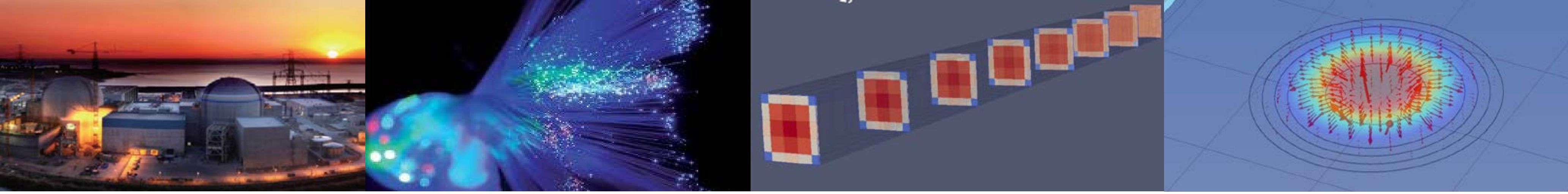

INTRODUCTION TO NUMERICAL ANALYSIS

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5. EIGENVALUES AND EIGENVECTORS

Eigenvalue

- The word eigenvalue is derived from the German word *eigenwert*, which means "proper or characteristic value."

$$[a][u] = \lambda[u]$$

- Generalized form

$$Lu = \lambda u$$

  - L : operator that can represent multiplication by a matrix, differentiation, integration, and so on.
  - Ex) : second differentiation with respect to x, y

$$\frac{d^2y}{dx^2} = k^2y$$

λ: eigenvalue associated by the operator L

u: eigenvector or eigenfunction corresponding to the eigenvalue λ and operator L

Importance of eigenvalues and eigenvectors in science and engineering

- Ex) vibration
  - Eigenvalues represent the natural frequencies of a system or component.
  - Eigenvectors represent the modes of these vibrations.
  - Important to identify these natural frequencies
    - Periodic external loads at or near these frequencies, resonance can cause the motion of the structure to be amplified.
    - Leading to failure of the component

- Mechanics of materials
  - The principal stresses are the eigenvalues of the stress matrix.
  - The principal directions are the directions of the associated eigenvectors.

Importance of eigenvalues and eigenvectors in science and engineering

- Quantum mechanics
  - In Heisenberg's formulation of quantum mechanics

$$L\Psi = c\Psi$$

  - Ψ : any quantity that can be measured or inferred experimentally, wave function
    - Such as position, velocity, or energy
  - c : eigenvalue
- Ex)

$$\frac{ih}{2\pi}\frac{\partial\Psi}{\partial t} = E\Psi \qquad -i\frac{h}{2\pi}\vec{\nabla}\Psi = \vec{p}\Psi$$

$$\frac{ih}{2\pi}\frac{\partial}{\partial t}(\ )$$: energy operator

$$-i\frac{h}{2\pi}\vec{\nabla}(\ )$$: momentum operator

Importance of eigenvalues and eigenvectors in science and engineering

- Link between eigenvalue problems involving differential equations and eigenvalue problems involving matrices ?

- Numerical solution of eigenvalue problems involving ODEs
  - Results in systems of simultaneous equations

$$[a][u] = \lambda[u]$$

- Eigenvalues of a matrix can also provide useful information
  - Jacobi and Gauss-Siedel iterative methods

$$x_i^{(k+1)} = b'_i - [a]x_i^{(k)}$$

  - It turns out that whether or not these iterative methods converge to a solution depends on the eigenvalues of the matrix $[a]$.
  - How quickly the iterations converge depends on the magnitudes of the eigenvalues of $[a]$.

5.2 The Characteristic Equation

Eigenvalues of a matrix

$$[a - \lambda I][u] = 0$$

- $[a - \lambda I]$: non-singular (if it has an inverse) $\Rightarrow [u] = 0$ (trivial solution)
- $[a - \lambda I]$: singular (if it does not have an inverse) $\Rightarrow [u] = 0$ (non-trivial solution is possible.)

Characteristic equation

$$\det[a - \lambda I] = 0$$

- Polynomial equation for λ
- For a small matrix $[a]$
  - The eigenvalues can be determined directly by calculating the determinant and solving for the roots of the characteristic equation.
- For a large matrix
  - Difficult to determine
  - Various numerical methods: power method and QR factorization method

Power method

- Iterative procedure for determining the **largest real eigenvalue** and the corresponding eigenvector of a matrix
- For a $(n \times n)$ matrix $[a]$, n distinct real eigenvalues and n associated eigenvectors

$$\lambda_1, \lambda_2, \ldots, \lambda_n \qquad [u]_1, [u]_2, \ldots, [u]_n$$

$$|\lambda_1| > |\lambda_2| > \ldots > |\lambda_n|$$

- Eigenvectors are linearly independent!
  - Any vector can be written as a linear combination of the basis vectors.

