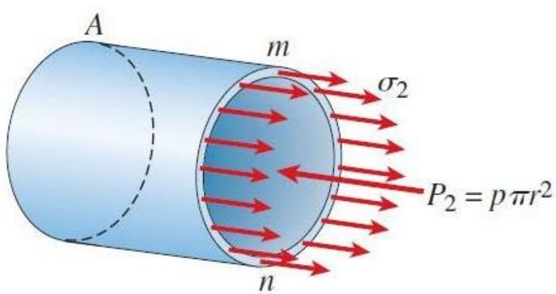
Membrane stress

❖ **Cylindrical Pressure vessel**





(b)



(c)

σ_1 : **circumferential stress**

hoop stress

$$\sigma_1(2bt) - 2pbr = 0$$

$$\therefore \sigma_1 = \frac{pr}{t}$$

σ_2 : **longitudinal stress**

$$\sigma_2(2\pi rt) - p\pi r^2 = 0$$

$$\therefore \sigma_2 = \frac{pr}{2t}$$

$$\sigma_1 = 2\sigma_2$$

Membrane stress

❖ Ship & Offshore Structure

Global Stress

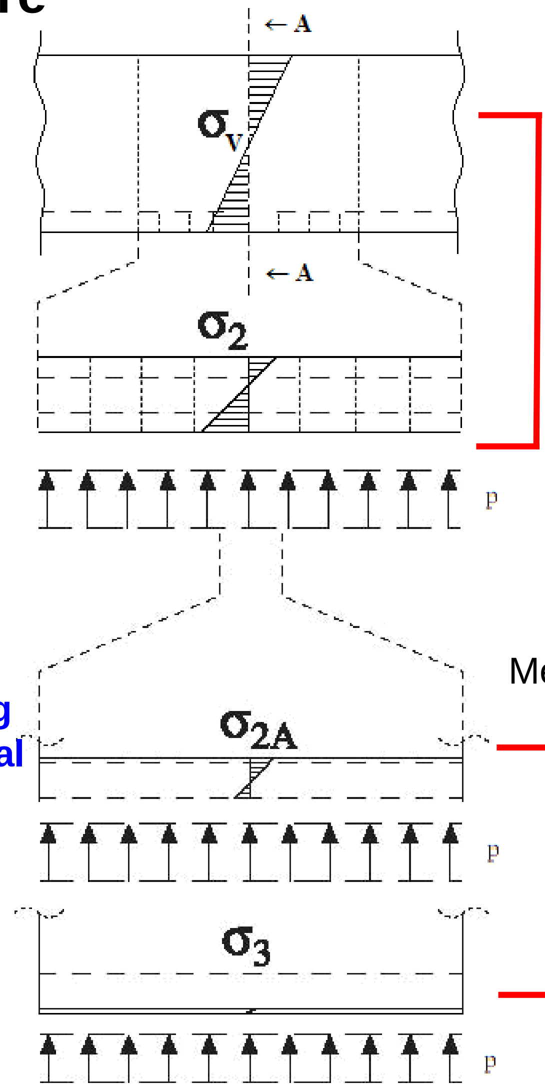
Hull Girder Bending

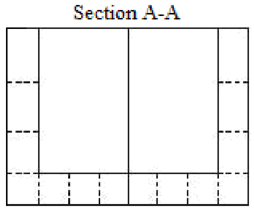
Local Stress

Double Bottom Bending

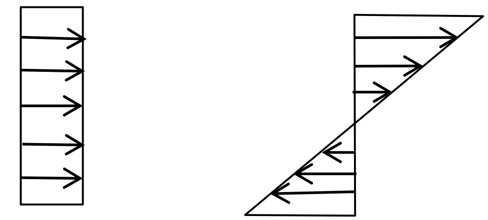
Local Bending of Longitudinal

Plate Bending





Membrane stress on plating



