재료의 기계적 거동
(Mechanical Behavior of Materials)

Lecture 7 – Elastic Behavior

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Elasticity

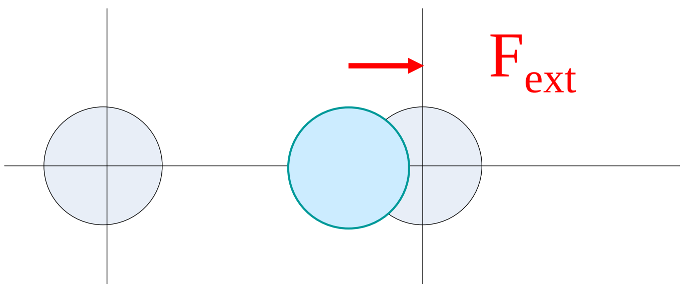
- Elasticity are extremely important because engineering design is done in the elastic region.

- Material fracture is related to elastic properties because the elastic energy release is one of driving force for fracture.

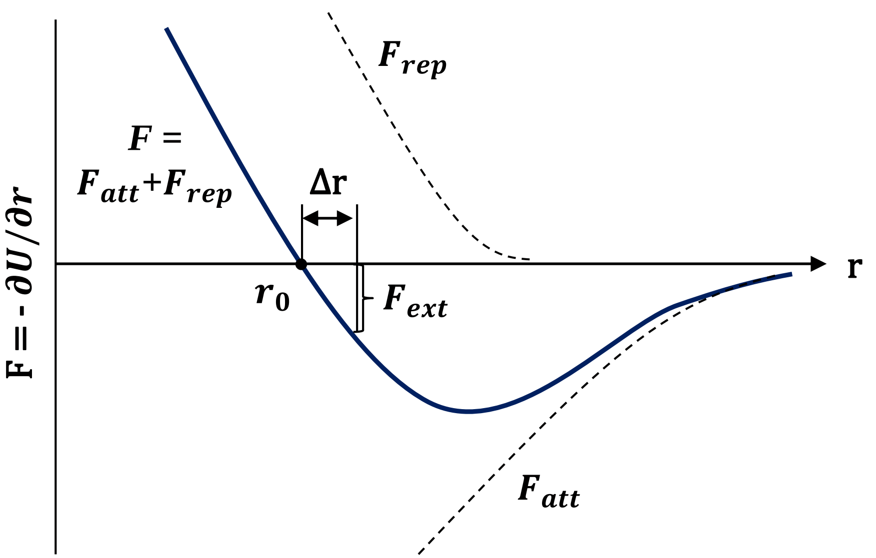
- Elastic behavior is inherently anisotropic for individual grains. However, most polycrystalline materials are elastically isotropic. Polycrystalline materials can be anisotropic if they are textured.