$$[x] = c_1[u]_1 + c_2[u]_2 + \ldots + c_n[u]_n \qquad c_i \neq 0$$

Power method

- Successive iteration

$$[a][x]_1 = c_1[a][u]_1 + c_2[a][u]_2 + \dots + c_n[a][u]_n = \lambda_1 c_1 [x]_2$$

$$[a][x]_2 = \lambda_1[u]_1 + \frac{c_2}{c_1}\frac{\lambda_2^2}{\lambda_1}[u]_2 + \dots + \frac{c_n}{c_1}\frac{\lambda_n^2}{\lambda_1}[u]_n = \lambda_1[x]_3$$

$$[a][x]_3 = \lambda_1[u]_1 + \frac{c_2}{c_1}\frac{\lambda_2^3}{\lambda_1^2}[u]_2 + \dots + \frac{c_n}{c_1}\frac{\lambda_n^3}{\lambda_1^2}[u]_n = \lambda_1[x]_4$$

$$[x]_2 = [u]_1 + \frac{c_2}{c_1}\frac{\lambda_2}{\lambda_1}[u]_2 + \dots + \frac{c_n}{c_1}\frac{\lambda_n}{\lambda_1}[u]_n$$

$$[x]_3 = [u]_1 + \frac{c_2}{c_1}\frac{\lambda_2^2}{\lambda_1^2}[u]_2 + \dots + \frac{c_n}{c_1}\frac{\lambda_n^2}{\lambda_1^2}[u]_n$$

$$[x]_{k+1} = [u]_1 + \frac{c_2}{c_1}\frac{\lambda_2^k}{\lambda_1^k}[u]_2 + \dots + \frac{c_n}{c_1}\frac{\lambda_n^k}{\lambda_1^k}[u]_n$$

5.3 Basic Power Method

Power method

- Successive iteration

$$[x]_{k+1} = [u]_1 + \frac{c_2}{c_1}\frac{\lambda_2^k}{\lambda_1^k}[u]_2 + \ldots + \frac{c_n}{c_1}\frac{\lambda_n^k}{\lambda_1^k}[u]_n$$

  - When k is sufficiently large,

$$[x]_{k+1} = [u]_1 + \frac{c_2}{c_1}\frac{\lambda_2^k}{\lambda_1^k}[u]_2 + \ldots + \frac{c_n}{c_1}\frac{\lambda_n^k}{\lambda_1^k}[u]_n$$

$$[x]_{k+1} = [u]_1$$

$$[a][x]_{k+1} \rightarrow \lambda_1[u]_1$$

Power method

- Algorithm of the power method
  - Start with a column eigenvector $[x]_i$ of length n. (the vector can be any non zero vector)
  - Multiply the vector $[x]_i$ by the matrix $[a] \Rightarrow [x]_{i+1}$

$$[x]_{i+1} = [a][x]_i$$

  - Normalizing $[x]_{i+1}$
  - Assign the normalized vector and go back to the first step

- Convergence criteria

$$\|[x]_{i+1} - [x]_i\|_\infty \leq Tolerance$$

Power method

- Example

Example 5-2: Using the power method to determine the largest eigenvalue of a matrix.

Determine the largest eigenvalue of the following matrix:

$$\begin{bmatrix} 4 & 2 & -2 \\ -2 & 8 & 1 \\ 2 & 4 & -4 \end{bmatrix} \quad (5.21)$$

Use the power method and start with the vector $x = [1, 1, 1]^T$.

$$[x]_2 = [a][x]_1 = \begin{bmatrix} 4 & 2 & -2 \\ -2 & 8 & 1 \\ 2 & 4 & -4 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 2 \end{bmatrix} = 7\begin{bmatrix} 0.5714 \\ 1 \\ 0.2857 \end{bmatrix}$$

$$[x]_3 = [a][x]_2 = \begin{bmatrix} 4 & 2 & -2 \\ -2 & 8 & 1 \\ 2 & 4 & -4 \end{bmatrix}\begin{bmatrix} 0.5714 \\ 1 \\ 0.2857 \end{bmatrix} = \begin{bmatrix} 3.7143 \\ 7.1429 \\ 4 \end{bmatrix} = 7.1429\begin{bmatrix} 0.52 \\ 1 \\ 0.56 \end{bmatrix}$$

$$[x]_4 = [a][x]_3 = \begin{bmatrix} 4 & 2 & -2 \\ -2 & 8 & 1 \\ 2 & 4 & -4 \end{bmatrix}\begin{bmatrix} 0.52 \\ 1 \\ 0.56 \end{bmatrix} = \begin{bmatrix} 2.96 \\ 7.52 \\ 2.8 \end{bmatrix} = 7.52\begin{bmatrix} 0.3936 \\ 1 \\ 0.3723 \end{bmatrix}$$

$$[x]_5 = [a][x]_4 = \begin{bmatrix} 4 & 2 & -2 \\ -2 & 8 & 1 \\ 2 & 4 & -4 \end{bmatrix}\begin{bmatrix} 0.3936 \\ 1 \\ 0.3723 \end{bmatrix} = \begin{bmatrix} 2.8298 \\ 7.5851 \\ 3.2979 \end{bmatrix} = 7.5851\begin{bmatrix} 0.3731 \\ 1 \\ 0.4348 \end{bmatrix}$$

