Dynamics (동역학)

Lecture 1: Particle Kinematics

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Course Information

- Lecture: M/W 11:00-12:15am Zoom Online Lecture

- Instructor: Prof. Dongjun Lee (이동준)
  Office: 1517@301 djlee@snu.ac.kr 880-1724
  Interactive & Networked Robotics Lab
  https://www.inrol.snu.ac.kr/

- Office hour: M/W 1-2pm or by appointment (email me beforehand)

- Teaching assistants: Minji Lee (이민지) mingg8@snu.ac.kr
  (R211, 880-1690) Minhyeong Lee (이민형) minhyeong@snu.ac.kr
  Undergraduate Course Assistants (TBA)

- TA session: 5-6 sessions during the semester (1 absence = -0.5%)
  problem solving (HW, previous exams) + computer SW
  start from the week of 9/14 (TAs will announce)

Course Information

- **Grading**: quiz 15%; HW 15%; mid-exam 30%; final-exam 40%
  mid-term: 10/29 (Th) 7-9:00pm[1]/on-site (one A4 paper allowed)
  final: 12/14 (M) 7:00-9:30pm[2]/on-site (two A4 papers allowed)
  students with other exam/class: [1]can start late if before 9pm; [2]can start earlier if stay until 7pm (or arranged w/ TA)

- **Attendance** mandatory: more than or equal to 5 unjustified absences = F
  1 absence = -2%; 1 tardiness (more than 15min late) = -0.5%
  1 attendance cheating = 3 absences

- **HW** assigned almost every other week
  should be turned in before the class on the due data
  0.0/0.5/1.0-scale; 50% on the same day / 0% otherwise
  all the quizzes will be on the due-date class and from the HW

- **Any academic dishonesty is strictly prohibited**;
  if caught = F + academic disciplinary action;
  work-ethics/integrity for better society w/o excessive societal cost

- This is English-based course, yet, Korean will (and can) be used whenever deemed necessary or more efficient (question, summary, emphasis, etc)

Course Information

- Text: F. P. Beer, E. R. Johnston & P. J. Cornwell, "Vector Mechanics for Engineers: Dynamics", McGraw-Hill, 12th Ed
  * Edition doesn't matter: all HW problems will be scanned/posted
  * Supplementary materials may also be uploaded from time to time

- Course Objectives:
  * able to formulate, solve and analyze the kinematics and dynamics of a particle (and particles) in 2D and 3D
  * able to formulate, solve and analyze the kinematics and dynamics of a rigid body in 2D and 3D
  * able to apply the concepts/tools of dynamics for the analysis/design of real/new engineering systems

Course Information

Relevant Systems


KUKA VR-ride


RC-car drifting

Gyrobike


Deep drone acrobatic


CUBI

Atlas - Parkour

Course Topics

Ch 11. Particle kinematics

Ch 12. Particle dynamics: Newton's second law

Ch 13. Particle dynamics: energy, momentum, impulse, impact

Ch 14. Systems of particles

Ch 15. Rigid body kinematics in 2D and 3D

Ch.16. Rigid body dynamics in 2D: force & acceleration

Ch.17. Rigid body dynamics in 2D: energy & momentum method

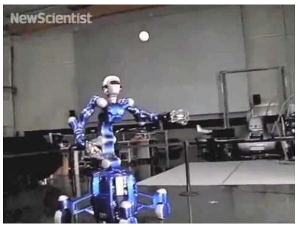
Ch.18. Rigid body dynamics in 3D, gyroscopic motion

+

Brief introduction on Lagrangian (analytical) dynamics if time permits

Kinematics vs Dynamics

- **Kinematics:** study of the geometry of motion; used to relate displacement, velocity, acceleration of objects without reference to the cause of motion $\Rightarrow$ how to describe the object's motion?
- **Dynamics (kinetics):** study of the relation between the forces/torques acting on an object and its motion; used to predict the motion caused by given force or to determine the force required to produce a given motion $\Rightarrow$ how the object's motion evolves w/ actuation or interaction?



- Robot Dynamics: $M(x)a + C(x,v)v + G(x) = u + f$

- $x \in \mathrm{SE}(3)$
- $\dot{x} = v$
- $\dot{v} = a$
- $\dot{a} = j$

kinematics

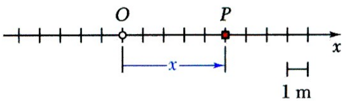
actuation/interaction

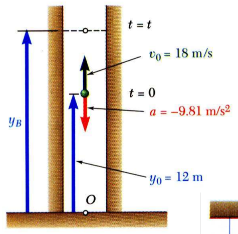
$= M^{-1}(x)[u + f - C(x,v)v - G(x)]$

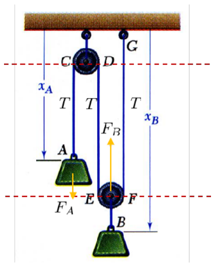
$= \frac{d}{dt}\left(M^{-1}(x)[u + f - C(x,v)v - G(x)]\right)$

dynamics

Rectilinear Motion







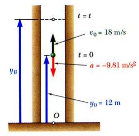
- Particle moving along a straight line is said to be in ***rectilinear motion***.
- Motion of a particle is given by *its scalar position coordinate* $x \in \Re$ from a fixed origin on the line.
- Kinematics of rectilinear motion (scalar):

$$\dot{x}(t) = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t} = v(t)$$

$$\dot{v}(t) = \lim_{\Delta t \to 0} \frac{\Delta v}{\Delta t} = a(t)$$

- ***Curvilinear* motion**: motion of a particle along a curved line in two or three dimensions (i.e., x is not a scalar, but a vector (e.g., $\vec{x} \in \Re^3$)).

Kinematics Integration

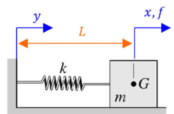


- Acceleration given as a function of *time*, $a = f(t)$, $t_f \Rightarrow x, v$

$$\frac{dv}{dt} = a = f(t) \qquad dv = f(t)dt \qquad \int_{v_0}^{v(t)} dv = \int_0^t f(t)dt \qquad \boxed{v(t) - v_0 = \int_0^t f(t)dt}$$

$$\frac{dx}{dt} = v(t) \qquad dx = v(t)dt \qquad \int_{x_0}^{x(t)} dx = \int_0^t v(t)dt \qquad \boxed{x(t) - x_0 = \int_0^t v(t)dt}$$

- Acceleration given as a function of *position*, $a = f(x)$, $t_f \Rightarrow x, v$



$$v = \frac{dx}{dt} \text{ or } dt = \frac{dx}{v} \qquad a = \frac{dv}{dt} \text{ or } \boxed{a = v\frac{dv}{dx} = f(x)}$$

$$v\,dv = f(x)dx \qquad \int_{v_0}^{v(x)} v\,dv = \int_{x_0}^{x} f(x)dx \qquad \boxed{\tfrac{1}{2}v(x)^2 - \tfrac{1}{2}v_0^2 = \int_{x_0}^{x} f(x)dx}$$

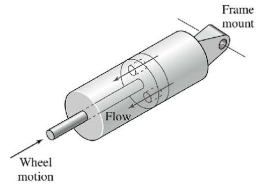
$$\boxed{t = \int_{x_o}^{x(t)} \frac{dx}{v(x)}}$$

Kinematics Integration

- Acceleration given as a function of velocity, $a = f(v)$, $t_f \Rightarrow x, v$

$$\frac{dv}{dt} = a = f(v) \qquad \boxed{\frac{dv}{f(v)} = dt} \qquad \int_{v_0}^{v(t)} \frac{dv}{f(v)} = \int_0^t dt$$

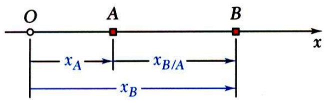
$$\boxed{\int_{v_0}^{v(t)} \frac{dv}{f(v)} = t}$$



$$\boxed{v\frac{dv}{dx} = a = f(v)} \qquad dx = \frac{v\,dv}{f(v)} \qquad \int_{x_0}^{x(t)} dx = \int_{v_0}^{v(t)} \frac{v\,dv}{f(v)}$$

$$\boxed{x(t) - x_0 = \int_{v_0}^{v(t)} \frac{v\,dv}{f(v)}}$$

Motion of Several Particles: Relative Motion



- For particles moving along the same line, if time be recorded from the same starting instant and displacements be measured from the same origin in the same direction.