Basis for linear elasticity

- **Consider two atoms**

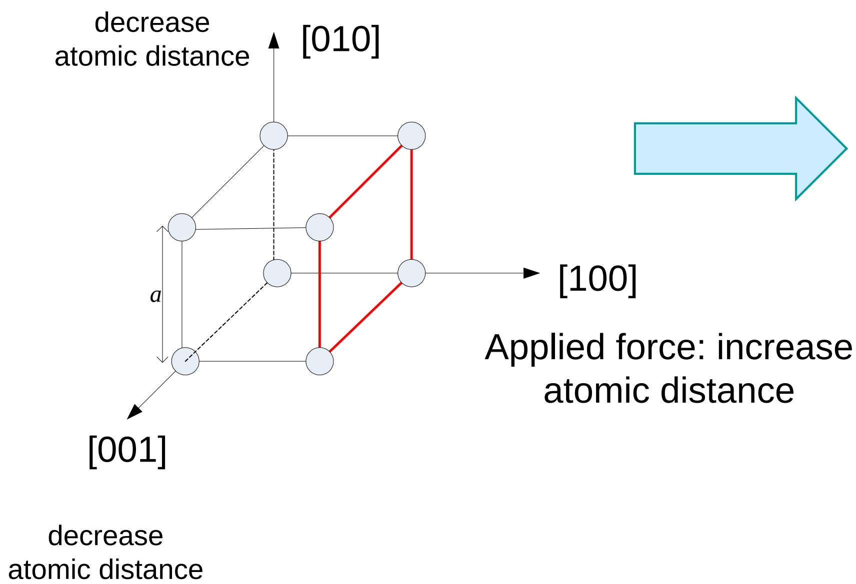


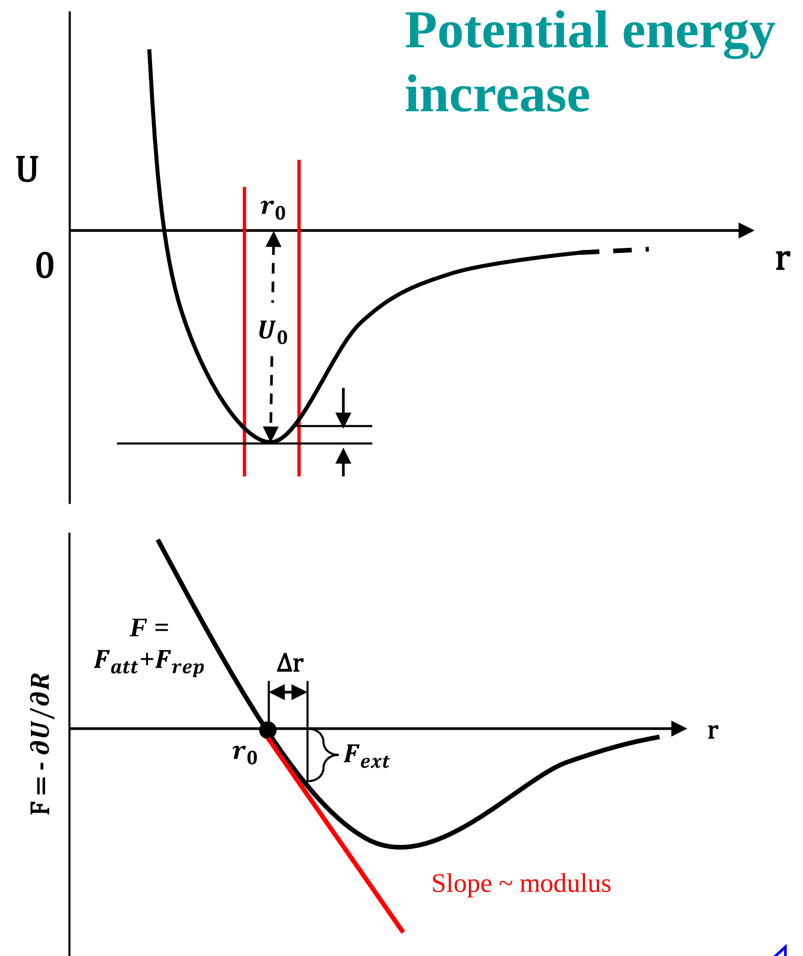
F_{ext} is a force that should be applied to separate the atom from r_o position ; external force



Basis for linear elasticity
(Young's modulus)

- **Consider cubic crystal material**

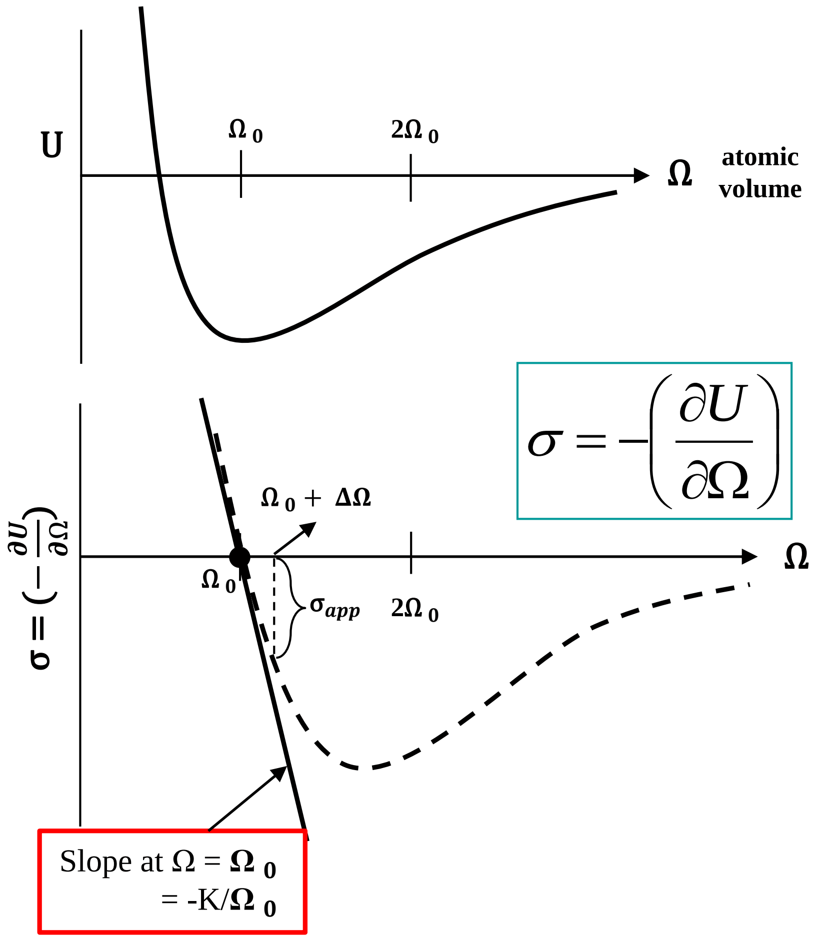




- **Slope : modulus**

Basis for linear elasticity
(Bulk modulus)

- **Relate elastic modulus to volume change**



$$F = -\left(\frac{\partial U}{\partial \Omega}\right) \times \text{area}$$

$$\sigma = -\left(\frac{\partial U}{\partial \Omega}\right)$$

- **Bulk modulus**

$$K = -\Omega_0 \left(\frac{\partial \sigma}{\partial \Omega}\right)_{\Omega_0} = \Omega_0 \left(\frac{\partial^2 U}{\partial \Omega^2}\right)_{\Omega_0}$$