$$[x]_6 = [a][x]_5 = \begin{bmatrix} 4 & 2 & -2 \\ -2 & 8 & 1 \\ 2 & 4 & -4 \end{bmatrix}\begin{bmatrix} 0.3731 \\ 1 \\ 0.4348 \end{bmatrix} = \begin{bmatrix} 2.6227 \\ 7.6886 \\ 3.0070 \end{bmatrix} = 7.6886\begin{bmatrix} 0.3411 \\ 1 \\ 0.3911 \end{bmatrix}$$

$$[x]_9 = [a][x]_8 = \begin{bmatrix} 4 & 2 & -2 \\ -2 & 8 & 1 \\ 2 & 4 & -4 \end{bmatrix}\begin{bmatrix} 0.3272 \\ 1 \\ 0.3946 \end{bmatrix} = \begin{bmatrix} 2.5197 \\ 7.7401 \\ 3.0760 \end{bmatrix} = 7.7401\begin{bmatrix} 0.3255 \\ 1 \\ 0.3974 \end{bmatrix}$$

Power method

- Convergence of the power method
  - Converges very slowly unless the starting vector $[x]$ is close to the eigenvector $[u]$
  - A problem can arise when c_1 is zero.

$$[x]_{k+1} = [u]_1 + \frac{c_2}{c_1}\frac{\lambda_2^k}{\lambda_1^k}[u]_2 + \dots + \frac{c_n}{c_1}\frac{\lambda_n^k}{\lambda_1^k}[u]_n$$

- When can the power method be used?
  - Only the largest eigenvalue is desired.
  - The largest eigenvalue cannot be a repeated root of the characteristic equation.
    - Other eigenvalues with the same magnitude
  - The largest eigenvalue must be real.
    - Two eigenvalues with the same magnitude.

❖ Inverse Power Method

- To determine the **smallest eigenvalue**
- Power method: for the **largest eigenvalue**
- Apply the power method to the inverse of the given matrix $[a]$
  - This works because the eigenvalues of the inverse matrix $[a]^{-1}$
    - Reciprocals of the eigenvalues of $[a]$

$$[a][x] = \lambda[x] \Rightarrow$$

$$[a]^{-1}[a] = [I] \Rightarrow$$

  - **$1/\lambda$: eigenvalue of the inverse matrix $[a]^{-1}$**

- Application of the power method $\Rightarrow$ the largest value of $1/\lambda$

  $\Rightarrow$ **the smallest value of λ**

Procedure

- Starting vector $[x]_i$
- Multiplied by $[a]^{-1}$
- Normalization

$$[x]_{i+1} = [a]^{-1}[x]_i$$

- Inverse matrix $[a]^{-1}$ has to be calculated before iterations !
  - Calculating the inverse of a matrix is computationally inefficient and not desirable.

$$[x]_{i+1} = [a]^{-1}[x]_i \quad \Longrightarrow \quad [a][x]_{i+1} = [x]_i$$

  - By solving systems of linear equations, $[x]_{i+1}$ can be obtained.
  - This can best be done by using the LU decomposition method.

- With the power method and inverse power method, the largest and the smallest eigenvalues of a matrix can be found.
- In some instances, it is necessary to find all the eigenvalues.
  - **Shifted power method, QR factorization method**

Shifted Power Method

- Once the largest or the smallest eigenvalue is known, the shifted power method can be used for finding the other eigenvalues.

$$[a][x] = \lambda[x]$$

  - λ_1 : the largest or smallest eigenvalue obtained by using the power method or inverse power method

- Shifted matrix

$$[a][x] = \lambda[x] \quad \Rightarrow \quad (\lambda - \lambda_1)[x] = \alpha[x]$$

  - α : eigenvalues of the shifted matrix $\alpha = (\lambda - \lambda_1)$
  - $\lambda = \lambda_1, \lambda_2, \lambda_3, \dots, \lambda_n$

$$\alpha = 0, \lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \dots, \lambda_n - \lambda_1$$

Shifted Power Method

- Shifted matrix

$[a - \lambda_1 I]$ $\alpha = 0, \lambda_2 - \lambda_1, \lambda_3 - \lambda_1, \lambda_4 - \lambda_1, \ldots, \lambda_n - \lambda_1$

  - Apply the basic power method !
  - The largest eigenvalue of the shifted matrix, α_k can be determined.
  - The eigenvalue λ_k can be determined. $\alpha_k = \lambda_k - \lambda_1$
  - Repeat this process together with shifted inverse power method!

- Comment
  - Tedious and inefficient process!
  - QR factorization

5.6 QR Factorization and Iteration Method

QR factorization and iteration method

- Popular means for finding all the eigenvalues of a matrix
- Based on the facts
  - **Similar matrices** have the same eigenvalues and associated eigenvectors.
  - The eigenvalues of an upper triangular matrix are the elements along the diagonal.
- Strategy
  - Transform the matrix into a **similar matrix** that is **upper triangular**.
  - Iterative process is required.