$x_{B/A} = x_B - x_A =$ relative position of B from A

$x_B = x_A + x_{B/A}$

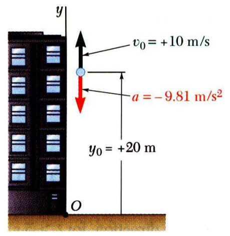
$v_{B/A} = v_B - v_A =$ relative velocity of B from A

$v_B = v_A + v_{B/A}$

$a_{B/A} = a_B - a_A =$ relative acceleration of B from A

$a_B = a_A + a_{B/A}$

Sample Problem 11.2



Ball tossed with 10 m/s vertical velocity from window 20 m above ground.

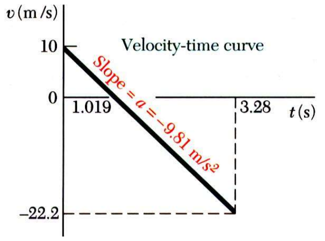
Determine:

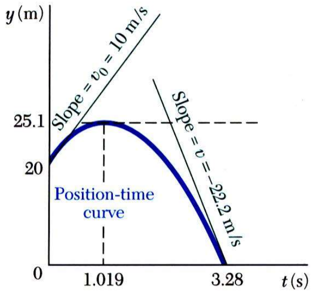
- velocity and elevation above ground at time *t*,
- highest elevation reached by ball and corresponding time, and
- time when ball will hit the ground and corresponding velocity.

SOLUTION:

- Integrate twice to find $v(t)$ and $y(t)$, each being first and second order polynomials w.r.t. t.
- Solve for t at which velocity equals zero (time for maximum elevation) and evaluate corresponding altitude.
- Solve for t at which altitude equals zero (time for ground impact) and evaluate corresponding velocity.

Sample Problem 11.2





SOLUTION:

- Integrate twice to find $v(t)$ and $y(t)$.

$$\frac{dv}{dt} = a = -9.81\,\mathrm{m/s^2}$$

$$\int_{v_0}^{v(t)} dv = -\int_0^t 9.81\,dt \qquad v(t) - v_0 = -9.81t$$

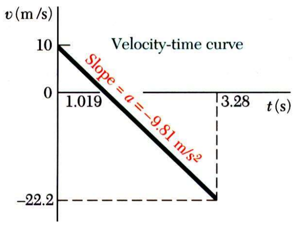
$$\boxed{v(t) = 10\frac{\mathrm{m}}{\mathrm{s}} - \left(9.81\frac{\mathrm{m}}{\mathrm{s}^2}\right)t}$$

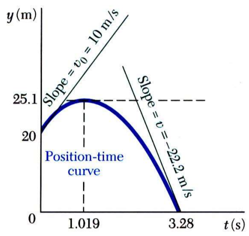
$$\frac{dy}{dt} = v = 10 - 9.81t$$

$$\int_{y_0}^{y(t)} dy = \int_0^t (10 - 9.81t)dt \qquad y(t) - y_0 = 10t - \tfrac{1}{2}9.81t^2$$

$$\boxed{y(t) = 20\,\mathrm{m} + \left(10\frac{\mathrm{m}}{\mathrm{s}}\right)t - \left(4.905\frac{\mathrm{m}}{\mathrm{s}^2}\right)t^2}$$

Sample Problem 11.2





- Solve for t at which velocity equals zero and evaluate corresponding altitude.

$$v(t) = 10\frac{\text{m}}{\text{s}} - \left(9.81\frac{\text{m}}{\text{s}^2}\right)t = 0$$

$$\boxed{t = 1.019\,\text{s}}$$

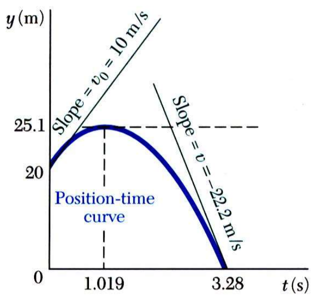
- Solve for y at which velocity becomes zero.

$$y(t) = 20\,\text{m} + \left(10\frac{\text{m}}{\text{s}}\right)t - \left(4.905\frac{\text{m}}{\text{s}^2}\right)t^2$$

$$y = 20\,\text{m} + \left(10\frac{\text{m}}{\text{s}}\right)(1.019\,\text{s}) - \left(4.905\frac{\text{m}}{\text{s}^2}\right)(1.019\,\text{s})^2$$

$$\boxed{y = 25.1\,\text{m}}$$

Sample Problem 11.2



- Solve for t at which altitude equals zero and evaluate corresponding velocity.

$$y(t) = 20\,\text{m} + \left(10\frac{\text{m}}{\text{s}}\right)t - \left(4.905\frac{\text{m}}{\text{s}^2}\right)t^2 = 0$$

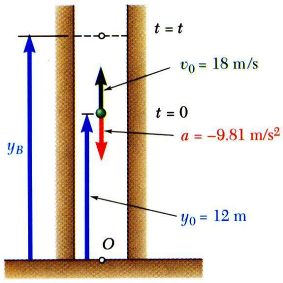
$$\boxed{\begin{array}{l} t = -1.243\,\text{s}\ \text{(meaningless)} \\ t = 3.28\,\text{s} \end{array}}$$

$$v(t) = 10\frac{\text{m}}{\text{s}} - \left(9.81\frac{\text{m}}{\text{s}^2}\right)t$$

$$v(3.28\,\text{s}) = 10\frac{\text{m}}{\text{s}} - \left(9.81\frac{\text{m}}{\text{s}^2}\right)(3.28\,\text{s})$$

$$\boxed{v = -22.2\frac{\text{m}}{\text{s}}}$$

Sample Problem 11.4



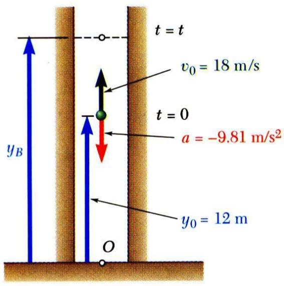
Ball thrown vertically from 12 m level in elevator shaft with initial velocity of 18 m/s. At same instant, open-platform elevator passes 5 m level moving upward at 2 m/s.

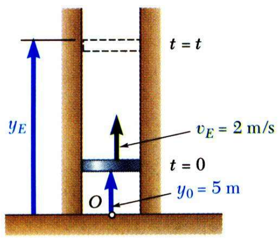
Determine *(a)* when and where ball hits elevator and *(b)* relative velocity of ball and elevator at contact.

SOLUTION:

- Compute y_B (quadratic in *t*)
- Compute y_E (linear in *t*)
- Find time t_c s.t. $y_B(t_c) = y_E(t_c)$
- Find $\dot{y}_B(t_c) - \dot{y}_E(t_c)$

Sample Problem 11.3





SOLUTION:

- Substitute initial position and velocity and constant acceleration of ball into general equations for uniformly accelerated rectilinear motion.

$$v_B = v_0 + at = 18\frac{\text{m}}{\text{s}} - \left(9.81\frac{\text{m}}{\text{s}^2}\right)t$$

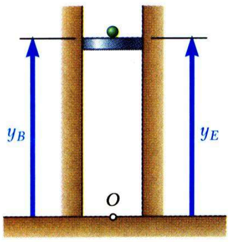
$$y_B = y_0 + v_0 t + \tfrac{1}{2}at^2 = 12\,\text{m} + \left(18\frac{\text{m}}{\text{s}}\right)t - \left(4.905\frac{\text{m}}{\text{s}^2}\right)t^2$$

- Substitute initial position and constant velocity of elevator into equation for uniform rectilinear motion.

$$v_E = 2\frac{\text{m}}{\text{s}}$$

$$y_E = y_0 + v_E t = 5\,\text{m} + \left(2\frac{\text{m}}{\text{s}}\right)t$$

Sample Problem 11.3



- Write equation for relative position of ball with respect to elevator and solve for zero relative position, i.e., impact.

$$y_{B/E} = \left(12 + 18t - 4.905t^2\right) - (5 + 2t) = 0$$

$$\boxed{\begin{aligned} t &= -0.39\,\mathrm{s}\ \ (\text{meaningless}) \\ t &= 3.65\,\mathrm{s} \end{aligned}}$$

- Substitute impact time into equations for position of elevator and relative velocity of ball with respect to elevator.