Membrane stress

Bending stress

Bending stress on plating

# of elements should be more than thee (3) between boundaries.

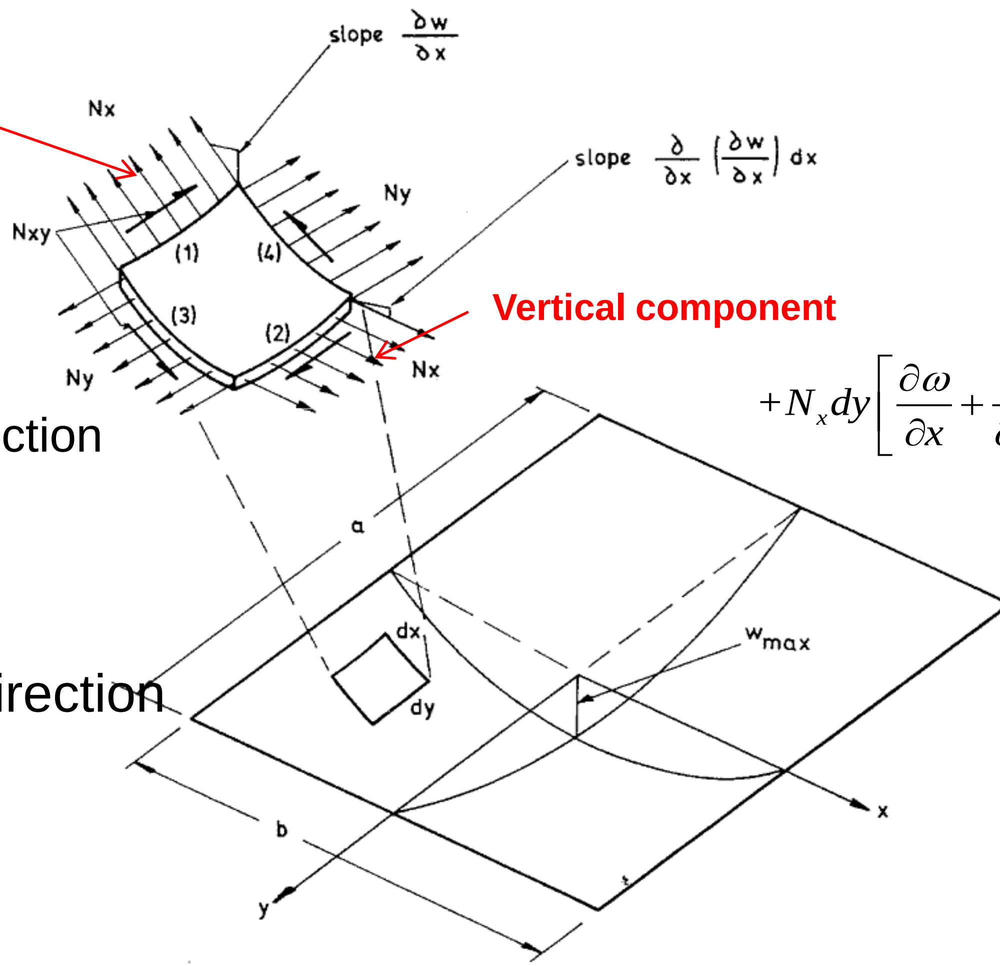
Large-Deflection Plate Theory by von Karman

- N_x, N_y : membrane tension per unit length
- N_{xy} : membrane shear force per unit length
- Vertical component of the tension force at side (1)

$$-N_x dy \sin(slope) \approx -N_x dy \tan(slope) = -N_x dy(\partial w / \partial x)$$

Vertical component

$$-N_x dy(\frac{\partial w}{\partial x})$$



Vertical component

$$+N_x dy\left[\frac{\partial \omega}{\partial x} + \frac{\partial}{\partial x}\left(\frac{\partial \omega}{\partial x}\right)dx\right]$$