- The QR factorization method finds all the eigenvalues of a matrix, but cannot find the corresponding eigenvectors.

- For real eigenvalues
  - QR factorization method eventually factors the given matrix into an orthogonal matrix and an upper triangular matrix
- For complex eigenvalues (not covered in this course)

5.6 QR Factorization and Iteration Method

Similar matrices

- Two matrices $[a]$ and $[b]$ are similar if

  : similarity transformation

  - Similar matrices have the same eigenvalues and associated eigenvectors.

Orthogonal matrix

- Whose inverse is the same as its transpose

$$\Rightarrow \quad [Q]^T[Q] = [Q]^{-1}[Q] = [I]$$

QR factorization and iteration procedure

- Start with the matrix $[a]_1$

  - $[Q]_1$: orthogonal matrix, $[R]_1$: upper triangular matrix

5.6 QR Factorization and Iteration Method

QR factorization and iteration procedure

- The first iteration

$$[a]_1 = [Q]_1[R]_1 \qquad [R]_1 = [Q_1]^{-1}[a]_1$$

$$[a] = [c]^{-1}[b][c]$$

  - $[R]_1$ is multiplied by $[Q]_1$ from the right

$$[R]_1[Q]_1 = [Q_1]^{-1}[a]_1[Q]_1 = [Q_1]^T[a]_1[Q]_1$$

  - $[a]_1$ and $[a]_2$ are similar; have the same eigenvalues.

- The second iteration

$$=[Q_2]^T[a]_2[Q]_2$$

  - $[a]_2$ and $[a]_3$ are similar; have the same eigenvalues.

- Iterations continue until an upper triangular matrix is resulted in.

The eigenvalues of an upper triangular matrix are the elements along the diagonal.

$$\begin{bmatrix} \lambda_1 & X & X & X \\ 0 & \lambda_2 & X & X \\ 0 & 0 & \lambda_3 & X \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

5.6 QR Factorization and Iteration Method

QR factorization

$$[a] = [Q][R]$$

- Householder matrix $[H]$ for factorization

$$[H] = [I] - \frac{2}{[v]^T[v]}[v][v]^T$$

  - $[v]$: n-element column vector $\quad [v] = [c] + \|c\|_2[e] \quad \|c\|_2 = \sqrt{c_1^2 + c_2^2 + c_3^2 + \ldots + c_n^2}$
  - $[v]^T$: row vector
  - $[v]^T[v]$: scalar
  - $[v][v]^T$: $(n \times n)$ matrix

  - Special properties of $[H]$
    - Symmetric
    - Orthogonal
    - $[H][a][H]$: similar to $[a]$

$$[H]^{-1} = [H]^T = [H]$$

$$[a] = [c]^{-1}[b][c]$$

$$[H]^{-1}[H][a][H][H] = [a]$$

QR factorization

$$[a] = [Q][R]$$

- Step 1: identify the vector $[c]$ and $[e]$

$$[c] = \begin{bmatrix} a_{11} \\ a_{21} \\ \dots \\ a_{n1} \end{bmatrix} \qquad [e] = \begin{bmatrix} \pm 1 \\ 0 \\ 0 \\ \dots \\ 0 \end{bmatrix}$$

  - In $[e]$, the first element is
    - $+1$: if the first element of $[c]$ $(a_{11})$is positive.
    - -1 : if the first element of $[c]$ $(a_{11})$is negative.

  - $[H]^{(1)}$ can be constructed.
  - $$[Q]^{(1)} = [H]^{(1)}$$

$$[R]^{(1)} = [H]^{(1)}[a]$$

$$[H] = [I] - \frac{2}{[v]^T[v]}[v][v]^T$$

$$[v] = [c] + \|c\|_2[e]$$

$$\|c\|_2 = \sqrt{c_1^2 + c_2^2 + c_3^2 + \dots + c_n^2}$$

$$\begin{bmatrix} R_{11}^{(1)} & R_{12}^{(1)} & R_{13}^{(1)} & R_{14}^{(1)} & R_{15}^{(1)} \\ 0 & R_{22}^{(1)} & R_{23}^{(1)} & R_{24}^{(1)} & R_{25}^{(1)} \\ 0 & R_{32}^{(1)} & R_{33}^{(1)} & R_{34}^{(1)} & R_{35}^{(1)} \\ 0 & R_{42}^{(1)} & R_{43}^{(1)} & R_{44}^{(1)} & R_{45}^{(1)} \\ 0 & R_{52}^{(1)} & R_{53}^{(1)} & R_{54}^{(1)} & R_{55}^{(1)} \end{bmatrix}$$