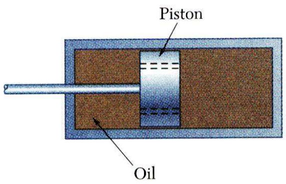
$$y_E = 5 + 2(3.65)$$

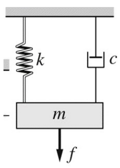
$$\boxed{y_E = 12.3\,\mathrm{m}}$$

$$\begin{aligned} v_{B/E} &= (18 - 9.81t) - 2 \\ &= 16 - 9.81(3.65) \end{aligned}$$

$$\boxed{v_{B/E} = -19.81\,\frac{\mathrm{m}}{\mathrm{s}}}$$

Sample Problem 11.3





$a = -kv$

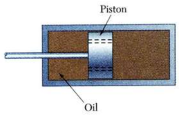
Brake mechanism used to reduce gun recoil consists of piston attached to barrel moving in fixed cylinder filled with oil. As barrel recoils with initial velocity v_0, piston moves and oil is forced through orifices in piston, causing piston and cylinder to decelerate at rate proportional to their velocity.

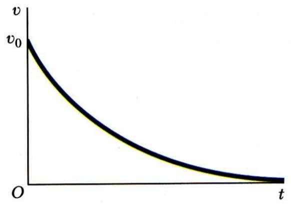
Determine $v(t)$, $x(t)$, and $v(x)$.

SOLUTION:

- Integrate $a = dv/dt = -kv$ to find $v(t)$.
- Integrate $v(t) = dx/dt$ to find $x(t)$.
- Integrate $a = v\, dv/dx = -kv$ to find relation between $v(x)$ as a function of x.

Sample Problem 11.3



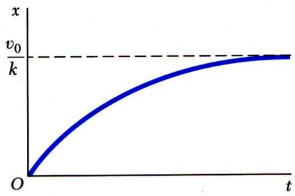


SOLUTION:

- **Integrate $a = dv/dt = -kv$ to find $v(t)$.**

$$a = \frac{dv}{dt} = -kv \qquad \int_{v_0}^{v(t)} \frac{dv}{v} = -k\int_0^t dt \qquad \ln\frac{v(t)}{v_0} = -kt$$

$$\boxed{v(t) = v_0 e^{-kt}}$$



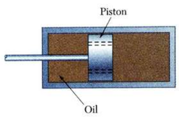
- **Integrate $v(t) = dx/dt$ to find $x(t)$.**

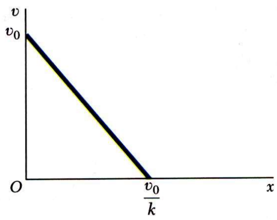
$$v(t) = \frac{dx}{dt} = v_0 e^{-kt}$$

$$\int_0^{x(t)} dx = v_0\int_0^t e^{-kt}\,dt \qquad x(t) = v_0\left[-\frac{1}{k}e^{-kt}\right]_0^t$$

$$\boxed{x(t) = \frac{v_0}{k}\left(1 - e^{-kt}\right)}$$

Sample Problem 11.3





- **Integrate $a = v\,dv/dx = -kv$ to find $v(x)$.**

$$a = v\frac{dv}{dx} = -kv \qquad dv = -k\,dx \qquad \int_{v_0}^{v} dv = -k\int_0^x dx$$

$$v - v_0 = -kx$$

$$\boxed{v = v_0 - kx}$$

- **Alternatively,**

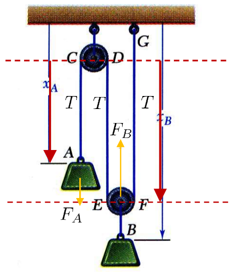
with $x(t) = \frac{v_0}{k}\left(1 - e^{-kt}\right)$

and $v(t) = v_0 e^{-kt}$ or $e^{-kt} = \frac{v(t)}{v_0}$

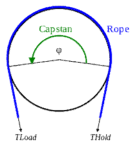
then $x(t) = \frac{v_0}{k}\left(1 - \frac{v(t)}{v_0}\right)$

$$\boxed{v = v_0 - kx}$$

Particles with Constraints: Pulleys



$$x_A(t) + 2x_B(t) = L$$



$$T_{\text{load}} = T_{\text{hold}} e^{\mu\varphi}$$

- Position of a particle may *depend* on position of one or more other particles.
- Position of block B depends on position of block A. Since rope is of constant length,

$$\boxed{x_A + 2x_B = \text{ constant (one degree of freedom)}}$$

- For linearly related positions, similar relations hold between velocities and accelerations.

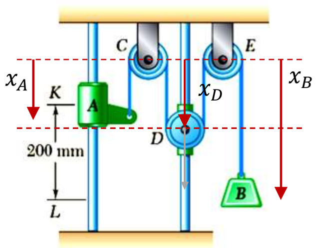
$$\frac{dx_A}{dt} + 2\frac{dx_B}{dt} = 0 \quad \text{or} \quad v_A + 2v_B = 0$$

$$\frac{dv_A}{dt} + 2\frac{dv_B}{dt} = 0 \quad \text{or} \quad a_A + 2a_B = 0$$

- If no slack, no slip and no bearing friction, tension T along the (taut) string is all the same (vs capstan)
- Thus, force is amplified, while motion scaled-down

$$F_B = 2F_A = 2T, \quad v_B = -\frac{1}{2}v_A \quad F_A v_A + F_B v_B = 0$$

Sample Problem 11.5



Pulley D is attached to a collar which is pulled down at 75mm/s. At $t = 0$, collar A starts moving down from K with constant acceleration and zero initial velocity. Knowing that velocity of collar A is 300 mm/s as it passes L, determine the change in elevation, velocity, and acceleration of block B when block A is at L.

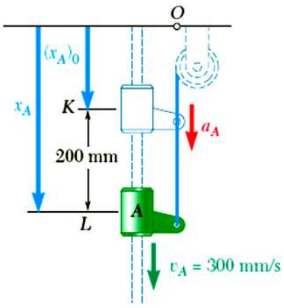
SOLUTION:

- Draw x_A, x_D, x_B.

$$x_A(t) + 2x_D(t) + x_B(t) = L_{string}$$

- Compute $x_A(t)$; Find t s.t. $x_A(t) = L$
- Compute $x_D(t)$
- Identify constraint among x_A, x_D, x_B
- Use the constraint to compute $x_B(t), v_B(t), a_B(t)$

Sample Problem 11.5



SOLUTION:

- Define origin at upper horizontal surface with positive displacement downward.
- Collar *A* has uniformly accelerated rectilinear motion. Solve for acceleration and time *t* to reach *L*.