Basis for linear elasticity
(Temperature effect)

- **Bulk (Young's) moduli relates to**
  - **Curvature of bonding energy**
- **Bonding energy correlates with the melting temperature**

$$U_0 \propto kT_m \qquad k = 1.38 \times 10^{-3}\, J/atom\, K$$

$$E \propto \frac{kT_m}{\Omega}$$

- **Temperature (heat) increases atomic vibration**
  - **Thermal energy added**
  - **Potential increased**
  - **Curvature of bonding energy decreases**

Basis for linear elasticity
(anisotropy)

- The **forces** between atoms, molecules, or ions in crystals **depends on the distances between them**. Thus, they also vary with crystallographic direction so it should not be surprising that **crystalline moduli are anisotropic**.

Hooke's Law in One Dimension

Robert Hooke [1635-1702] first drew attention to the linear relation between the impressed force and the resulting displacement, and in recognition of this we have *Hooke's Law*. By definition, this holds for all linear elastic solids, and for the example of the wire it simply states that the applied uniaxial stress σ is linearly related to the longitudinal strain ε. In one dimension, this relation can be written either as

$$\sigma = C\,\varepsilon \qquad \text{or} \qquad \varepsilon = S\,\sigma$$

where C is known as the *stiffness* and S as the *compliance*. In one dimension, the stiffness is also referred to as *Young's modulus or elastic modulus*.

Hooke's Law in Three Dimensions

The alternative forms of ***Hooke's law*** are best written in the repeated suffix notation

$$\sigma_{ij} = C_{ijkl}\varepsilon_{kl} \quad \text{and} \quad \varepsilon_{ij} = S_{ijkl}\sigma_{kl}$$

Each of these statements of ***Hooke's law*** stands for **9 equations each** having nine terms on the right-hand side, altogether making **81 components** of the stiffness or compliance.

$$\varepsilon_{ij} = S_{ijkl}\sigma_{kl} \;=\; \varepsilon_{ji} = S_{jikl}\sigma_{kl}$$

The number of independent components of compliance is reduced to 36.

An exactly parallel argument can be used to conclude that the stiffness, C_{ijkl} **, also has just 36 components.**

Hooke's Law in Three Dimensions

Changing Reference Axes.

The *compliance* or *stiffness* constants defined by these equations are themselves tensors and consequently they obey the transformation law for a *fourth-rank tensor*

$$S'_{ijkl} = a_{im} a_{jn} a_{ko} a_{lp} S_{mnop} \qquad \textit{and} \qquad C'_{ijkl} = a_{im} a_{jn} a_{ko} a_{lp} C_{mnop}$$

Contracted or Matrix Notation.

The compliance (or stiffness) is a fourth rank tensor and so its components have four subscripts. A more economical notation has been devised for the components of compliance (or stiffness) having only two subscripts; this is called *the contracted or matrix notation.* Each pair of subscripts of the tensor components is replaced by a single subscript according to the following table;

Tensor	11	22	33	23 or 32	13 or 31	12 or 21
Contracted	1	2	3	4	5	6

Hooke's Law in Three Dimensions

$$\begin{bmatrix} \sigma_1 & \sigma_6 & \sigma_5 \\ \sigma_6 & \sigma_2 & \sigma_4 \\ \sigma_5 & \sigma_4 & \sigma_3 \end{bmatrix} \qquad \begin{bmatrix} \varepsilon_{12} & \varepsilon_{12} & \varepsilon_{13} \\ \varepsilon_{21} & \varepsilon_{22} & \varepsilon_{23} \\ \varepsilon_{31} & \varepsilon_{32} & \varepsilon_{33} \end{bmatrix} = \begin{bmatrix} \varepsilon_1 & \varepsilon_6/2 & \varepsilon_5/2 \\ \varepsilon_6/2 & \varepsilon_2 & \varepsilon_4/2 \\ \varepsilon_5/2 & \varepsilon_4/2 & \varepsilon_3 \end{bmatrix}$$

$$\sigma_{ij} = C_{ijkl}\,\varepsilon_{kl} \quad \Rightarrow \quad \sigma_i = C_{ij}\,\varepsilon_j$$

$$\varepsilon_{ij} = S_{ijkl}\,\sigma_{kl} \quad \Rightarrow \quad \varepsilon_i = S_{ij}\,\sigma_j$$

pS_{ijkl} (in the tensor notation) is equal to S_{mn} (in the matrix notation) where m and n correspond to ij and kl, respectively

$$pS_{ijkl} = S_{mn}$$

where $p = 1$ **when both m and n are 1, 2 or 3** ($S_{1111} = S_{11}$, $S_{1122} = S_{12}$ ……)

$p = 2$ **when either m or n are 1, 2 or 3** ($2\,S_{1123} = S_{14}$, $2\,S_{1113} = S_{15}$ ……)