QR factorization

$$[a] = [Q][R]$$

- Step 2: identify the vector $[c]$ and $[e]$

$$[H] = [I] - \frac{2}{[v]^T[v]}[v][v]^T$$

$$[v] = [c] + \|c\|_2[e]$$

$$\|c\|_2 = \sqrt{c_1^2 + c_2^2 + c_3^2 + \ldots + c_n^2}$$

$$\begin{bmatrix} R_{11}^{(2)} & R_{12}^{(2)} & R_{13}^{(2)} & R_{14}^{(2)} & R_{15}^{(2)} \\ 0 & R_{22}^{(2)} & R_{23}^{(2)} & R_{24}^{(2)} & R_{25}^{(2)} \\ 0 & 0 & R_{33}^{(2)} & R_{34}^{(2)} & R_{35}^{(2)} \\ 0 & 0 & R_{43}^{(2)} & R_{44}^{(2)} & R_{45}^{(2)} \\ 0 & 0 & R_{53}^{(2)} & R_{54}^{(2)} & R_{55}^{(2)} \end{bmatrix}$$

$$[c] = \begin{bmatrix} 0 \\ R_{22}^{(1)} \\ R_{32}^{(1)} \\ \cdots \\ R_{n2}^{(1)} \end{bmatrix} \qquad [e] = \begin{bmatrix} 0 \\ \pm 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

- In $[e]$, the second element is
  - $+1$: if the first element of $[c]$ ($R_{22}^{(1)}$)is positive.
  - -1 : if the first element of $[c]$ ($R_{22}^{(1)}$)is negative.

$$[a] = [H]^{(1)}[R]^{(1)} = [Q]^{(1)}[R]^{(1)}$$

$$[R]^{(1)} = [H]^{(2)}[R]^{(2)}$$

$$[a] = [Q]^{(1)}[H]^{(2)}[R]^{(2)} = [Q]^{(2)}[R]^{(2)}$$

- $[H]^{(2)}$ can be constructed.

$$[Q]^{(2)} = [Q]^{(1)}[H]^{(2)} \qquad [R]^{(2)} = [H]^{(2)}[R]^{(1)}$$

QR factorization

$$[a] = [Q][R]$$

- Step 3: identify the vector $[c]$ and $[e]$

$$[c] = \begin{bmatrix} 0 \\ 0 \\ R_{33}^{(2)} \\ R_{34}^{(2)} \\ \cdots \\ R_{n3}^{(2)} \end{bmatrix} \qquad [e] = \begin{bmatrix} 0 \\ 0 \\ \pm 1 \\ 0 \\ \cdots \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} R_{11}^{(3)} & R_{12}^{(3)} & R_{13}^{(3)} & R_{14}^{(3)} & R_{15}^{(3)} \\ 0 & R_{22}^{(3)} & R_{23}^{(3)} & R_{24}^{(3)} & R_{25}^{(3)} \\ 0 & 0 & R_{33}^{(3)} & R_{34}^{(3)} & R_{35}^{(3)} \\ 0 & 0 & 0 & R_{44}^{(3)} & R_{45}^{(3)} \\ 0 & 0 & 0 & R_{54}^{(3)} & R_{55}^{(3)} \end{bmatrix}$$

  - In $[e]$, the third element is
    - $+1$: if the first element of $[c]$ ($R_{33}^{(1)}$)is positive.
    - -1 : if the first element of $[c]$ ($R_{33}^{(1)}$)is negative.

  - $[H]^{(3)}$ can be constructed.

$$[Q]^{(3)} = [Q]^{(2)}[H]^{(3)} \qquad [R]^{(3)} = [H]^{(3)}[R]^{(2)}$$

QR factorization

$$[a] = [Q][R]$$

- Step 4: last step (n-1)

$$[Q]^{(i)} = [Q]^{(i-1)}[H]^{(i)}$$

$$[R]^{(i)} = [H]^{(i)}[R]^{(i-1)}$$

⇒

Orthogonal matrix

$$[a] = [Q]^{(n-1)}[R]^{(n-1)}$$

Upper triangular matrix

  - Eigenvalue ?
    - $[a]_n = [R]_{(n-1)}\,[Q]_{(n-1)}$
    - $[a]_n = [Q]_{(n)}\,[R]_{(n)}$
    - $[a]_{n+1} = [R]_{(n)}\,[Q]_{(n)}$

Step for the factorization!

QR factorization

- Example

Example 5-3: QR factorization of a matrix.

Factor the following matrix $[a]$ into an orthogonal matrix $[Q]$ and an upper triangular matrix $[R]$:

$$[a] = \begin{bmatrix} 6 & -7 & 2 \\ 4 & -5 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad (5.56)$$

Step 1: The vector $[c]$ is defined as the first column of the matrix $[a]$:

$$[c] = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix}$$

The vector $[e]$ is defined as the following three-element column vector:

$$[e] = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$$

QR factorization

- Example

Using Eq. (5.40), the Euclidean norm, $\|c\|_2$, of $[c]$ is:

$$\|c\|_2 = \sqrt{c_1^2 + c_2^2 + c_3^2} = \sqrt{6^2 + 4^2 + 1^2} = 7.2801$$

Using Eq. (5.39), the vector $[v]$ is:

$$[v] = [c] + \|c\|_2[e] = \begin{bmatrix} 6 \\ 4 \\ 1 \end{bmatrix} + 7.2801 \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 13.2801 \\ 4 \\ 1 \end{bmatrix}$$