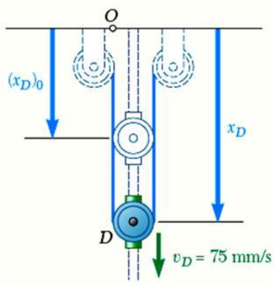
$$v_A^2 = (v_A)_0^2 + 2a_A[x_A - (x_A)_0]$$

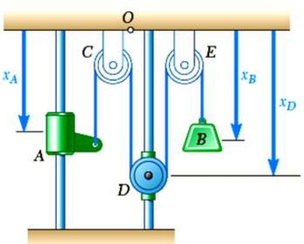
$$\left(300\frac{\text{mm}}{\text{s}}\right)^2 = 2a_A(200\,\text{mm}) \qquad a_A = 225\frac{\text{mm}}{\text{s}^2}$$

$$v_A = (v_A)_0 + a_A t$$

$$300\frac{\text{mm}}{\text{s}} = 225\frac{\text{mm}}{\text{s}^2}t \qquad t = 1.333\,\text{s}$$

Sample Problem 11.5





- Pulley *D* has uniform rectilinear motion. Calculate change of position at time *t*.

$$x_D = (x_D)_0 + v_D t$$

$$x_D - (x_D)_0 = \left(75\frac{\text{mm}}{\text{s}}\right)(1.333\,\text{s}) = 100\,\text{mm}$$

- Block *B* motion is dependent on motions of collar *A* and pulley *D*. Write motion relationship and solve for change of block *B* position at time *t*.

Total length of cable remains constant,

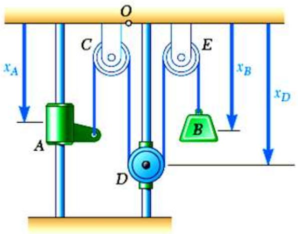
$$x_A + 2x_D + x_B = (x_A)_0 + 2(x_D)_0 + (x_B)_0$$

$$[x_A - (x_A)_0] + 2[x_D - (x_D)_0] + [x_B - (x_B)_0] = 0$$

$$(200\text{mm}) + 2(100\,\text{mm}) + [x_B - (x_B)_0] = 0$$

$$\boxed{x_B - (x_B)_0 = -400\,\text{mm}}$$

Sample Problem 11.5



- Differentiate motion relation twice to develop equations for velocity and acceleration of block *B*.

$$x_A + 2x_D + x_B = \text{constant}$$

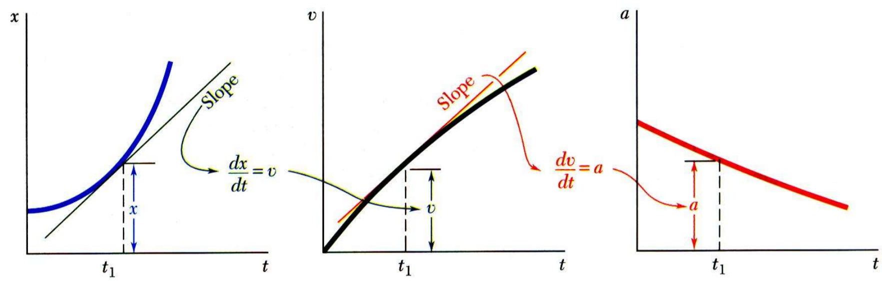
$$v_A + 2v_D + v_B = 0$$

$$\left(300\frac{\text{mm}}{\text{s}}\right) + 2\left(75\frac{\text{mm}}{\text{s}}\right) + v_B = 0 \qquad \boxed{v_B = 450\frac{\text{mm}}{\text{s}}}$$

$$a_A + 2a_D + a_B = 0$$

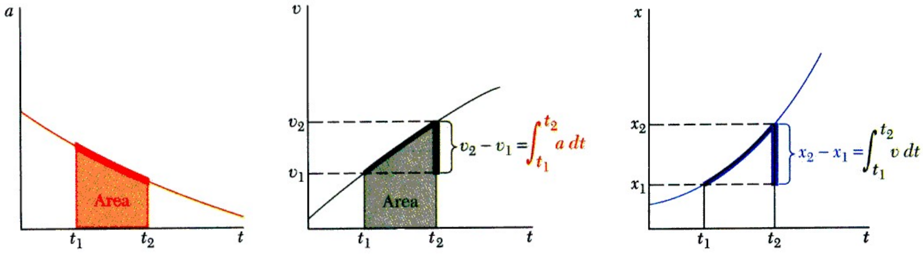
$$\left(225\frac{\text{mm}}{\text{s}^2}\right) + v_B = 0 \qquad \boxed{a_B = -225\frac{\text{mm}}{\text{s}^2}}$$

Graphical Solution of Rectilinear Motion



- **Given the *x-t* curve, the *v-t* curve is equal to the *x-t* curve slope.**
- **Given the *v-t* curve, the *a-t* curve is equal to the *v-t* curve slope.**

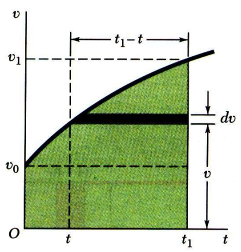
Graphical Solution of Rectilinear Motion

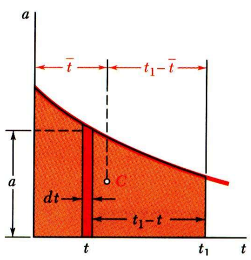


- **Given the *a-t* curve, the change in velocity between t_1 and t_2 is equal to the area under the *a-t* curve between t_1 and t_2.**

- **Given the *v-t* curve, the change in position between t_1 and t_2 is equal to the area under the *v-t* curve between t_1 and t_2.**

Other Graphical Methods





- *Moment-area method* to determine particle position at time *t* directly from the *a-t* curve:

$$x_1 - x_0 = \text{area under } v - t \text{ curve}$$
$$= v_0 t_1 + \int_{v_0}^{v_1} \left(t_1(v) - t(v)\right) dv$$

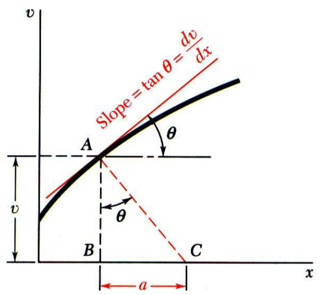
using $dv = a(t)\,dt$,

$$x_1 - x_0 = v_0 t_1 + \int_0^{t_1} (t_1 - t)\,a(t)\,dt$$

$\int_0^{t_1} (t_1 - t)\,a\,dt =$ first moment of area under *a-t* curve with respect to $t = t_1$ line.

$$x_1 = x_0 + v_0 t_1 + (\text{area under } a\text{-}t \text{ curve})(t_1 - \bar{t})$$
$\bar{t}$ = abscissa of centroid C

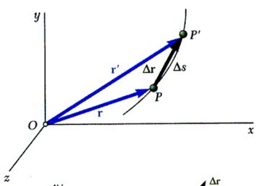
Other Graphical Methods

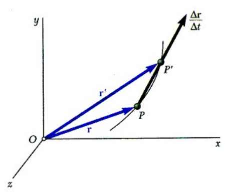


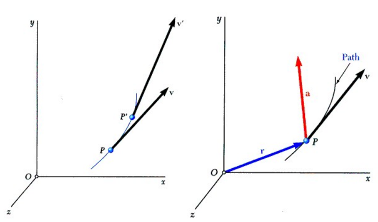
- **Method to determine particle acceleration from *v-x* curve:**

$$a = v\frac{dv}{dx}$$
$$= AB\tan\theta$$
$$= BC = \textbf{\textit{subnormal}} \textbf{ to } \textbf{\textit{v-x}} \textbf{ curve}$$

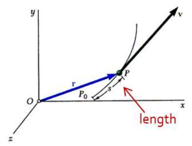
Curvilinear Motion







- Particle moving along a curve other than a straight line is in ***curvilinear motion***.
- *Position vector* $\boxed{\vec{r}(t) = OP \in \Re^3}$
- Kinematics relations:

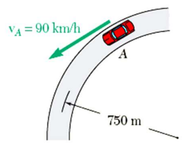


velocity (vector)

$$\vec{v} = \lim_{\Delta t \to 0}\frac{\Delta\vec{r}}{\Delta t} = \frac{d\vec{r}}{dt} = \dot{\vec{r}}$$

speed (scalar)

$$v = \lim_{\Delta t \to 0}\frac{\Delta s}{\Delta t} = \frac{ds}{dt}$$

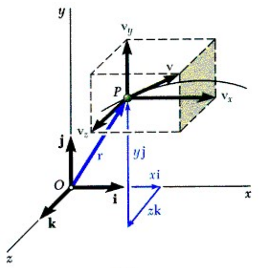


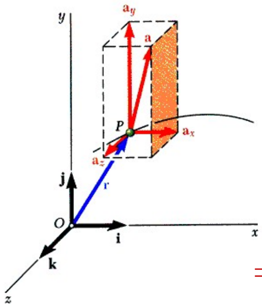
acceleration (vector)

$$\vec{a} = \lim_{\Delta t \to 0}\frac{\Delta\vec{v}}{\Delta t} = \frac{d\vec{v}}{dt}$$

- In general, acceleration vector is not tangent to particle path and velocity vector.
- How to describe the vector $\vec{r}$?