$p = 4$ **when both m and n are 4, 5 or 6** ($4\,S_{1223} = S_{64}$, $4\,S_{1212} = S_{66}$ ……)

Hooke's Law in Three Dimensions

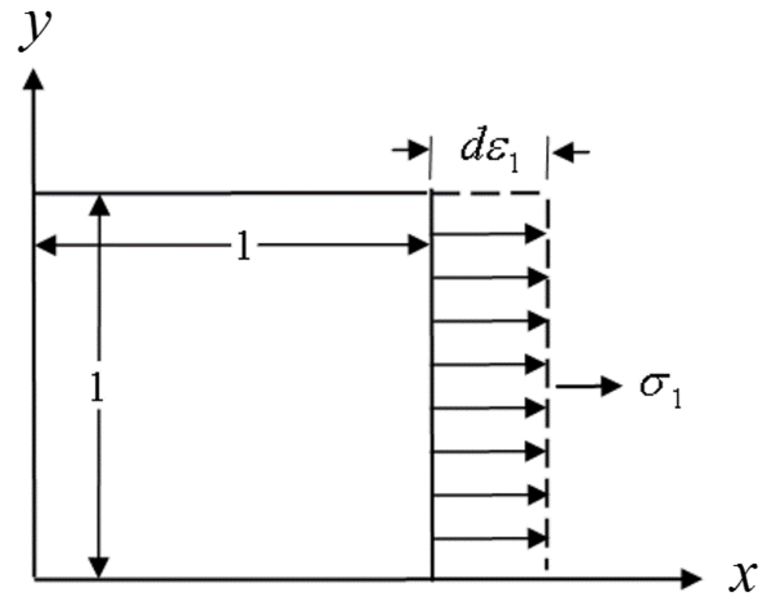
$$\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix} = \begin{bmatrix} C_{11} & C_{12} & C_{13} & C_{14} & C_{15} & C_{16}\\ C_{21} & C_{22} & C_{23} & C_{24} & C_{25} & C_{26}\\ C_{31} & C_{32} & C_{33} & C_{34} & C_{35} & C_{36}\\ C_{41} & C_{42} & C_{43} & C_{44} & C_{45} & C_{46}\\ C_{51} & C_{52} & C_{53} & C_{54} & C_{55} & C_{56}\\ C_{61} & C_{62} & C_{63} & C_{64} & C_{65} & C_{66}\end{bmatrix}\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix}$$

$$\begin{bmatrix}\varepsilon_1\\ \varepsilon_2\\ \varepsilon_3\\ \varepsilon_4\\ \varepsilon_5\\ \varepsilon_6\end{bmatrix} = \begin{bmatrix} S_{11} & S_{12} & S_{13} & S_{14} & S_{15} & S_{16}\\ S_{21} & S_{22} & S_{23} & S_{24} & S_{25} & S_{26}\\ S_{31} & S_{32} & S_{33} & S_{34} & S_{35} & S_{36}\\ S_{41} & S_{42} & S_{43} & S_{44} & S_{45} & S_{46}\\ S_{51} & S_{52} & S_{53} & S_{54} & S_{55} & S_{56}\\ S_{61} & S_{62} & S_{63} & S_{64} & S_{65} & S_{66}\end{bmatrix}\begin{bmatrix}\sigma_1\\ \sigma_2\\ \sigma_3\\ \sigma_4\\ \sigma_5\\ \sigma_6\end{bmatrix}$$

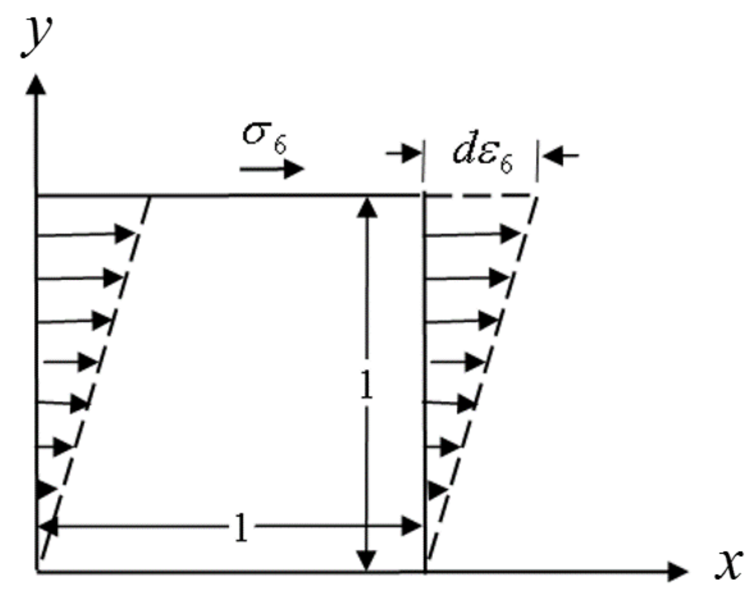
$[\sigma] = [C][\varepsilon]$ **and** $[\varepsilon] = [S][\sigma]$

$[C] = [S]^{-1}$ **or** $[S] = [C]^{-1}$

Elastic Strain Energy



Normal strain $d\varepsilon_1$ due to normal stress σ_1.