- Net force in the x-direction

$$-N_x dy(\frac{\partial^2 w}{\partial x^2})$$

- Net force in the y-direction

$$-N_y dx(\frac{\partial^2 w}{\partial y^2})$$

Large-Deflection Plate Theory by von Karman

- The vertical component of the shear force along sides (3) and (4) is

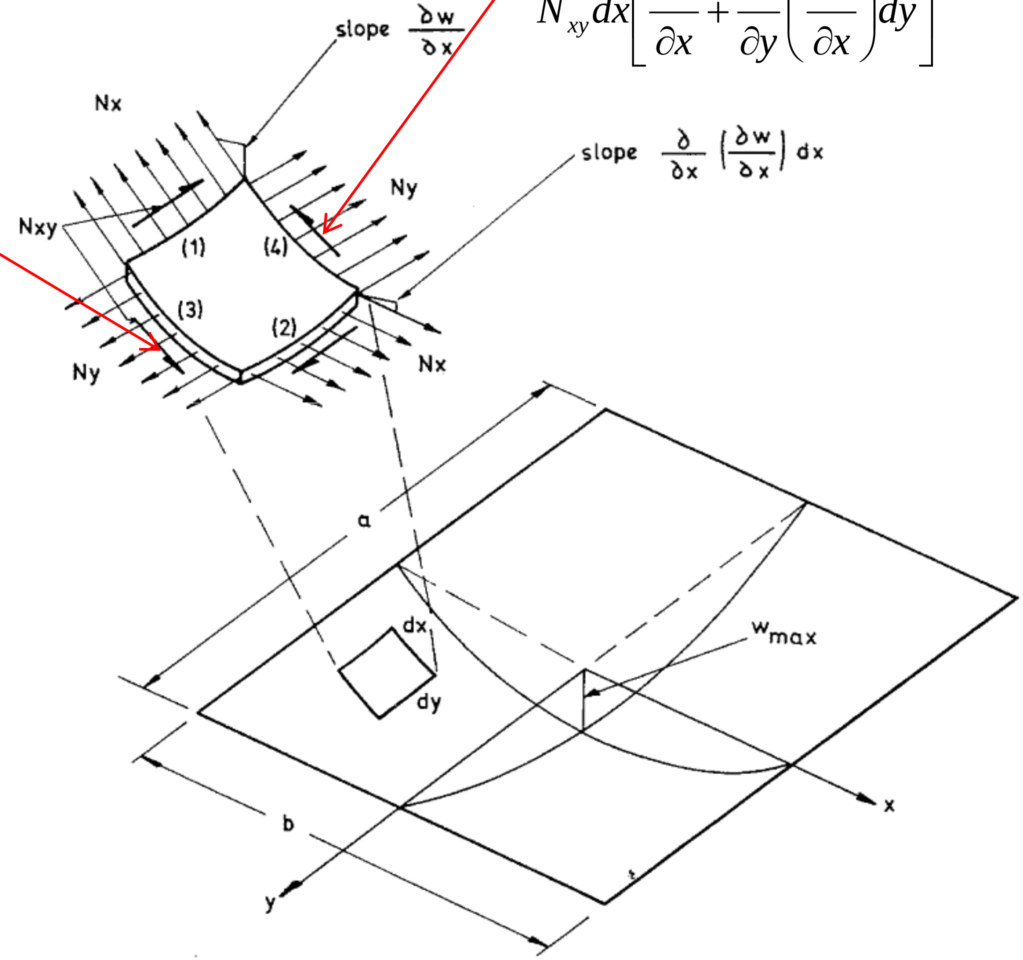
$$N_{xy}dx\left[\frac{\partial\omega}{\partial x}+\frac{\partial}{\partial y}\left(\frac{\partial\omega}{\partial x}\right)dy\right]-N_{xy}dx\left(\frac{\partial\omega}{\partial x}\right)$$

Vertical component

$$N_{xy}dx\left[\frac{\partial w}{\partial x}+\frac{\partial}{\partial y}\left(\frac{\partial w}{\partial x}\right)dy\right]$$

Vertical component

$$N_{xy}dx(\frac{\partial w}{\partial x})$$



Large-Deflection Plate Theory by von Karman

- Total vertical forces

$$\left(N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}\right) dxdy$$

- New terms are added

$$\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{1}{D}\left(p + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}\right) \Leftarrow \boxed{\frac{\partial^4 w}{\partial x^4} + 2\frac{\partial^4 w}{\partial x^2 \partial y^2} + \frac{\partial^4 w}{\partial y^4} = \frac{p}{D}}$$

$$\boxed{\nabla^4 w = \frac{1}{D}\left(p + N_x \frac{\partial^2 w}{\partial x^2} + 2N_{xy} \frac{\partial^2 w}{\partial x \partial y} + N_y \frac{\partial^2 w}{\partial y^2}\right)}$$

- N_x, N_y, N_{xy} are functions of x and y.
- Except for a few simple cases, precise mathematical solutions are very difficult.

Membrane Tension(Edges Restrained Against Pull-in)

- The relative magnitude of membrane effects depends on
  : the degree of lateral deflection or "curvature" of the plate surface
  the degree to which the edges are restrained form pulling in.

- If the edges are restrained and large deflection (w>1.5 t)
  the contribution of membrane tension > bending, vice versa.

- In ship plating : little restraint against edge pull-in
  for large deflections, supporting stiffeners or beams fails earlier.