Cartesian Coordinates





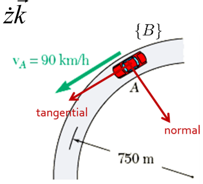
- When position vector of particle P is given by its (fixed) rectangular (or Cartesian) components,

$$\vec{r} = x\vec{i} + y\vec{j} + z\vec{k}$$

- Velocity vector,

$$\vec{v} = \frac{dx}{dt}\vec{i} + \frac{dy}{dt}\vec{j} + \frac{dz}{dt}\vec{k} = \dot{x}\vec{i} + \dot{y}\vec{j} + \dot{z}\vec{k}$$

$$= v_x\vec{i} + v_y\vec{j} + v_z\vec{k}$$



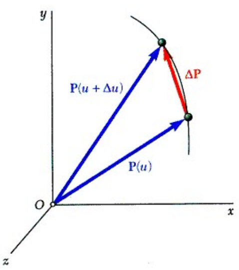
- Acceleration vector,

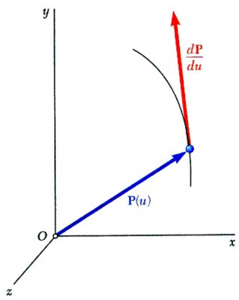
$$\vec{a} = \frac{d^2x}{dt^2}\vec{i} + \frac{d^2y}{dt^2}\vec{j} + \frac{d^2z}{dt^2}\vec{k} = \ddot{x}\vec{i} + \ddot{y}\vec{j} + \ddot{z}\vec{k}$$

$$= a_x\vec{i} + a_y\vec{j} + a_z\vec{k}$$

$\Rightarrow$ with each i, j, k fixed (i.e., , $\frac{di}{dt} = \frac{dj}{dt} = \frac{dk}{dt} = 0$), component in each direction can be analyzed separately

Derivatives of Vector Functions





- Let $\vec{P}(u)$ be a vector function of scalar variable u,

$$\frac{d\vec{P}}{du} = \lim_{\Delta u \to 0} \frac{\Delta \vec{P}}{\Delta u} = \lim_{\Delta u \to 0} \frac{\vec{P}(u+\Delta u) - \vec{P}(u)}{\Delta u}$$

- Derivative of vector sum,

$$\frac{d(\vec{P}+\vec{Q})}{du} = \frac{d\vec{P}}{du} + \frac{d\vec{Q}}{du}$$

- Derivative of product of scalar and vector functions,

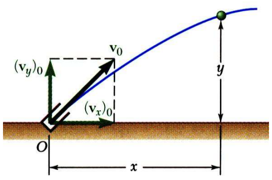
$$\frac{d(f\vec{P})}{du} = \frac{df}{du}\vec{P} + f\frac{d\vec{P}}{du}$$

- Derivative of *scalar product* and *vector product,*

$$\frac{d(\vec{P}^T\vec{Q})}{du} = \frac{d\vec{P}^T}{du}\vec{Q} + \vec{P}^T\frac{d\vec{Q}}{du}$$

$$\frac{d(\vec{P}\times\vec{Q})}{du} = \frac{d\vec{P}}{du}\times\vec{Q} + \vec{P}\times\frac{d\vec{Q}}{du}$$

Projectile Motion



- Rectangular components particularly effective when component accelerations can be integrated independently, e.g., motion of a projectile,

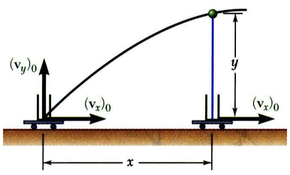
$$a_x = \ddot{x} = 0 \qquad a_y = \ddot{y} = -g \qquad a_z = \ddot{z} = 0$$

with initial conditions,

$$x_0 = y_0 = z_0 = 0 \qquad (v_x)_0, (v_y)_0, (v_z)_0 = 0$$

Integrating twice yields

$$v_x = (v_x)_0 \quad v_y = (v_y)_0 - gt \qquad v_z = 0$$
$$x = (v_x)_0 t \quad y = (v_y)_0 y - \tfrac{1}{2} g t^2 \quad z = 0$$



- Motion in horizontal direction is uniform.
- Motion in vertical direction is uniformly accelerated.
- Motion of projectile could be replaced by two independent rectilinear motions.

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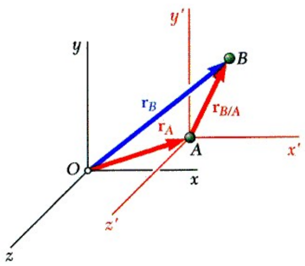
Motion Relative to a Frame in Translation



- Designate one frame as the *fixed (inertia) frame of reference*. All other frames not rigidly attached to the fixed frame are *moving (body) frames of reference*.
- Position vectors for particles *A* and *B* with respect to the fixed frame of reference *Oxyz* are $\vec{r}_A$ and $\vec{r}_B \in \Re^3$.
- Vector $\vec{r}_{B/A}$ joining *A* and *B* defines the relative position of *B* from *A* w.r.t. *Ax'y'z'*

$$\boxed{\vec{r}_B = \vec{r}_A + \vec{r}_{B/A}}$$

- Differentiating twice,

$\vec{v}_B = \vec{v}_A + \vec{v}_{B/A}$ $\quad \vec{v}_{B/A} =$ relative velocity of *B* from *A* w.r.t. *Ax'y'z'*.

$\vec{a}_B = \vec{a}_A + \vec{a}_{B/A}$ $\quad \vec{a}_{B/A} =$ relative accel. of *B* from *A* w.r.t. *Ax'y'z'*.

- Absolute motion of *B* can be obtained by combining the motion of *A* and the relative motion of *B* from *A w.r.t. A'x'y'z'*.

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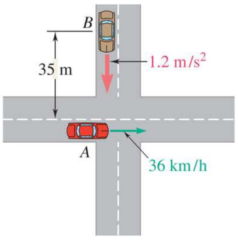
Sample Problem 11.4



Automobile *A* is traveling east at the constant speed of 36 km/h. As automobile *A* crosses the intersection shown, automobile *B* starts from rest 35 m north of the intersection and moves south with a constant acceleration of $1.2\ \mathrm{m/s^2}$. Determine the position, velocity, and acceleration of *B* relative to *A* 5 *s* after *A* crosses the intersection.

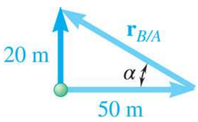
Strategy:

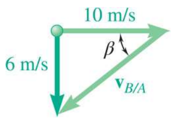
- Define inertial axes for the system.
- Determine the position, speed, and acceleration of car A at $t = 5$ s.
- Determine the position, speed, and acceleration of car B at $t = 5$ s.
- Using vectors (Equation 11.30, 11.32, and 11.33) or a graphical approach, determine the relative position, velocity, and acceleration.