Next, the products $[v]^T[v]$ and $[v][v]^T$ are calculated:

$$[v]^T[v] = \begin{bmatrix} 13.2801 & 4 & 1 \end{bmatrix} \begin{bmatrix} 13.2801 \\ 4 \\ 1 \end{bmatrix} = 193.3611$$

$$[v][v]^T = \begin{bmatrix} 13.2801 \\ 4 \\ 1 \end{bmatrix} \begin{bmatrix} 13.2801 & 4 & 1 \end{bmatrix} = \begin{bmatrix} 176.3611 & 53.1204 & 13.2801 \\ 53.1204 & 16 & 4 \\ 13.2801 & 4 & 1 \end{bmatrix}$$

QR factorization

- Example

The Householder matrix $[H]^{(1)}$ is then:

$$[H]^{(1)} = [I] - \frac{2}{[v]^T[v]}[v][v]^T = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{2}{193.3611}\begin{bmatrix} 176.3611 & 53.1204 & 13.2801 \\ 53.1204 & 16 & 4 \\ 13.2801 & 4 & 1 \end{bmatrix} = \begin{bmatrix} -0.8242 & -0.5494 & -0.1374 \\ -0.5494 & 0.8345 & -0.0414 \\ -0.1374 & -0.0414 & 0.9897 \end{bmatrix}$$

Once the Housholder matrix $[H]^{(1)}$ is constructed, $[a]$ can be factored into $[Q]^{(1)}[R]^{(1)}$, where:

$$[Q]^{(1)} = [H]^{(1)} = \begin{bmatrix} -0.8242 & -0.5494 & -0.1374 \\ -0.5494 & 0.8345 & -0.0414 \\ -0.1374 & -0.0414 & 0.9897 \end{bmatrix}$$

and

$$[R]^{(1)} = [H]^{(1)}[a] = \begin{bmatrix} -0.8242 & -0.5494 & -0.1374 \\ -0.5494 & 0.8345 & -0.0414 \\ -0.1374 & -0.0414 & 0.9897 \end{bmatrix}\begin{bmatrix} 6 & -7 & 2 \\ 4 & -5 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -7.2801 & 8.6537 & -2.8846 \\ 0 & -0.2851 & 0.5288 \\ 0 & 0.1787 & 0.6322 \end{bmatrix}$$

This completes the first step.

❖ QR factorization

- Step 2

$$[c] = \begin{bmatrix} 0 \\ R_{22}^{(1)} \\ R_{32}^{(1)} \end{bmatrix} = \begin{bmatrix} 0 \\ -0.2851 \\ 0.1787 \end{bmatrix}$$

$$[e] = \begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix}$$

Using Eq. (5.40), the Euclidean norm, $\|c\|_2$, of $[c]$ is:

$$\|c\|_2 = \sqrt{c_1^2 + c_2^2 + c_3^2} = \sqrt{0^2 + (-0.2851)^2 + 0.1787^2} = 0.3365$$

Using Eq. (5.39), the vector $[v]$ is:

$$[v] = [c] + \|c\|_2[e] = \begin{bmatrix} 0 \\ -0.2851 \\ 0.1787 \end{bmatrix} + 0.3365\begin{bmatrix} 0 \\ -1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ -0.6215 \\ 0.1787 \end{bmatrix}$$

Next, the products $[v]^T[v]$ and $[v][v]^T$ are calculated:

$$[v]^T[v] = \begin{bmatrix} 0 & -0.6215 & 0.1787 \end{bmatrix}\begin{bmatrix} 0 \\ -0.6215 \\ 0.1787 \end{bmatrix} = 0.4183$$

$$[v][v]^T = \begin{bmatrix} 0 \\ -0.6215 \\ 0.1787 \end{bmatrix}\begin{bmatrix} 0 & -0.6215 & 0.1787 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0.3864 & -0.1111 \\ 0 & 0.1111 & 0.0319 \end{bmatrix}$$

❖ QR factorization

- Step 2

The Householder matrix $[H]^{(2)}$ is then:

$$[H]^{(2)}=[I]-\frac{2}{[v]^T[v]}[v][v]^T=\begin{bmatrix}1&0&0\\0&1&0\\0&0&1\end{bmatrix}-\frac{2}{0.4183}\begin{bmatrix}0&0&0\\0&0.3864&-0.1111\\0&0.1111&0.0319\end{bmatrix}=\begin{bmatrix}1&0&0\\0&-0.8474&0.5311\\0&0.5311&0.8473\end{bmatrix}$$

Once the Housholder matrix $[H]^{(2)}$ is constructed, $[a]$ can be factored into $[Q]^{(2)}[R]^{(2)}$, where:

$$[Q]^{(2)}=[Q]^{(1)}[H]^{(2)}=\begin{bmatrix}-0.8242&-0.5494&-0.1374\\-0.5494&0.8345&-0.0414\\-0.1374&-0.0414&0.9897\end{bmatrix}\begin{bmatrix}1&0&0\\0&-0.8474&0.5311\\0&0.5311&0.8473\end{bmatrix}=\begin{bmatrix}-0.8242&0.3927&-0.4082\\-0.5494&-0.7291&0.4082\\-0.1374&0.5607&0.8166\end{bmatrix}$$

and

$$[R]^{(2)}=[H]^{(2)}[R]^{(1)}=\begin{bmatrix}1&0&0\\0&-0.8474&0.5311\\0&0.5311&0.8473\end{bmatrix}\begin{bmatrix}-7.2801&8.6537&-2.8846\\0&-0.2851&0.5288\\0&0.1787&0.6322\end{bmatrix}=\begin{bmatrix}-7.2801&8.6537&-2.8846\\0&0.3365&-0.1123\\0&0&0.8165\end{bmatrix}$$

QR factorization

- Example

Example 5-3: QR factorization of a matrix.

Factor the following matrix $[a]$ into an orthogonal matrix $[Q]$ and an upper triangular matrix $[R]$:

$$[a] = \begin{bmatrix} 6 & -7 & 2 \\ 4 & -5 & 2 \\ 1 & -1 & 1 \end{bmatrix} \quad (5.56)$$

$$[a] = [Q]^{(2)}[R]^{(2)} \quad \text{or} \quad \begin{bmatrix} 6 & -7 & 2 \\ 4 & -5 & 2 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} -0.8242 & 0.3927 & -0.4082 \\ -0.5494 & -0.7291 & 0.4082 \\ -0.1374 & 0.5607 & 0.8166 \end{bmatrix} \begin{bmatrix} -7.2801 & 8.6537 & -2.8846 \\ 0 & 0.3365 & -0.1123 \\ 0 & 0 & 0.8165 \end{bmatrix}$$

$$q = \begin{pmatrix} 0.824 & 0.393 & 0.408 \\ 0.549 & -0.729 & -0.408 \\ 0.137 & 0.561 & -0.816 \end{pmatrix}, \quad q = \begin{pmatrix} 0.992 & -0.113 & 0.047 \\ 0.102 & 0.976 & 0.192 \\ -0.068 & -0.186 & 0.98 \end{pmatrix}, \quad q = \begin{pmatrix} 0.999 & 0.04 & 0.026 \\ -0.042 & 0.997 & 0.072 \\ -0.023 & -0.073 & 0.997 \end{pmatrix}, \quad q = \begin{pmatrix} 1 & -0.028 & 0.01 \\ 0.028 & 0.999 & 0.032 \\ -0.011 & -0.032 & 0.999 \end{pmatrix},$$

5.6 QR Factorization and Iteration Method

Iteration

- Repeat the factorization until the last matrix in the sequence is upper triangular.

$$A_n = [R]_{(n-1)}\,[Q]_{(n-1)}$$

$$\begin{bmatrix} \lambda_1 & X & X & X \\ 0 & \lambda_2 & X & X \\ 0 & 0 & \lambda_3 & X \\ 0 & 0 & 0 & \lambda_4 \end{bmatrix}$$

Example

Example 5-4: Calculating eigenvalues using the QR factorization and iteration method.

The three-dimensional state of stress at a point inside a loaded structure is given by:

$$\sigma_{ij} = \begin{bmatrix} 45 & 30 & -25 \\ 30 & -24 & 68 \\ -25 & 68 & 80 \end{bmatrix} \text{ MPa}$$

Determine the principal stresses at this point by determining the eigenvalues of the stress matrix, using the QR factorization method.

Example

```
function [Q R] = QRFactorization(R)
% The function factors a matrix [A] into an orthogonal matrix [Q]
% and an upper-triangular matrix [R].
% Input variables:
% A  The (square) matrix to be factored.
% Output variables:
% Q  Orthogonal matrix.
% R  Upper-triangular matrix.

nmatrix = size(R);
n = nmatrix(1);
I = eye(n);
Q = I;
for j = 1:n-1
    c = R(:,j);
    c(1:j-1) = 0;
    e(1:n,1)=0;
    if c(j) > 0
        e(j) = 1;
    else
        e(j) = -1;
    end
    clength = sqrt(c'*c);
    v = c + clength*e;
    H = I - 2/(v'*v)*v*v';
    Q = Q*H;
    R = H*R;
end
```

```
clear all
A = [45 30 -25; 30 -24 68; -25 68 80]
for i = 1:100
    [q R] = QRFactorization(A);
    A = R*q;
end
A
e = diag(A)
```

QR factorization

- Householder matrix transformation

$$H = I - \frac{2uu^T}{u^T u}$$

- Characteristics

$$Hu = \left(I - 2uu^T\right)u = Iu - 2u\left(u^T u\right) = u - 2u = -u$$

  - Eigenvalue: -1

- Orthogonal vector v

$$u^T v = uv = 0$$

$$Hv = \left(I - 2uu^T\right)v = v - 2u\left(u^T v\right) = v$$

  - Eigenvector of householder matrix
  - Eigenvalue: 1

QR factorization

- Characteristics

$$H^T = \left(I - 2uu^T\right)^T = I^T - 2\left(uu^T\right)^T = I^T - 2\left(u^T\right)^T u^T = I - 2uu^T = H$$