- Nevertheless, some situations in which large deflections can be permitted,
  → the use of membrane tension can give substantial weight savings.

Membrane Tension(Edges Restrained Against Pull-in)

❖ The case of a unit-width strip

- Laterally loaded
- Edges are prevented from approaching
- the difference between the arc length of the deflected strip and the original straight length

$$\delta=\int_0^b\left(\sqrt{1+\left(\frac{d\omega}{dx}\right)^2}-1\right)dx \qquad \delta\cong\int_0^b\frac{1}{2}\left(\frac{d\omega}{dx}\right)^2dx$$

- Membrane force is unidirectional (N_y=N_{xy}=0) and constant over the length.
- Extension due to the tension

$$T=\frac{Et}{2b}\int_0^b\left(\frac{d\omega}{dx}\right)^2dx$$

Membrane Tension(Edges Restrained Against Pull-in)

- w_0 initial deflection, w_1 due to the load

$$\omega(x) = (\omega_0 + \omega_1)\sin\frac{\pi x}{b}$$

- The total change in length is

$$\delta = \int_0^b \frac{1}{2}(\omega_0 + \omega_1)^2 \frac{\pi^2}{b^2}\cos^2\frac{\pi x}{b}dx$$

$$= \frac{\pi^2}{4b}(\omega_0 + \omega_1)^2$$

- The change in length due to the initial deflection

$$\frac{\pi^2 w_0^2}{4b}$$

Membrane Tension(Edges Restrained Against Pull-in)

- The change in length due to the loading is the difference between these two:

$$\delta_1 = \frac{\pi^2}{4b}(2\omega_0\omega_1 + \omega_1^{\ 2})$$

- Strain energy due to bending

$$= \int_0^b \frac{E'I}{2}\left(\frac{d^2w}{dx^2}\right)dx = \int_0^b \frac{E'I}{2}(w_1)^2\frac{\pi^4}{b^4}\sin^2\frac{\pi x}{b}dx = \frac{\pi^4 E'I(w_1)^2}{4b^3}$$

- Strain energy due to tension:

$$\frac{1}{2}T\delta_p = \frac{1}{2}\frac{AE}{b}\delta_1^2$$

- The work done

$$W = \int_0^b \frac{1}{2}pwdx = \int_0^b \frac{1}{2}pw_1\sin\left(\frac{\pi x}{b}\right)dx = \frac{pw_1 b}{\pi}$$

- The work done by the load is equal to the total strain energy, hence:

$$\frac{pw_1b}{\pi} = \frac{AE\delta_1^2}{2b} + \frac{\pi^4E'I(w_1^2)}{4b^3} = \frac{\pi^4}{32}\frac{AE}{b^3}(2w_0w_1 + w_1^2)^2 + \frac{\pi^4E'I(w_1^2)}{4b^3}$$