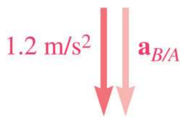
Sample Problem 11.4

$$\mathbf{a}_A = 0 \qquad \mathbf{a}_B = 1.2\ \mathrm{m/s^2} \downarrow$$
$$\mathbf{v}_A = 10\,\mathrm{m/s} \rightarrow \qquad \mathbf{v}_B = 6\ \mathrm{m/s} \downarrow$$
$$\mathbf{r}_A = 50\ \mathrm{m} \rightarrow \qquad \mathbf{r}_B = 20\,\mathrm{m} \downarrow$$

We can solve the problems geometrically, and apply the arctangent relationship:







$r_{B/A} = 53.9\ \mathrm{m} \quad \alpha = 21.8^\circ$ $\qquad v_{B/A} = 11.66\,\mathrm{m/s} \quad \beta = 31.0^\circ$

$\mathbf{r}_{B/A} = 53.9\ \mathrm{m} \measuredangle 21.8^\circ \qquad \mathbf{v}_{B/A} = 11.66\ \mathrm{m/s} \measuredangle 31.0^\circ \qquad \mathbf{a}_{B/A} = 1.2\ \mathrm{m/s^2} \downarrow$

Or we can solve the problems using vectors to obtain equivalent results:

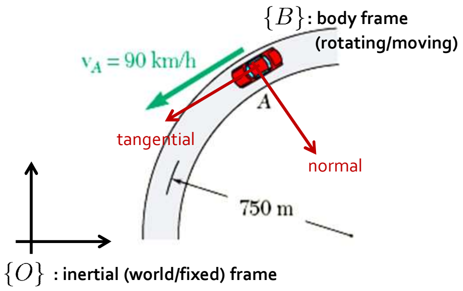
$$\mathbf{r}_B = \mathbf{r}_A + \mathbf{r}_{B/A} \qquad \mathbf{v}_B = \mathbf{v}_A + \mathbf{v}_{B/A} \qquad \mathbf{a}_B = \mathbf{a}_A + \mathbf{a}_{B/A}$$

$$20\mathbf{j} = 50\mathbf{i} + \mathbf{r}_{B/A} \qquad -6\mathbf{j} = 10\mathbf{i} + \mathbf{v}_{B/A} \qquad -1.2\mathbf{j} = 0\mathbf{i} + \mathbf{a}_{B/A}$$

$$\mathbf{r}_{B/A} = 20\mathbf{j} - 50\mathbf{i}\ \text{(m)} \qquad \mathbf{v}_{B/A} = -6\mathbf{j} - 10\mathbf{i}\ \text{(m/s)} \qquad \mathbf{a}_{B/A} = -1.2\mathbf{j}\ (\mathrm{m/s^2})$$

Physically, a rider in car A would "see" car B travelling south and west.

Sample Problem 11.10



A motorist is traveling on curved section of highway at 90 km/h. The motorist applies brakes causing a constant deceleration rate (tangential).

Knowing that after 8 s the speed has been reduced to 72 km/h, determine the acceleration of the automobile immediately after the brakes are applied.

SOLUTION:

- Calculate tangential (forward) and normal (sideway) components of acceleration.

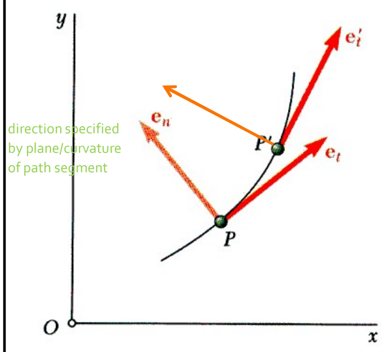
  $\Rightarrow$ convenient to describe motion along longitudinal and sideway for car/driver

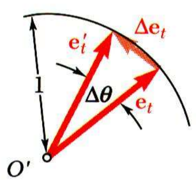
  $\Rightarrow$ basis (unit) vectors e_t, e_n rotate

  $\Rightarrow$ not Cartesian anymore

- Determine acceleration magnitude and direction with respect to tangent to curve.

Tangential and Normal Components





- Velocity vector of particle is tangent to the path of particle. In general, acceleration vector is not. Given a path, often convenient to express motion in terms of tangential and normal components.

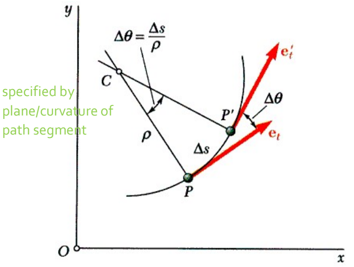
- $\vec{e}_t$ and $\vec{e}_t'$ are tangential unit vectors for the particle path at P and P'. When drawn with respect to the same origin, $\Delta\vec{e}_t = \vec{e}_t' - \vec{e}_t$ and $\Delta\theta$ is the angle between them.

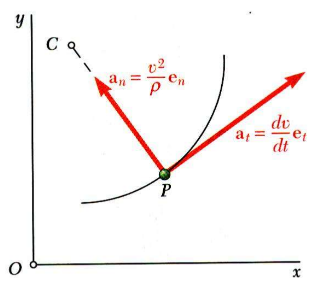
$$\Delta e_t = 2\sin(\Delta\theta/2)\vec{e}_n$$

$$\lim_{\Delta\theta\to 0}\frac{\Delta\vec{e}_t}{\Delta\theta} = \lim_{\Delta\theta\to 0}\frac{\sin(\Delta\theta/2)}{\Delta\theta/2}\vec{e}_n = \vec{e}_n$$

$$\boxed{\vec{e}_n = \frac{d\vec{e}_t}{d\theta}}$$

Tangential and Normal Components





- with $\boxed{\vec{v} = v\vec{e}_t}$

$$\vec{a} = \frac{d\vec{v}}{dt} = \frac{dv}{dt}\vec{e}_t + v\frac{d\vec{e}_t}{dt} = \frac{dv}{dt}\vec{e}_t + v\frac{d\vec{e}_t}{d\theta}\frac{d\theta}{ds}\frac{ds}{dt}$$

where (or $\frac{d\theta}{dt} = \frac{v}{\rho}$)

$$\frac{d\vec{e}_t}{d\theta} = \vec{e}_n \qquad \rho\, d\theta = ds \qquad \frac{ds}{dt} = v$$

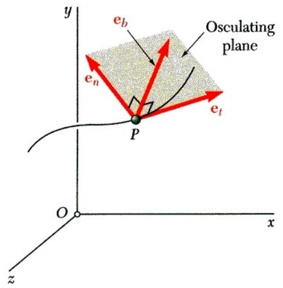
def. of curvature

After substituting,

$$\boxed{\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \qquad a_n = \frac{v^2}{\rho}}$$

- Tangential component of acceleration reflects change of speed and normal component reflects change of direction.
- Tangential component may be positive or negative. Normal component always points toward center C of path segment curvature.

Tangential and Normal Components

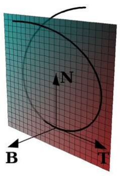


- Relations for tangential and normal acceleration also apply for particle moving along space curve.

$$\boxed{\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \qquad a_n = \frac{v^2}{\rho}}$$

- Plane containing tangential and normal unit vectors is called the *osculating plane*.