Shear strain $d\varepsilon_6$ due to shear stress σ_6.

The works done by these stresses is $\sigma_1\, d\varepsilon_1$ and $\sigma_6\, d\varepsilon_6$.

$$\boxed{dw = C_{ij}\varepsilon_j d\varepsilon_i}$$

Elastic Strain Energy

If the straining is carried out isothermally and reversibly, the energy expended is equal to the change in free energy ($d\phi$) of the body.

$$d\phi = C_{ij}\, \varepsilon_j\, d\varepsilon_i \qquad \text{or} \qquad \frac{\partial \phi}{\partial \varepsilon_i} = C_{ij}\, \varepsilon_j$$

$$\frac{\partial}{\partial \varepsilon_j}\left(\frac{\partial \phi}{\partial \varepsilon_i}\right) = C_{ij}$$

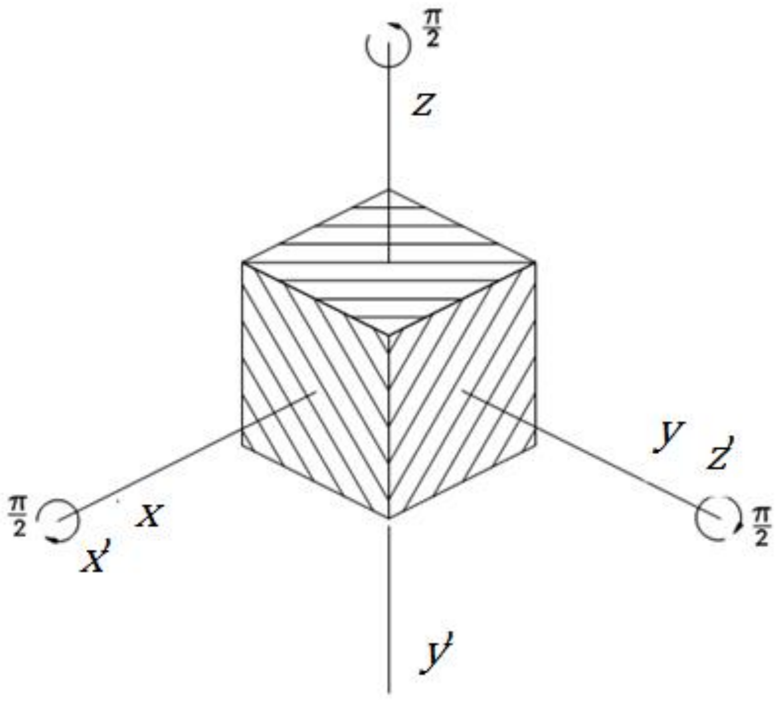
Since the free energy is a state property, this is a perfect differential and the order of differentiation is immaterial.

$$\boxed{C_{ij} = C_{ji}}$$

The matrix array of the components of stiffness is symmetrical. There can be no more than twenty-one independent components of stiffness.

$$\varphi = w = (1/2)\, C_{ij}\, \varepsilon_i\, \varepsilon_j = (1/2)\, \sigma_i\, \varepsilon_i = (1/2)\, S_{ij}\, \sigma_i\, \sigma_j$$

Effect of Materials Symmetry on Elastic Constants (*Cubic System*)



Three fourfold axes of rotation in cubic symmetry

If the crystal is rotated through $\pi/2$ about a fourfold axis,

	x	y	z
x'	1	0	0
y'	0	0	-1
z'	0	1	0

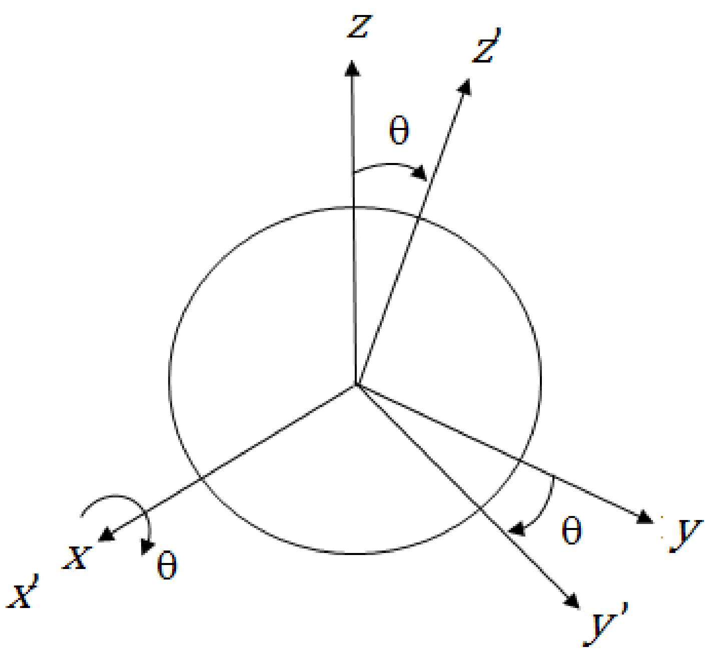
$$\begin{bmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{11} & C_{12} & 0 & 0 & 0 \\ C_{12} & C_{12} & C_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & C_{44} & 0 & 0 \\ 0 & 0 & 0 & 0 & C_{44} & 0 \\ 0 & 0 & 0 & 0 & 0 & C_{44} \end{bmatrix}$$

There are only three independent components of stiffness and three of compliance.