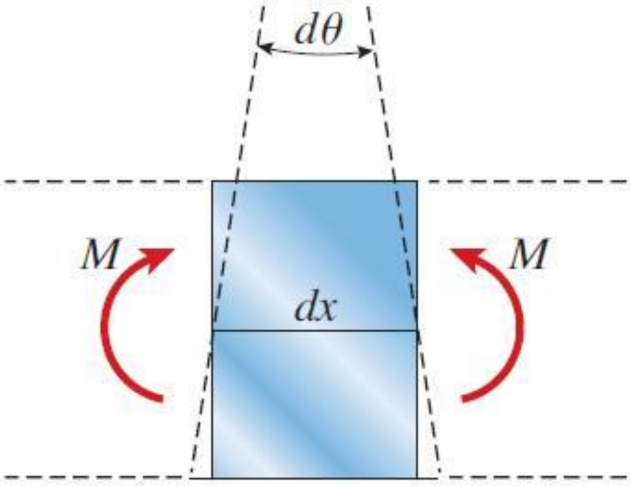
Strain Energy of Bending

- When bending moment M varies along its length

$$d\theta = \kappa dx = \frac{1}{\rho}dx = \frac{d^2v}{dx^2}dx$$

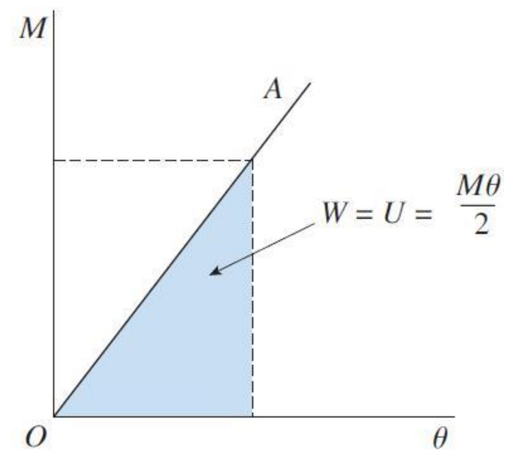
$$dU = \frac{Md\theta}{2} = \frac{M}{2}\frac{d^2v}{dx^2}dx = \frac{M}{2}\frac{M}{EI}dx = \frac{M^2dx}{2EI} \text{ or}$$

$$dU = \frac{EI}{2dx}\left(\frac{d^2v}{dx^2}dx\right)^2 = \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx$$



- if integrated

$$U = \int \frac{M^2dx}{2EI} \text{ or } U = \int \frac{EI}{2}\left(\frac{d^2v}{dx^2}\right)^2 dx$$



Membrane Tension(Edges Restrained Against Pull-in)

- The work done by the load is equal to the total strain energy, hence:

$$\frac{\pi^4 AE\omega_1^3}{32b^3}+\frac{\pi^4 AE\omega_0\omega_1^2}{8b^3}+\left(\frac{\pi^4 AE\omega_0^2}{8b^3}+\frac{\pi^4 E' I}{4b^3}\right)\omega_1-\frac{pb}{\pi}=0$$

- $A=t$, $E'=E/(1-v)$ and $I=t^3/12$ per unit width

$$\omega_1^3+4\omega_0\omega_1^2+\left(4\omega_0^2+\frac{2}{3}\frac{t^2}{1-\nu^2}\right)\omega_1-\frac{32pb^4}{\pi^5 Et}=0$$

Effect of Initial Deformation

- If there is no initial deflection:

$$\cancel{\omega_1^3} + \frac{2}{3}\frac{t^2}{1-\nu^2}\omega_1 = \frac{32pb^4}{\pi^5 Et}$$

- For the initial stages of loading the deflection w_1 will be small relative to the thickness, and hence the first term may be neglected:

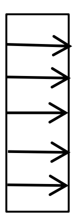
$$\frac{\omega_1}{t} = \frac{48(1-\nu^2)p}{\pi^5 E}\left(\frac{b}{t}\right)^4 = 0.143\frac{p}{E}\left(\frac{b}{t}\right)^4$$

- The result obtained from small-deflection theory, ignoring membrane action:

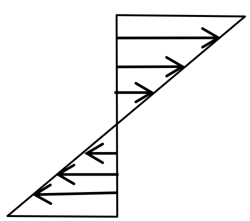
$$\frac{\omega_1}{t} = \frac{5(1-\nu^2)p}{32E}\left(\frac{b}{t}\right)^4 = 0.142\frac{p}{E}\left(\frac{b}{t}\right)^4$$

Effect of Initial Deformation

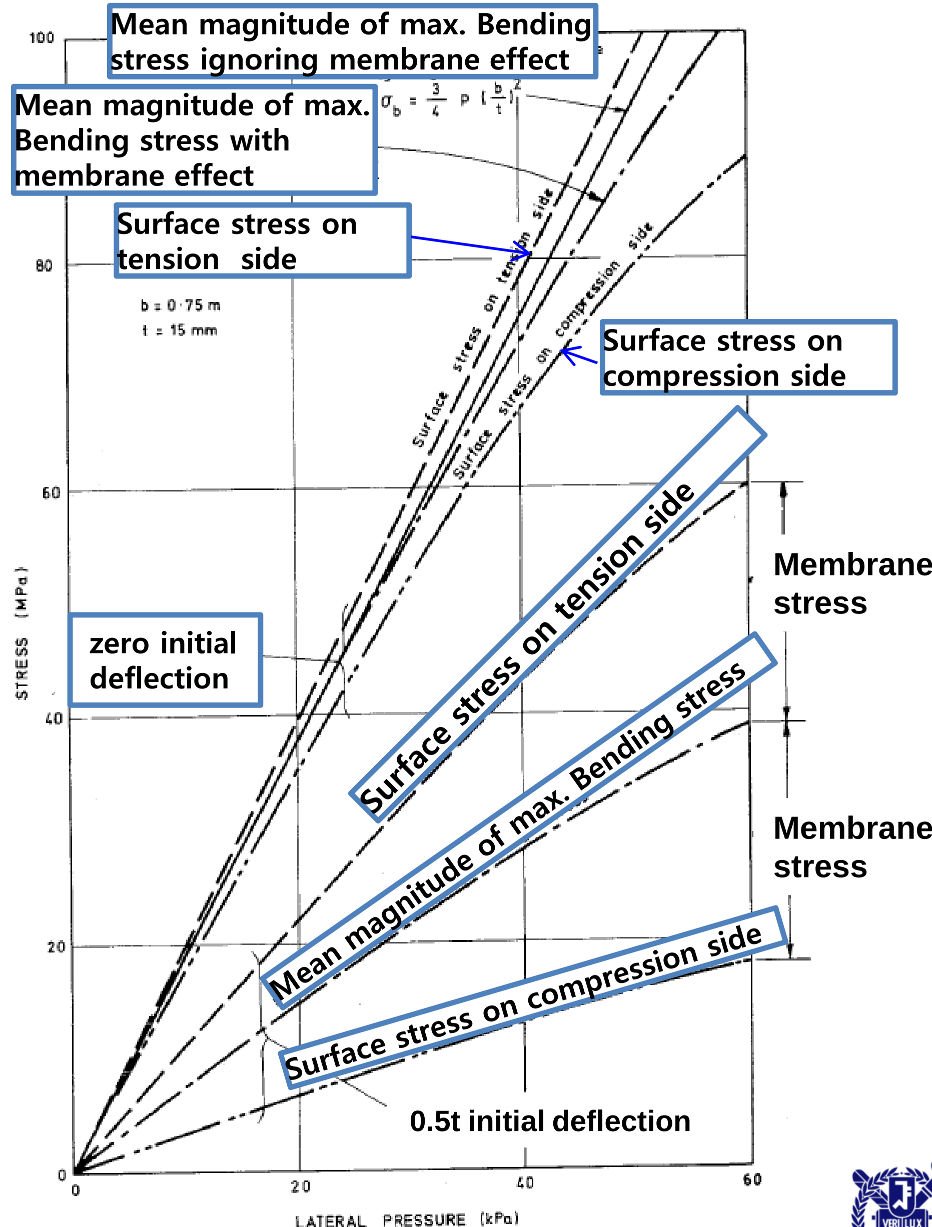
- Membrane action requires some deflection, either initial or due to load
- if there is no initial deflection → membrane action does not become significant until the deflection due to load approaches the plate thickness.
- Welding deformation & permanent set give a beneficial influence on elastic strength of plates.



Membrane stress



Bending stress

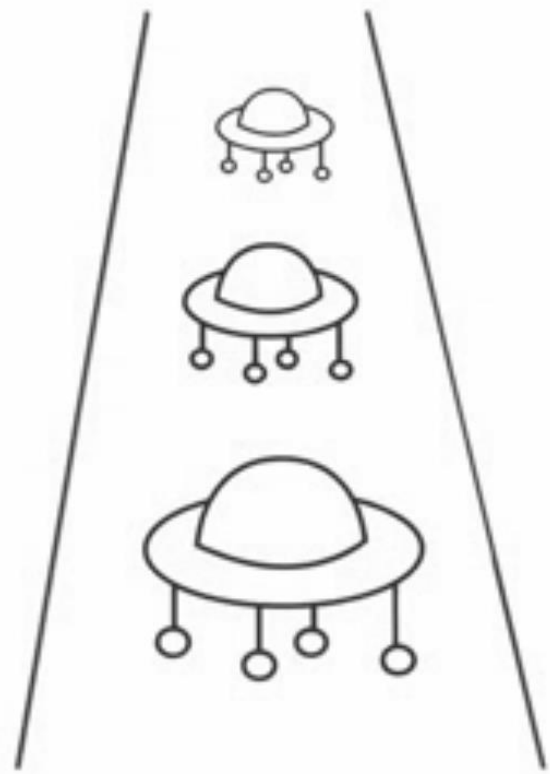


Effect of Initial Deformation

- The beneficial effect of initial deformation.

Type of Initial Deformation	Elastic Strength	Source of Increase in Elastic Strength
Flat plate	1.59	-
Initial deflection (stress-free) equal to plating thickness	2.58	Membrane action
Initially flat plate dished to a permanent set equal to plating thickness	4.50	Membrane action plus residual stresses

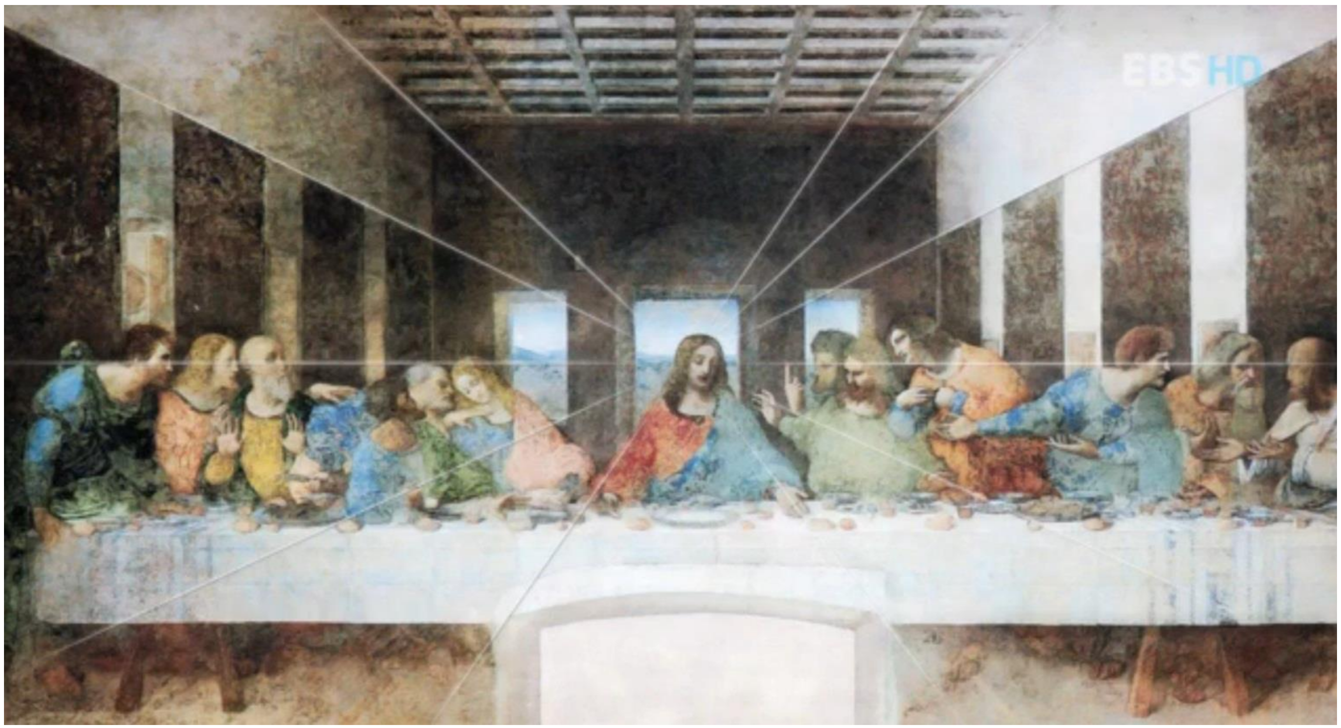
- Initial lateral deformation is beneficial only when the plate edges are at least partly restrained from pulling in, thus allowing the development of in-plane tension.



Which one is in front?

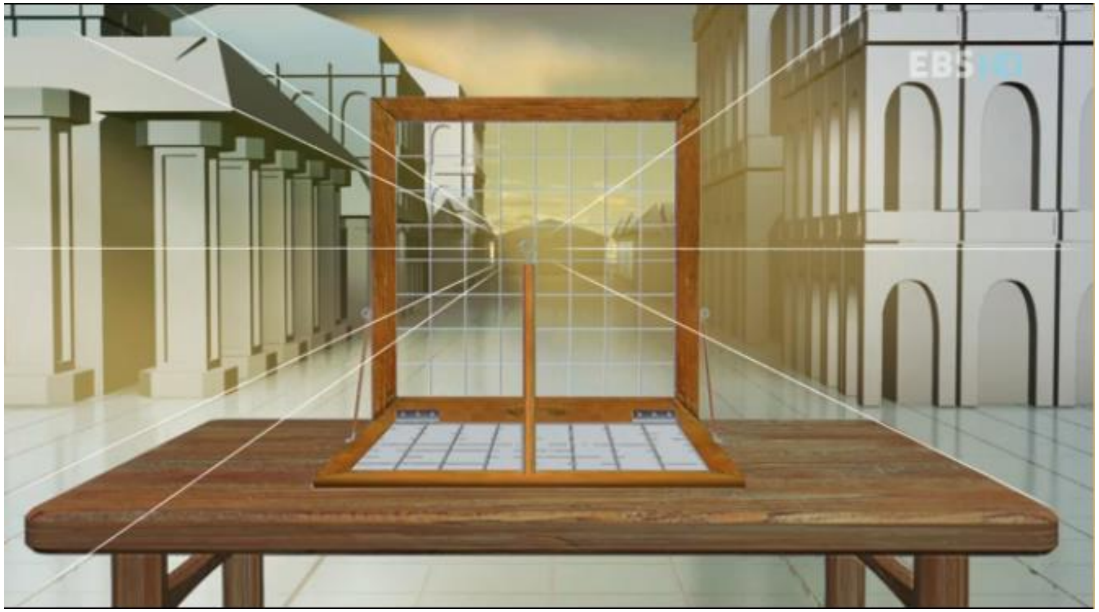
Di Vinci's ***Last Supper***

- Famous for not only artistic value but also fine command of perspective

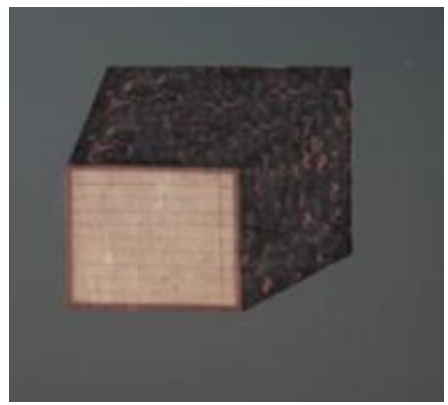


Westerner try to see and Asian try to be

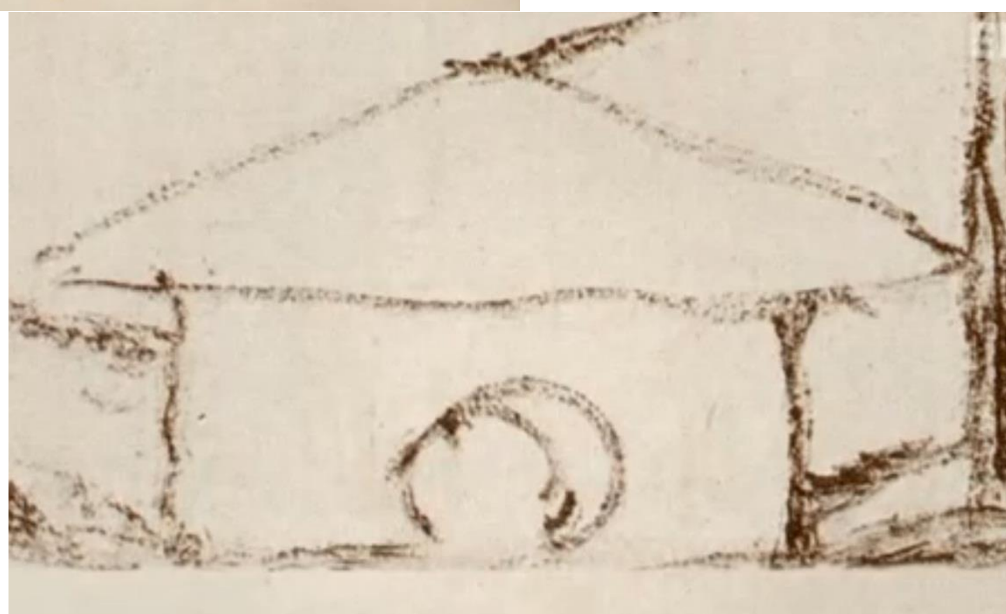
- The law of perspective is one of representative characteristics
- Object has the meaning of observation and all things
- Objective is not subjective.
- I see = I understand
- Seeing is believing, Westerner makes seeing very important.
- From the observer's perspective, farmost thing indicates that in front.



- Closer thing is drawn larger and the farther thing smaller. -> reverse perspective.
- Western artists draw picture seeing an object in front of him.
- Asian artists draw picture after seeing and feeling the object and returning.




歲寒圖
藕船是賞
阮堂
세한도(국보 180호)



- Universe is covered by Indra net. Each marble is mirroring others