  - Symmetric

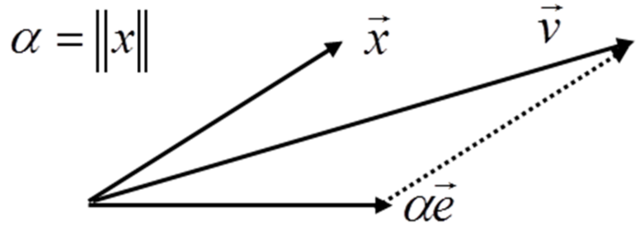
$$HH^T = \left(I - 2uu^T\right)\left(I - 2uu^T\right)^T = I - 4uu^T + 4uu^T uu^T = I$$

  - Orthogonal

QR factorization

- Characteristics

$$v = x + \alpha e \qquad \alpha = \|x\| \qquad \alpha^2 = x^T x$$



$$e^T x = x^T e, \;\; e^T e = 1 \qquad \alpha e = [\alpha, 0, \cdots, 0]^T$$

$$\|v\|^2 = (x + \alpha e)^T (x + \alpha e) = x^T x + \alpha e^T x + \alpha x^T e + \alpha^2 e^T e = 2\alpha^2 + 2\alpha e^T x = 2\alpha(\alpha + e^T x)$$

$$Hx = \left(I - \frac{2vv^T}{v^T v} \right) x = x - 2\frac{vv^T}{v^T v} x = x - 2\frac{vv^T}{\|v\|^2} x \qquad = x - \frac{1}{\alpha(\alpha + e^T x)} vv^T x = -\alpha e$$

$$\begin{aligned} vv^T x &= (x + \alpha e)(x + \alpha e)^T x = (xx^T + \alpha e x^T + \alpha x e^T + \alpha^2 e e^T) x \\ &= xx^T x + \alpha e x^T x + \alpha x e^T x + \alpha^2 e e^T x = \alpha^2 x + \alpha^3 e + \alpha(e^T x) x + \alpha^2 (e^T x) e \\ &= \alpha(\alpha + e^T x) x + \alpha^2 (\alpha + e^T x) e \quad = \alpha(\alpha + e^T x)(x + \alpha e) \end{aligned}$$

$$x = \begin{bmatrix} \chi_1 \\ \chi_2 \\ \vdots \\ \chi_n \end{bmatrix} \rightarrow Hx \rightarrow \begin{bmatrix} -\alpha \\ 0 \\ \vdots \\ 0 \end{bmatrix}$$

QR factorization

- Characteristics

$$Hx = \left(I - \frac{2uu^T}{u^T u} \right) x = (-\alpha, 0, \ldots, 0)^T$$

$$x = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} \qquad \alpha^2 = x^T x = 17 \qquad \alpha = 4.123$$

$$u = x + \alpha e = \begin{pmatrix} 0 \\ 4 \\ 1 \end{pmatrix} + \begin{pmatrix} 4.123 \\ 0 \\ 0 \end{pmatrix} = \begin{pmatrix} 4.123 \\ 4 \\ 1 \end{pmatrix}$$

$$H = \begin{pmatrix} 0 & -0.97 & -0.243 \\ -0.97 & 0.059 & -0.235 \\ -0.243 & -0.235 & 0.941 \end{pmatrix} \qquad H \cdot x = \begin{pmatrix} -4.123 \\ 0 \\ 0 \end{pmatrix}$$

QR factorization with Householder matrix

- Step 1

$$H_1 A = H_1 \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} * & * & \cdots & * \\ 0 & * & \cdots & * \\ \vdots & \vdots & & \vdots \\ 0 & * & \cdots & * \end{pmatrix}$$

$$H = \left(I - \frac{2uu^T}{u^T u} \right) \qquad \alpha = x^T x \qquad x = \begin{pmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{n1} \end{pmatrix}$$

$$u = x + \alpha e$$

- Step 2

$$H_2 A_1 = H_2 \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{n2} & \cdots & a'_{nn} \end{pmatrix} = \begin{pmatrix} 1 & 0 & \cdots & 0 \\ 0 & & \cdots & \\ \vdots & & H_2 & \vdots \\ 0 & & \cdots & \end{pmatrix} \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & a'_{22} & \cdots & a'_{2n} \\ \vdots & \vdots & & \vdots \\ 0 & a'_{n2} & \cdots & a'_{nn} \end{pmatrix} = \begin{pmatrix} a'_{11} & a'_{12} & \cdots & a'_{1n} \\ 0 & * & \cdots & * \\ \vdots & 0 & & \vdots \\ 0 & 0 & \cdots & * \end{pmatrix}$$

5.7 MATLAB Built-in Functions

❖ Eigenvalues and eigenvectors

```
d = eig(A)
```

d is a vector with the eigenvalues of A.

A is the matrix whose eigenvalues are to be determined.

```
[V,D] = eig(A)
```

V is a matrix whose columns are the eigenvectors of A. D is a diagonal matrix whose diagonal elements are the eigenvalues.

A is the matrix whose eigenvalues and eigenvectors are to be determined.

QR factorization