Effect of Materials Symmetry on Elastic Constants (*Cubic System*)

Material class	Material	C_{11} (10^{10} N/m^2)	C_{12} (10^{10} N/m^2)	C_{44} (10^{10} N/m^2)	Anisotropy ratio $(C_{11} - C_{12})/2C_{44}$
Metals	Ag	12.4	9.3	4.6	0.34
	Al	10.8	6.1	2.9	0.81
	Au	18.6	15.7	4.2	0.35
	Cu	16.8	12.1	7.5	0.31
	α-Fe	23.7	14.1	11.6	0.41
	Mo	46.0	17.6	11.0	1.29
	Na	0.73	0.63	0.42	0.12
	Ni	24.7	14.7	12.5	0.40
	Pb	5.0	4.2	1.5	0.27
	W	50.1	19.8	15.1	1.00
Covalent solids	Si	16.6	6.4	8.0	0.64
	Diamond	107.6	12.5	57.6	0.83
	TiC	51.2	11.0	17.7	1.14
Ionic solids	LiF	11.2	4.6	6.3	0.52
	MgO	29.1	9.0	15.5	0.65
	NaCl	4.9	1.3	1.3	1.38

Effect of Materials Symmetry on Elastic Constants (*Isotropic System*)



A rotation of θ about x-axis in isotropic material

Obviously, this includes cubic symmetry as a special case. Accordingly, let us transform the stiffness tensor of cubic material for a rotation of θ about x-axis,

	x	y	z
x'	1	0	0
y'	0	$\cos\theta$	$-\sin\theta$
z'	0	$\sin\theta$	$\cos\theta$

$$\begin{bmatrix} (\lambda+2\mu) & \lambda & \lambda & 0 & 0 & 0 \\ \lambda & (\lambda+2\mu) & \lambda & 0 & 0 & 0 \\ \lambda & \lambda & (\lambda+2\mu) & 0 & 0 & 0 \\ 0 & 0 & 0 & \mu & 0 & 0 \\ 0 & 0 & 0 & 0 & \mu & 0 \\ 0 & 0 & 0 & 0 & 0 & \mu \end{bmatrix}$$

We conclude that there are two independent components of stiffness.

Effect of Materials Symmetry on Elastic Constants (*Isotropic System*)

We can determine the compliances simply by taking the inverse of the matrix of stiffness components,

$$\begin{bmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{11} & S_{12} & 0 & 0 & 0 \\ S_{12} & S_{12} & S_{11} & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 & 0 \\ 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) & 0 \\ 0 & 0 & 0 & 0 & 0 & 2(S_{11}-S_{12}) \end{bmatrix}$$

$$S_{11} = \frac{\mu+\lambda}{\mu(3\lambda+2\mu)}$$

$$S_{12} = \frac{-\lambda}{2\mu(3\lambda+2\mu)}$$

Suppose that an elastically isotropic sample is acted on solely *uniaxial stress* along x-axis,

$\varepsilon_1 = S_{11}\,\sigma_1$ or $\sigma_1/\varepsilon_1 = 1/S_{11}$ ⇨ ***Young's modulus*, E=1/S_{11}**

$\varepsilon_2 = \varepsilon_3 = S_{12}\,\sigma_1$ $-\varepsilon_2/\varepsilon_1 = -\varepsilon_3/\varepsilon_1 = -S_{12}/S_{11}$ ⇨ ***Poisson's ratio*, ν=-S_{12}/S_{11}**

Effect of Materials Symmetry on Elastic Constants (*Isotropic System*)

Suppose now that the sole applied stress is *a shear stress σ_4*,

$$\sigma_4 = \mu\varepsilon_4 \qquad \varepsilon_4 = 2(S_{11} - S_{12})\sigma_4 \qquad \Rightarrow \qquad \textit{Shear modulus}, G = \frac{E}{2(1+\nu)}$$