Frenet-Serret formula*:



$$\frac{d\mathbf{T}}{ds} = \kappa\mathbf{N}, \qquad \frac{d\mathbf{N}}{ds} = -\kappa\mathbf{T} + \tau\mathbf{B}, \qquad \frac{d\mathbf{B}}{ds} = -\tau\mathbf{N},$$

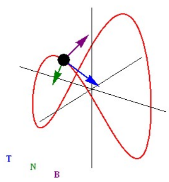
$$\begin{bmatrix}\mathbf{T}'\\ \mathbf{N}'\\ \mathbf{B}'\end{bmatrix} = \begin{bmatrix}0 & \kappa & 0\\ -\kappa & 0 & \tau\\ 0 & -\tau & 0\end{bmatrix}\begin{bmatrix}\mathbf{T}\\ \mathbf{N}\\ \mathbf{B}\end{bmatrix}$$

- Normal to the osculating plane is found from

$$\vec{e}_b = \vec{e}_t \times \vec{e}_n$$

$$\vec{e}_n = \textit{principal normal}$$

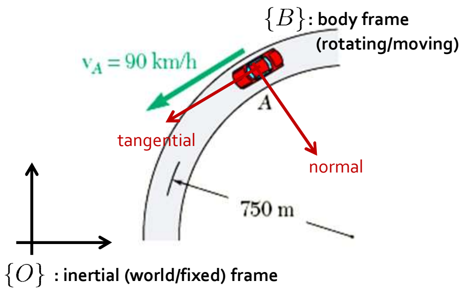
$$\vec{e}_b = \textit{binormal}$$



- Acceleration has no component along binormal.

https://en.wikipedia.org/wiki/Frenet%E2%80%93Serret_formulas

Sample Problem 11.10



A motorist is traveling on curved section of highway at 90 km/h. The motorist applies brakes causing a constant deceleration rate.

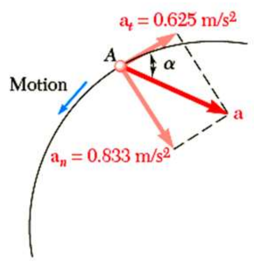
Knowing that after 8 s the speed has been reduced to 72 km/h, determine the acceleration of the automobile immediately after the brakes are applied.

SOLUTION:

- Calculate tangential and normal components of acceleration.
- Determine (total) acceleration magnitude and direction with respect to tangent to curve.

$$\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \qquad a_n = \frac{v^2}{\rho}$$

Sample Problem 11.10



$90\,\text{km/h} = 25\,\text{m/s}$

$72\,\text{km/h} = 20\,\text{m/s}$

SOLUTION:

- Calculate tangential and normal components of acceleration.

$$a_t = \frac{\Delta v}{\Delta t} = \frac{(25-20)\,\text{m/s}}{8\,\text{s}} = -0.625\,\frac{\text{m}}{\text{s}^2}$$

$$a_n = \frac{v^2}{\rho} = \frac{(25\,\text{m/s})^2}{750\,\text{m}} = 0.833\,\frac{\text{m}}{\text{s}^2}$$

- Determine acceleration magnitude and direction with respect to tangent to curve.

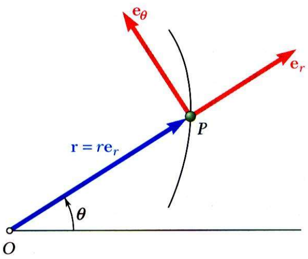
$$a = \sqrt{a_t^2 + a_n^2} = \sqrt{(-0.625)^2 + 0.833^2} \qquad \boxed{a = 1.041\,\frac{\text{m}}{\text{s}^2}}$$

$$\alpha = \tan^{-1}\frac{a_n}{a_t} = \tan^{-1}\frac{0.833}{0.625} \qquad \boxed{\alpha = 53.1^\circ}$$

polar coordinates?

straight outlet?

Polar Coordinates



$$\vec{r} = r\vec{e}_r$$

$$\frac{d\vec{e}_r}{d\theta} = \vec{e}_\theta \qquad \frac{d\vec{e}_\theta}{d\theta} = -\vec{e}_r$$

$$\Rightarrow \frac{d\vec{e}_r}{dt} = \frac{d\vec{e}_r}{d\theta}\frac{d\theta}{dt} = \dot{\theta}\cdot\vec{e}_\theta$$

$$\Rightarrow \frac{d\vec{e}_\theta}{dt} = \frac{d\vec{e}_\theta}{d\theta}\frac{d\theta}{dt} = -\dot{\theta}\cdot\vec{e}_r$$

- When particle position is given in polar coordinates, it is convenient to express velocity and acceleration with components parallel (radial ≠ normal) and perpendicular (transversal ≠ tangential) to <u>*OP*</u>.
- The particle velocity vector is

$$\vec{v} = \frac{d}{dt}(r\vec{e}_r) = \frac{dr}{dt}\vec{e}_r + r\frac{d\vec{e}_r}{dt} = \frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta$$

$$\boxed{= \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta}$$

- Similarly, the particle acceleration vector is

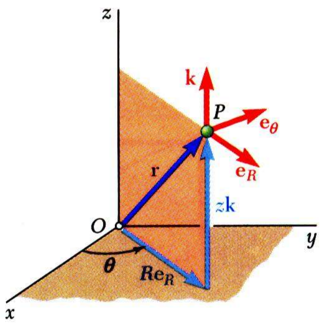
$$\vec{a} = \frac{d}{dt}\left(\frac{dr}{dt}\vec{e}_r + r\frac{d\theta}{dt}\vec{e}_\theta\right)$$

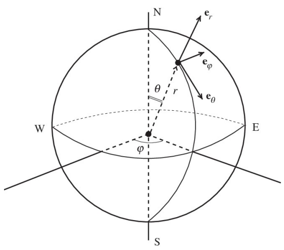
$$= \frac{d^2r}{dt^2}\vec{e}_r + \frac{dr}{dt}\frac{d\vec{e}_r}{dt} + \frac{dr}{dt}\frac{d\theta}{dt}\vec{e}_\theta + r\frac{d^2\theta}{dt^2}\vec{e}_\theta + r\frac{d\theta}{dt}\frac{d\vec{e}_\theta}{dt}$$

$$\boxed{= \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta}$$

centripetal acceleration   Coriolis acceleration

Cylindrical Coordinates





- When particle position is given in cylindrical coordinates, it is convenient to express the velocity and acceleration vectors using the unit vectors $\vec{e}_r, \vec{e}_\theta$ (rotating) and $\vec{k}$ (fixed).
- Position vector,

$$\vec{r} = r\vec{e}_r + z\vec{k}$$

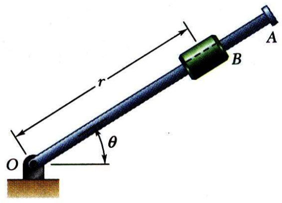
- Velocity vector,

$$\vec{v} = \frac{d\vec{r}}{dt} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + \dot{z}\vec{k}$$

- Acceleration vector,

$$\vec{a} = \frac{d\vec{v}}{dt} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta + \ddot{z}\vec{k}$$

Sample Problem 11.12



Rotation of the arm about O is defined by $\theta = 0.15t^2$ where θ is in radians and t in seconds. Collar B slides along the arm such that $r = 0.9 - 0.12t^2$ where r is in meters.

After the arm has rotated through 30°, determine *(a)* the total velocity of the collar, *(b)* the total acceleration of the collar, and *(c)* the relative acceleration of the collar with respect to (as observed in) the arm.

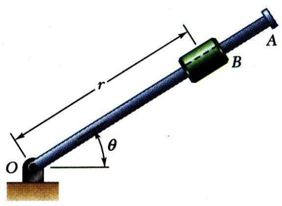
SOLUTION:

- Evaluate time t for $\theta = 30°$.
- Evaluate radial and angular positions, and first and second derivatives at time t.
- Calculate velocity and acceleration in cylindrical coordinates.
- Evaluate acceleration with respect to arm.

$$\vec{v} = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

Sample Problem 11.12



SOLUTION:

- Evaluate time t for $\theta = 30°$.

$$\theta = 0.15t^2$$
$$= 30° = 0.524\,\text{rad} \qquad t = 1.869\,\text{s}$$

- Evaluate radial and angular positions, and first and second derivatives at time t.