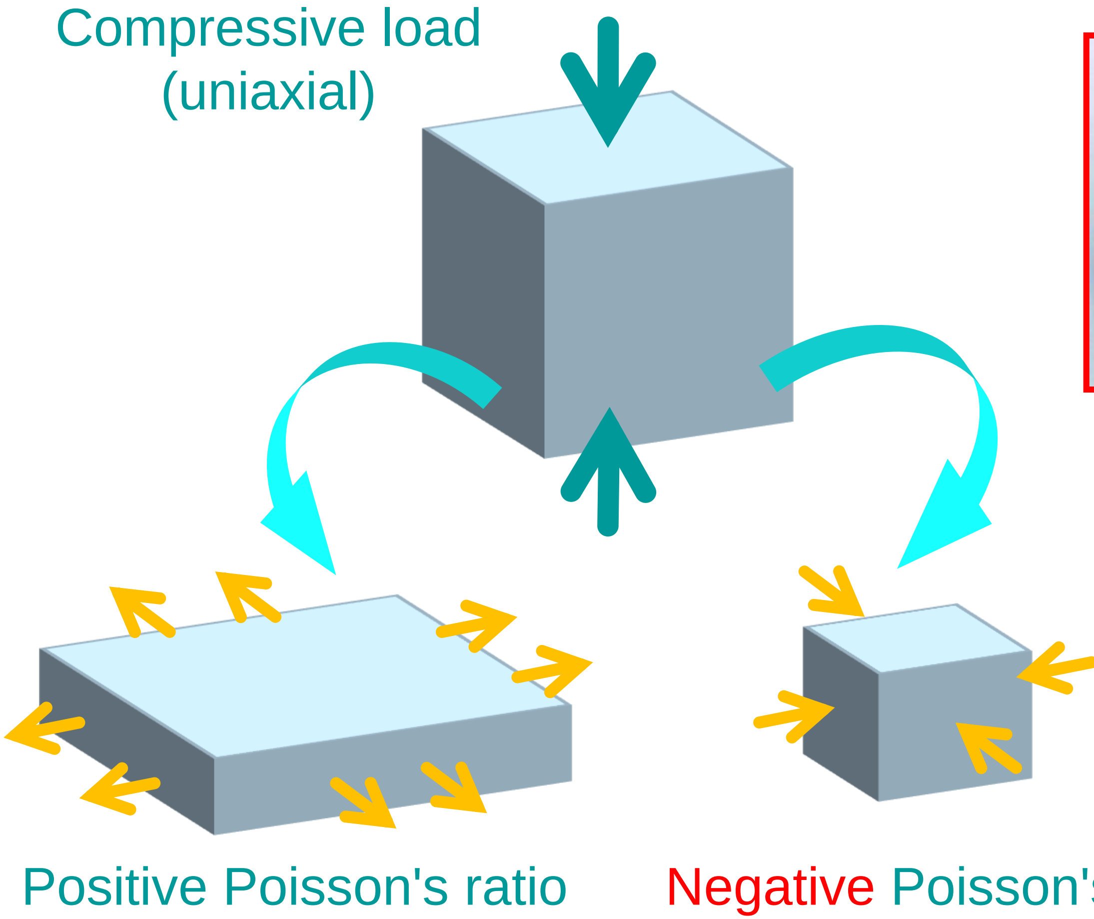
Let us consider the effect of *a hydrostatic stress σ_m*,

$$\Delta = \varepsilon_1 + \varepsilon_2 + \varepsilon_3 = 3\,\sigma_m (S_{11} + 2S_{12}) \qquad \Rightarrow \qquad \textit{Bulk modulus}, B = \frac{E}{3(1-2\nu)}$$

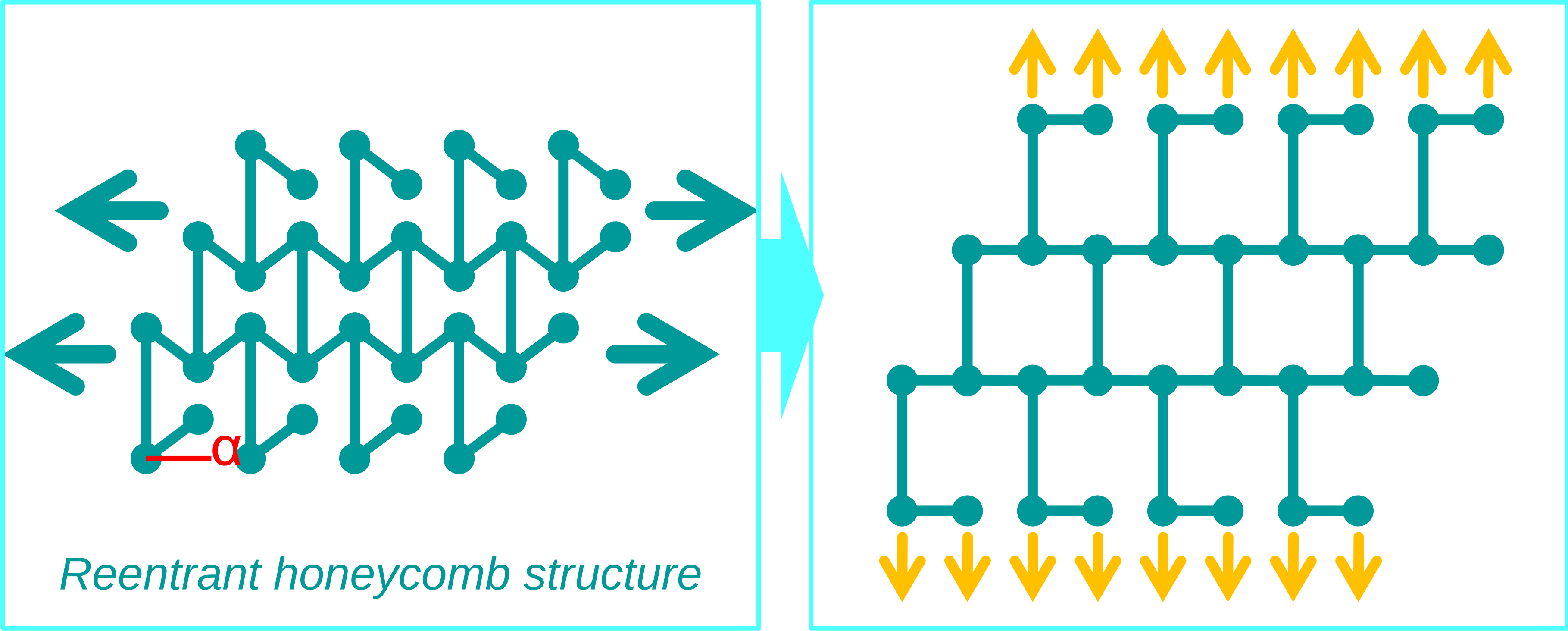
$$\frac{B}{G} = \frac{2(1+\nu)}{3(1-2\nu)} \qquad\qquad \nu = \frac{3(B/G) - 2}{6(B/G) + 2}$$

One extreme of properties is reached when *B* >> *G*, whereupon ν → 1/2. At the other extreme we have *B/G* → 0, with ν approaching a value of -1, and so the possible value range of Poisson's ratio is -1 < ν < 1/2. Poisson's ratio of zero arises when *B/G* = 2/3.

Negative Poisson's ratio

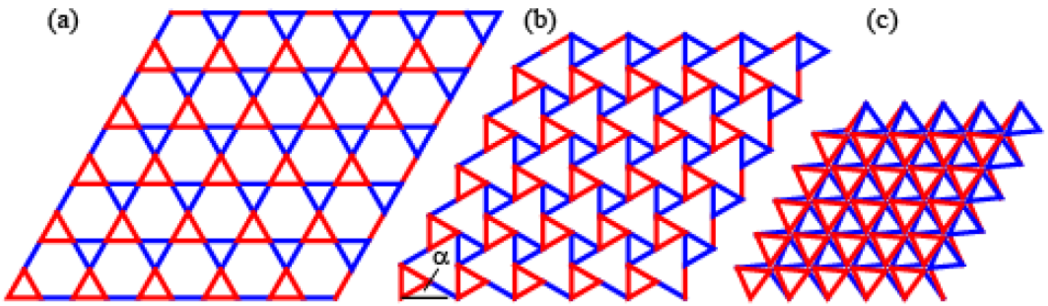


Mechanism: Negative Poisson's ratio



As the structure is stretched in the horizontal direction, angle α decreases forcing lengths along the vertical direction to increase. → Negative Poisson's ratio

Twisted kagome lattice



Twisted kagome lattice
Isotropic elasticity
with vanishing bulk modulus
(no deformation on lattice)

Regarding photonic feature,

(a) Periodic structure

(b) Small applied force - Huge conformation change

: Negative Poisson's ratio

: Extremely low bulk modulus

Elastic property - Photonic property

Isotropy considerations

$$\begin{pmatrix} C_{11} & C_{12} & C_{12} & 0 & 0 & 0 \\ \cdot & C_{11} & C_{12} & 0 & 0 & 0 \\ \cdot & \cdot & C_{11} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & C_{44} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & C_{44} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & C_{44} \end{pmatrix}$$

$$C_{44} = \frac{C_{11} - C_{12}}{2}$$

$$\begin{pmatrix} S_{11} & S_{12} & S_{12} & 0 & 0 & 0 \\ \cdot & S_{11} & S_{12} & 0 & 0 & 0 \\ \cdot & \cdot & S_{11} & 0 & 0 & 0 \\ \cdot & \cdot & \cdot & S_{44} & 0 & 0 \\ \cdot & \cdot & \cdot & \cdot & S_{44} & 0 \\ \cdot & \cdot & \cdot & \cdot & \cdot & S_{44} \end{pmatrix}$$

$$S_{44} = 2(S_{11} - S_{12})$$

For these systems, anisotropy is defined by the Zener ratio:

When the Zener ratio = 1, the material is isotropic.

$$(C_{11} - C_{12}) / 2C_{44}$$

$$or$$

$$S_{44} / 2(S_{11} - S_{12})$$

Elastic Moduli in Cubic Materials

We can use the different relations among elastic constants to ascertain elastic moduli along any orientation,

$$\frac{1}{E_{ijk}} = S_{11} - 2\left(S_{11} - S_{12} - \frac{1}{2}S_{44}\right)\left(l_{i1}{}^{2}l_{j2}{}^{2} + l_{j2}{}^{2}l_{k3}{}^{2} + l_{i1}{}^{2}l_{k3}{}^{2}\right)$$

where l_{i1}, l_{j2}, l_{k3} equal the direction cosines between the [ijk] direction and the [100], [010], and [001] directions.
(i.e., axes *x*, *y*, and *z*)