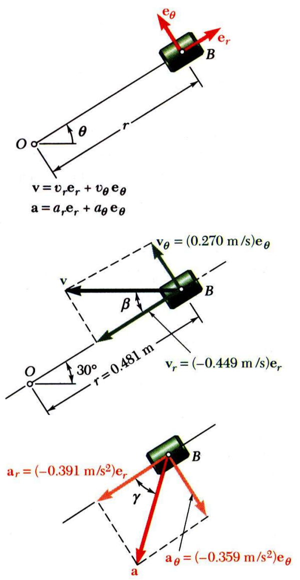
$$r = 0.9 - 0.12t^2 = 0.481\,\text{m}$$
$$\dot{r} = -0.24t = -0.449\,\text{m/s}$$
$$\ddot{r} = -0.24\,\text{m/s}^2$$

$$\theta = 0.15t^2 = 0.524\,\text{rad}$$
$$\dot{\theta} = 0.30t = 0.561\,\text{rad/s}$$
$$\ddot{\theta} = 0.30\,\text{rad/s}^2$$

$$\vec{v} = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

Sample Problem 11.12



- Calculate velocity and acceleration.

$$v_r = \dot{r} = -0.449\,\text{m/s}$$

$$v_\theta = r\dot{\theta} = (0.481\,\text{m})(0.561\,\text{rad/s}) = 0.270\,\text{m/s}$$

$$v = \sqrt{v_r^2 + v_\theta^2} \qquad \beta = \tan^{-1}\frac{v_\theta}{v_r}$$

$$\boxed{v = 0.524\,\text{m/s} \qquad \beta = 31.0°}$$

$$a_r = \ddot{r} - r\dot{\theta}^2$$
$$= -0.240\,\text{m/s}^2 - (0.481\,\text{m})(0.561\,\text{rad/s})^2$$
$$= -0.391\,\text{m/s}^2$$

$$a_\theta = r\ddot{\theta} + 2\dot{r}\dot{\theta}$$
$$= (0.481\,\text{m})(0.3\,\text{rad/s}^2) + 2(-0.449\,\text{m/s})(0.561\,\text{rad/s})$$
$$= -0.359\,\text{m/s}^2$$

$$a = \sqrt{a_r^2 + a_\theta^2} \qquad \gamma = \tan^{-1}\frac{a_\theta}{a_r}$$

$$\boxed{a = 0.531\,\text{m/s} \qquad \gamma = 42.6°}$$

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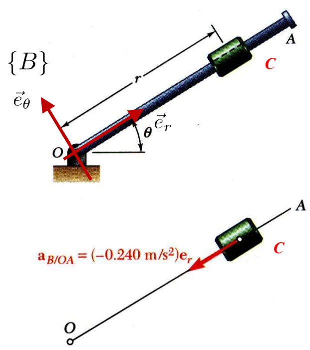
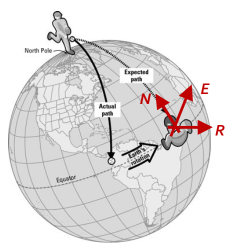
Sample Problem 11.12





- Evaluate relative acceleration with respect to arm.
- Relative motion of the collar with respect to the arm <u>as observed in/within</u> the (rotating) arm.

  = by observer sitting on arm tracking within motion in rod w/o knowing s/he is rotating

- Motion of collar with respect to arm is defined by the coordinates along $\vec{e}_r, \vec{e}_\theta$ and thus rectilinear. $\boxed{\ddot{r} = -0.240\,\text{m/s}^2\ (\approx (\ddot{\vec{r}})_B)}$

- Expressed in body-frame *{B} (rotating together)*: expressed in the body-frame coordinates $\vec{e}_r, \vec{e}_\theta$

  $\Rightarrow$ *relative acceleration <u>as expressed in</u> {B}*

$$\vec{a}_{C/O} = \vec{a}_C - \vec{a}_O = \vec{a}_C$$

$$\vec{a}_{C/O} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta \Rightarrow \vec{a}_{C/O}^{\,B} = (\ddot{r} - r\dot{\theta}^2; r\ddot{\theta} + 2\dot{r}\dot{\theta})$$

$\neq$ *relative acceleration <u>as observed in</u> {B}*

$$\vec{a}_{C/O}^{\,obs.B} = \ddot{r}\vec{e}_r = \frac{d^2}{dt^2}(r\vec{e}_r)_{\{B\}}$$

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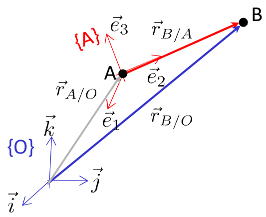
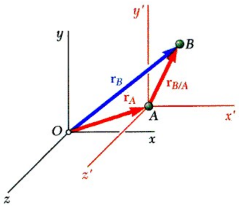
Relative Motion as Observed in {A}





- Position of B can be written by:

$$\boxed{\vec{r}_{B/O} = \vec{r}_{A/O} + \vec{r}_{B/A}}$$

where $\{O\}$ is a fixed frame with basis vectors $\vec{\imath}, \vec{\jmath}, \vec{k}$, $\{A\}$ is a moving frame w/ basis vectors $\vec{e}_1, \vec{e}_2, \vec{e}_3$ and $\vec{r}_{B/A}$ is the relative position of B from A (or w.r.t. $\{A\}$).

- Differentiating twice,

$\boxed{\vec{v}_{B/O} = \vec{v}_{A/O} + \vec{v}_{B/A}}$ $\quad \vec{v}_{B/A} = \dot{\vec{r}}_{B/A}$: relative velocity of B from A (or w.r.t. $\{A\}$)

$\boxed{\vec{a}_{B/O} = \vec{a}_{A/O} + \vec{a}_{B/A}}$ $\quad \vec{a}_{B/A} = \ddot{\vec{r}}_{B/A}$: relative acceleration of B from A (or w.r.t. $\{A\}$)

- All the vectors can be expressed in $\{O\}$ or in $\{A\}$ $\Rightarrow$

$\boxed{\vec{r}_{B/A} = r^x_{B/A}\vec{\imath} + r^y_{B/A}\vec{\jmath} + r^z_{B/A}\vec{k} = r^2_{B/A}\vec{e}_2}$ → they are real and the same

- Relative motions as observed in $\{A\}$ (cf. observed on Earth, *only along* $\vec{e}_1, \vec{e}_2, \vec{e}_3$):

position as observed in $\{A\}$: $\boxed{r^2_{B/A}\vec{e}_2}$

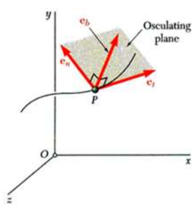
velocity as observed in $\{A\}$: $\boxed{\dot{r}^2_{B/A}\vec{e}_2}$ $(= \dot{r}\vec{e}_r \neq \dot{\vec{r}}_{B/A} = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta)$

acceleration as observed in $\{A\}$: $\boxed{\ddot{r}^2_{B/A}\vec{e}_2}$ $(\ddot{r}\vec{e}_r \neq \ddot{\vec{r}}_{B/A} = (\ddot{r} - r\dot{\theta}^2)\vec{e}_r + (r\ddot{\theta} + 2\dot{r}\dot{\theta})\vec{e}_\theta)$

they are not the same, since basis vectors are rotating (cf. Cartesian)

Key Equations

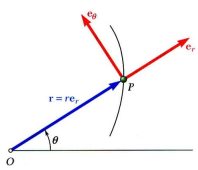
- Tangential and normal components given a path



$$\vec{v} = v\vec{e}_t$$

$$\vec{a} = \frac{dv}{dt}\vec{e}_t + \frac{v^2}{\rho}\vec{e}_n \qquad a_t = \frac{dv}{dt} \qquad a_n = \frac{v^2}{\rho}$$

- Radial and transversal of polar coordinates



$$\vec{v} = \dot{r}\,\vec{e}_r + r\dot{\theta}\,\vec{e}_\theta$$

$$\vec{a} = \left(\ddot{r} - r\dot{\theta}^2\right)\vec{e}_r + \left(r\ddot{\theta} + 2\dot{r}\dot{\theta}\right)\vec{e}_\theta$$

- Relative motion w.r.t. (as observed in) $\{B\}$: