445.204

Introduction to Mechanics of Materials

(재료역학개론)

Chapter 11: Transformation of stress

(2D only)

(Ch. 7.5 in Shames)

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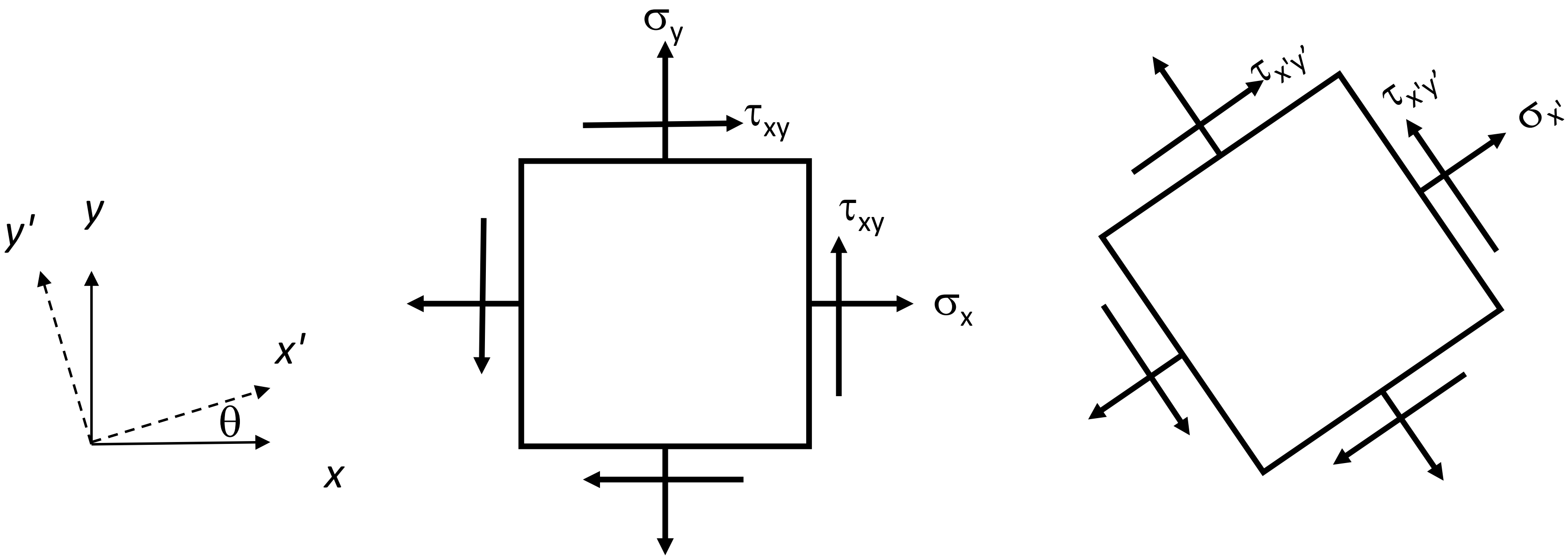
Objectives of the chapter

- The most common problems in engineering mechanics involve “transformation of axes”
- Question: stresses are known in x-y plane (or x-y coordinate). Then, what is the stress in the coordinate rotated about θ degrees?
- Stress (and strain) is not coordinate dependent, but they have different components if the coordinates are different!



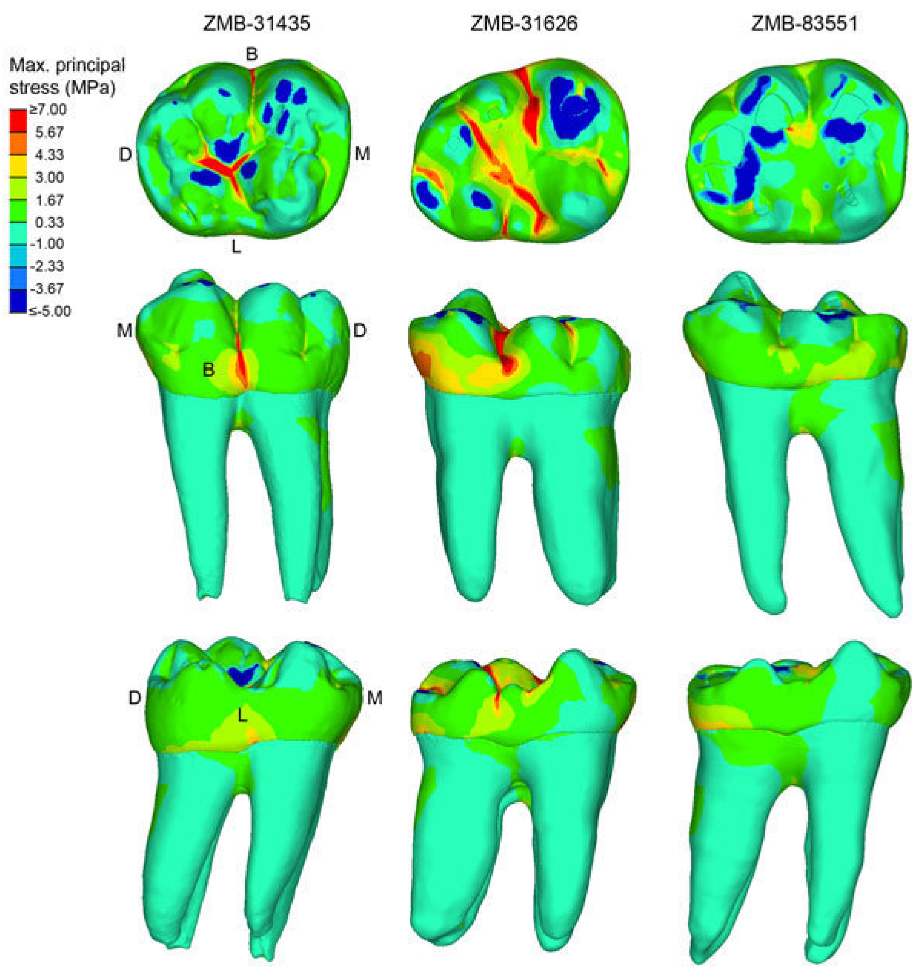
Objectives of the chapter

Two pieces of wood, cut at an angle, and glued together. The wood is being pulled apart by a tensile force *P*.

How do we know if the glued joint can sustain the resultant stress that this force produces?

(Assume that we know the tensile and shear properties of the glue)

Objectives of the chapter

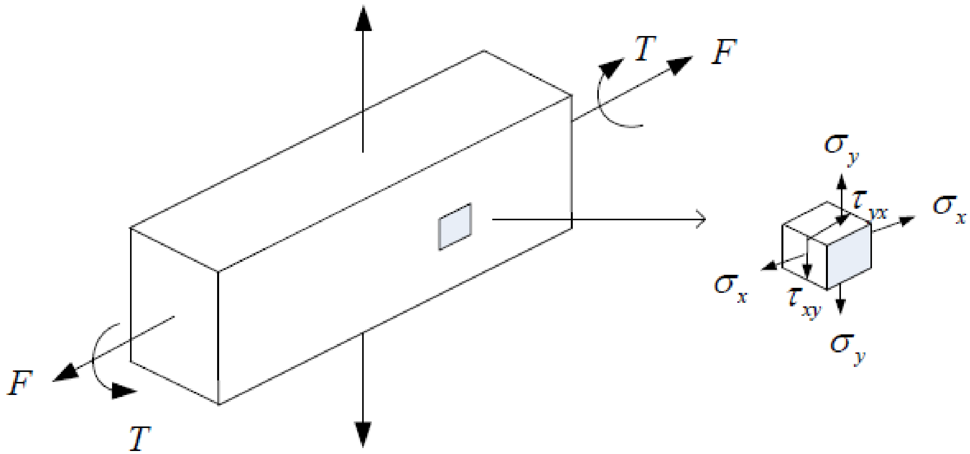


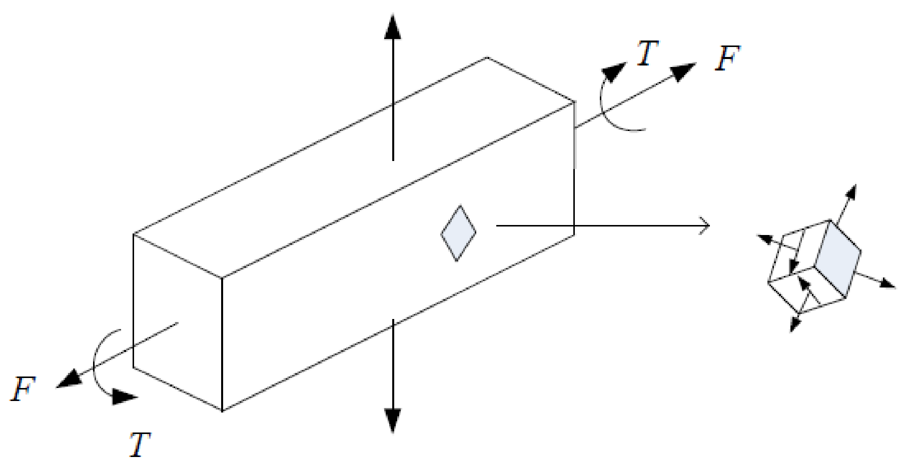
Maximal principal stress distribution observed in three gorilla teeth of an unworn (left), a lightly worn (middle) and a worn (right) condition

Researchers at the Max Planck Institute for Evolutionary Anthropology in Leipzig, Germany, and the Senckenberg Research Institute in Frankfurt am Main, Germany, have conducted stress analyses on gorilla teeth of differing wear stages. Their findings show that different features of the occlusal surface antagonize tensile stresses in the tooth to tooth contact during the chewing process. They further show that tooth wear with its loss of dental tissue and the reduction of the occlusal relief decreases tensile stresses in the tooth. The result, however, is that food processing becomes less effective. Thus, when the condition of the occlusal surface changes during an individual's lifetime due to tooth wear, the biomechanical requirements on the existing dental material change as well – an evolutionary compromise for longer tooth preservation.

Transformation of Stresses: 2D

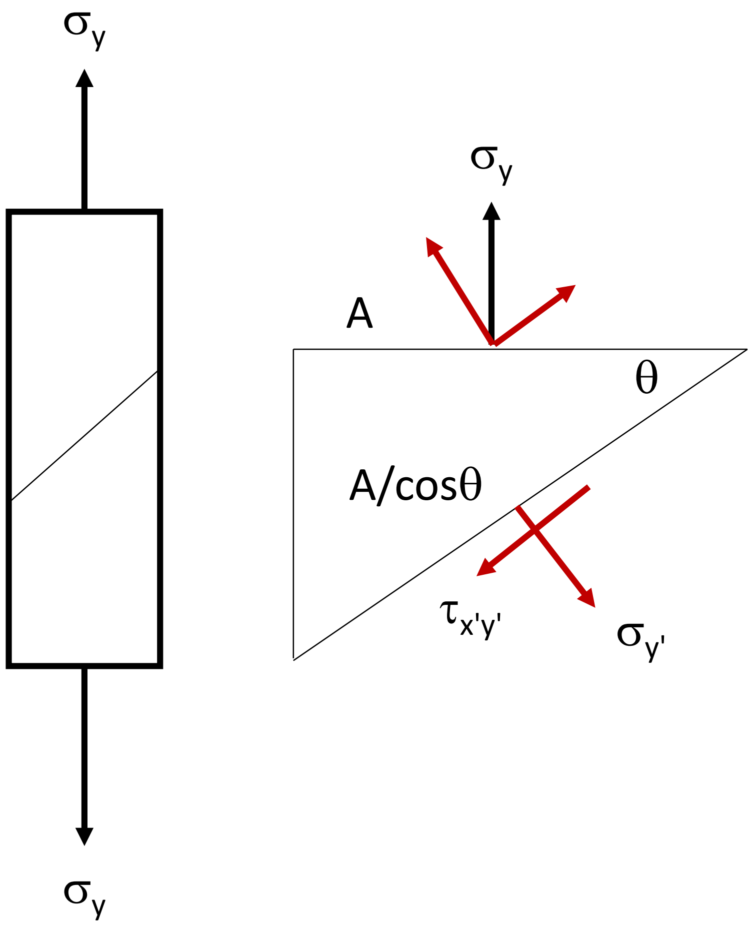
Consideration of static equilibrium





Transformation of Stresses: 2D

1D uniaxial tension



<u>Force equilibrium in the y' direction (normal to the y' plane)</u>

$$(\sigma_y A)\cos\theta - \sigma_{y'}(A/\cos\theta) = 0$$

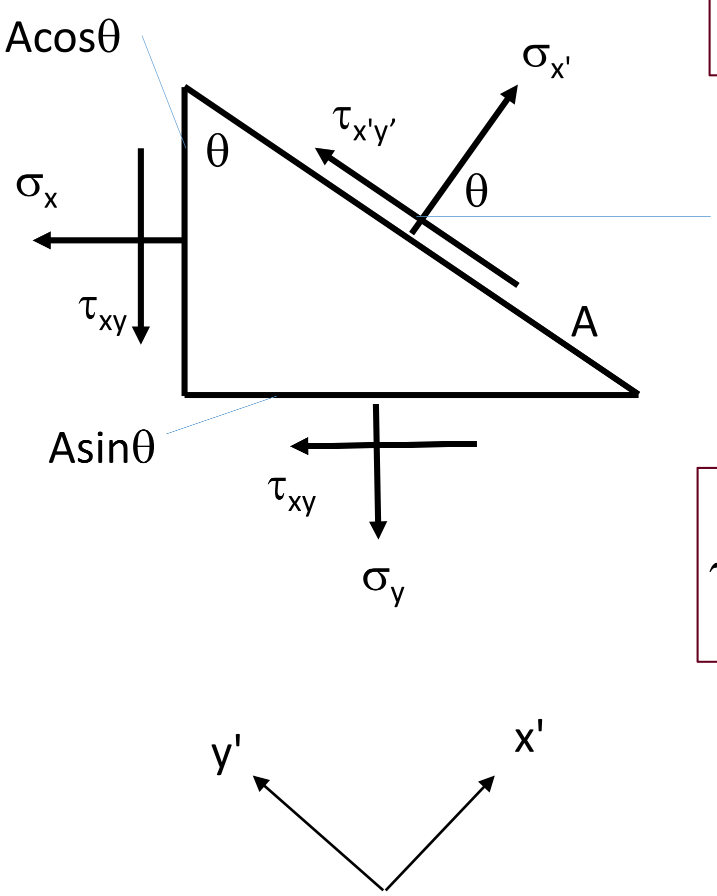
$$\boxed{\sigma_{y'} = \sigma_y \cos^2\theta}$$

<u>Force equilibrium in the tangential direction</u>

$$\boxed{\tau_{x'y'} = \sigma_y \sin\theta \cdot \cos\theta}$$

Transformation of Stresses: 2D, Direct approach

2D plane stress



Force equilibrium in the x' direction

$$\boxed{\sigma'_x = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta}$$

Force equilibrium in the y' direction

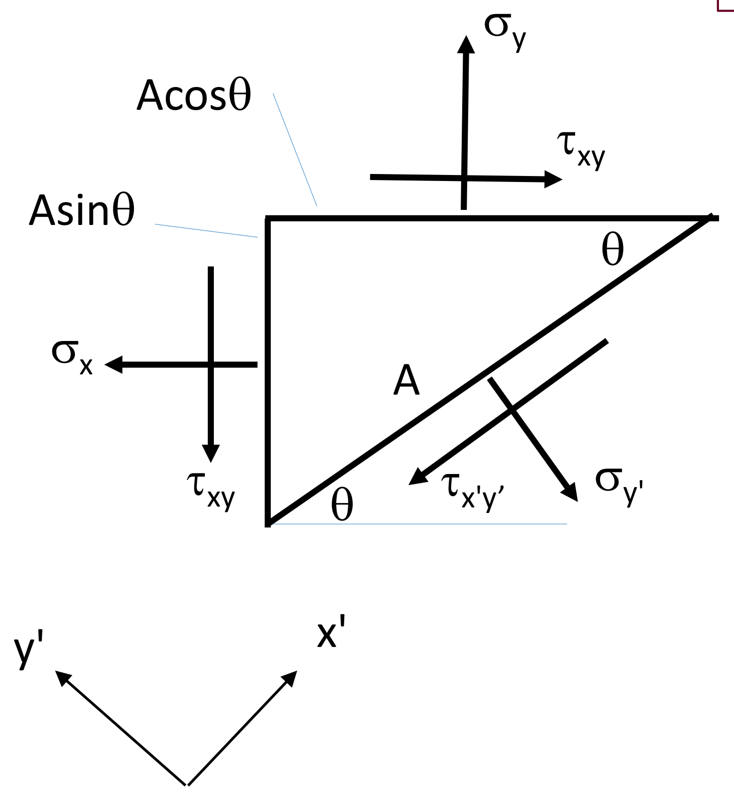
$$\boxed{\tau_{x'y'} = -\frac{\sigma_x}{2} \sin 2\theta + \frac{\sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta}$$

Transformation of Stresses: 2D, Direct approach

2D plane stress

Force equilibrium in the y' direction

$$\boxed{\sigma'_y = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta \cos\theta}$$



Transformation of Stresses: 2D, Direct approach

2D plane stress: summary

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta \cos\theta$$

$$\tau_{x'y'} = \left(\sigma_y - \sigma_x\right) \sin\theta \cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

Matrix form

$$\begin{pmatrix} \sigma_{x'} \\ \sigma_{y'} \\ \tau_{x'y'} \end{pmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{pmatrix} \sigma_x \\ \sigma_y \\ \tau_{xy} \end{pmatrix}$$

or $\boldsymbol{\sigma}' = \mathbf{A}\boldsymbol{\sigma}$

Transformation of Strain: 2D, Direct approach

$$\begin{pmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \frac{1}{2}\gamma_{x'y'} \end{pmatrix} = \begin{bmatrix} c^2 & s^2 & 2sc \\ s^2 & c^2 & -2sc \\ -sc & sc & c^2 - s^2 \end{bmatrix} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{pmatrix} = \mathbf{A} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{pmatrix}$$

$$\begin{pmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \gamma_{x'y'} \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{pmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \frac{1}{2}\gamma_{x'y'} \end{pmatrix} = \mathbf{R} \begin{pmatrix} \varepsilon_{x'} \\ \varepsilon_{y'} \\ \frac{1}{2}\gamma_{x'y'} \end{pmatrix} = \mathbf{RA} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \frac{1}{2}\gamma_{xy} \end{pmatrix} = \mathbf{RAR}^{-1} \begin{pmatrix} \varepsilon_x \\ \varepsilon_y \\ \gamma_{xy} \end{pmatrix}$$

or

$$\boldsymbol{\varepsilon}' = \mathbf{RAR}^{-1}\boldsymbol{\varepsilon}$$

$$\boxed{\begin{aligned} \varepsilon_{x'} &= \varepsilon_x \cos^2\theta + \varepsilon_y \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta \\ \varepsilon_{y'} &= \varepsilon_x \sin^2\theta + \varepsilon_y \cos^2\theta - \gamma_{xy} \sin\theta\cos\theta \\ \gamma_{x'y'} &= (\varepsilon_y - \varepsilon_x)\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta) \end{aligned}}$$

Note:

$\sigma \to \varepsilon$

$\tau \to \gamma/2$

Transformation of Stresses: 2D

$$\sigma_{x'} = \sigma_x \cos^2\theta + \sigma_y \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

$$\sigma_{y'} = \sigma_x \sin^2\theta + \sigma_y \cos^2\theta - 2\tau_{xy} \sin\theta \cos\theta$$

$$\tau_{x'y'} = \left(\sigma_y - \sigma_x\right) \sin\theta \cos\theta + \tau_{xy}\left(\cos^2\theta - \sin^2\theta\right)$$

$$\Downarrow$$

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2} \cos 2\theta - \tau_{xy} \sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2} \sin 2\theta + \tau_{xy} \cos 2\theta$$

Trigonometric Identities

$$\sin 2\theta = 2 \sin\theta \cos\theta$$

$$\sin^2\theta = \frac{1 - \cos 2\theta}{2}$$

$$\cos^2\theta = \frac{1 + \cos 2\theta}{2}$$

Transformation of Stresses: Example 7.1

Beams with rectangular cross sections subjected to loading in a plane of symmetry may be considered as being in a state of plane stress. The results of a given beam analysis indicate that, at a certain point P in the beam (Fig. 7.4) the nonzero stress components are

$$\tau_{xx} = 3000 \text{ psi} \qquad \tau_{yy} = 0 \qquad \tau_{xy} = -400 \text{ psi}$$

What are the nonzero stresses for axes $x'y'$ rotated 30° clockwise from xy at point P?

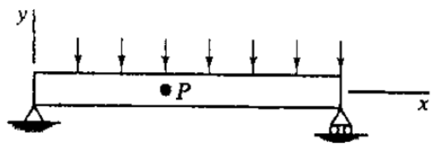


Figure 7.4. Beam with rectangular cross section.

An infinitesimal rectangle depicting interfaces for reference xy in the neighborhood of P is shown in Fig. 7.5(a), while an infinitesimal rectangle is shown in Fig. 7.5(b) depicting interfaces for $x'y'$ in the same neighborhood.

From Eqs. (7.6) and (7.7) we have

$$\tau_{x'x'} = \frac{3000 + 0}{2} + \frac{3000 - 0}{2} \cos(-60°) + (-400) \sin(-60°)$$

$$\tau_{x'x'} = 2596 \text{ psi}$$

$$\tau_{y'y'} = \frac{3000 + 0}{2} - \frac{3000 - 0}{2} \cos(-60°) - (-400) \sin(-60°)$$

$$\tau_{y'y'} = 404 \text{ psi}$$

Finally, from Eq. (7.8) we get

$$\tau_{x'y'} = \frac{0 - 3000}{2} \sin(-60°) + (-400) \cos(-60°)$$

$$\tau_{x'y'} = 1099 \text{ psi}$$

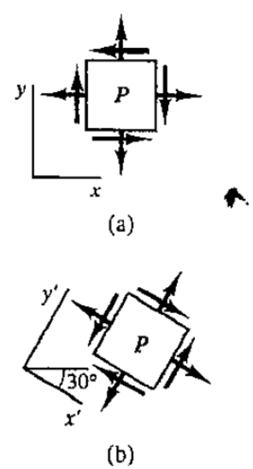
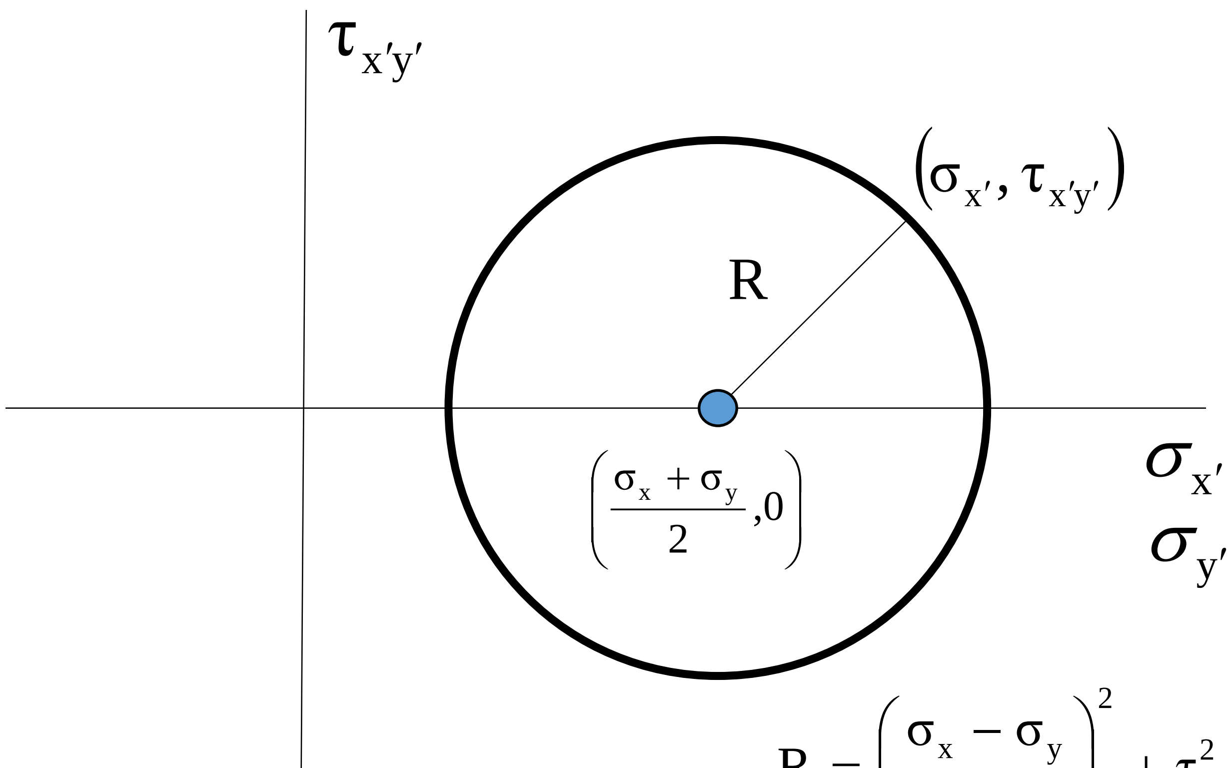


Figure 7.5. Stress at P for different axes.

Transformation of Stresses: 2D, Mohr Circle

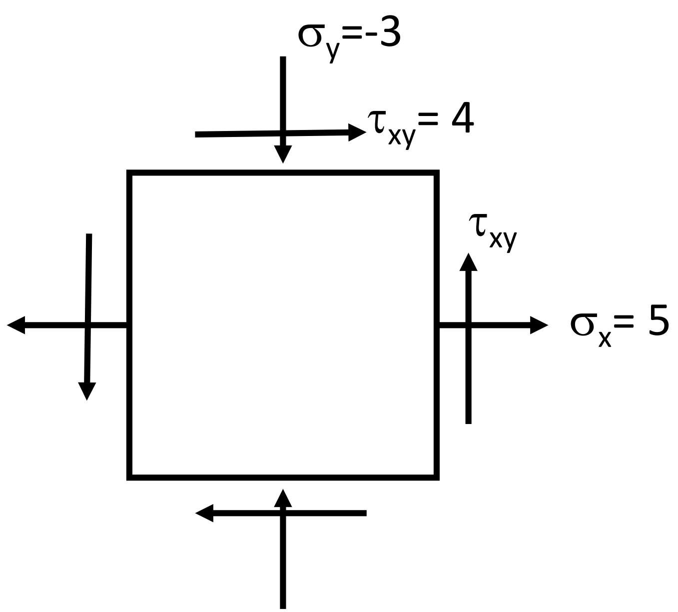
$$\left(\sigma_{x'} - \frac{\sigma_x + \sigma_y}{2}\right)^2 + \tau_{x'y'}{}^2 = \left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}{}^2$$



$$R = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \tau_{xy}^2}$$

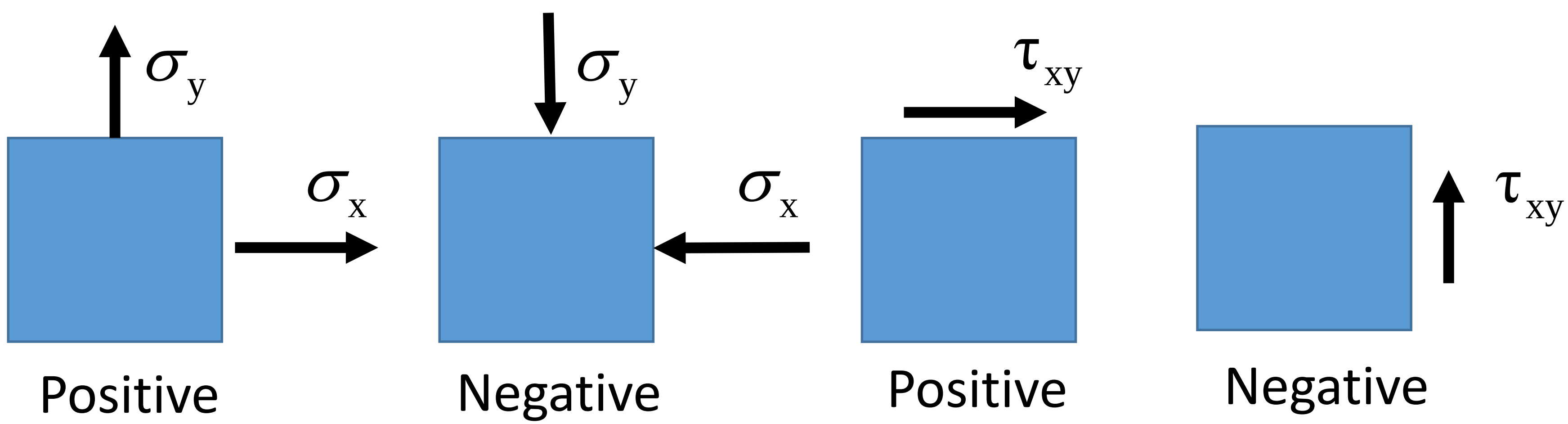
Transformation of Stresses: 2D, Mohr Circle

Steps to draw Mohr's circle: illustration



Step 1:

- Consider a shear stress acting in a clockwise-rotation sense as being positive (+), and counter-clockwise as negative (-)
- The shear stresses on the x and y faces have opposite signs
- The normal stresses are positive in tension and negative in compression as usual



Transformation of Stresses: 2D, Mohr Circle

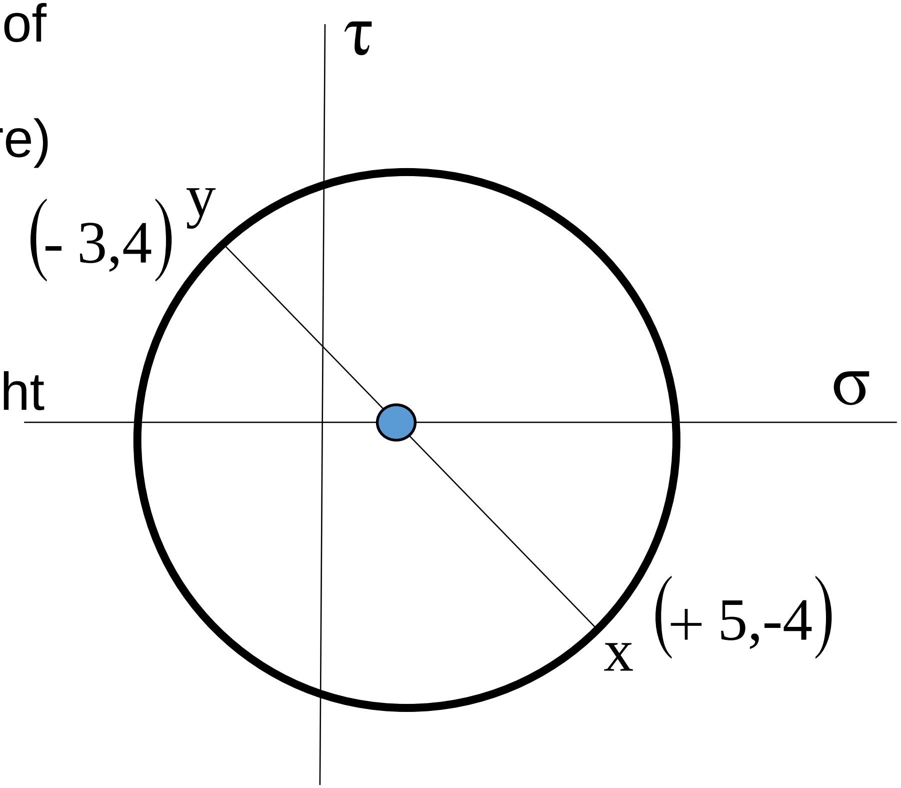
Steps to draw Mohr's circle: illustration

Step 2:

- Construct a graph with t as the ordinate (y axis) and σ as abscissa (x axis).
- Plot the stresses on the x and y faces of the stress as two points on this graph(follow the sign convention before)

Step 3:

- Connect these two points with a straight line
- Draw as circle with the line as a diameter

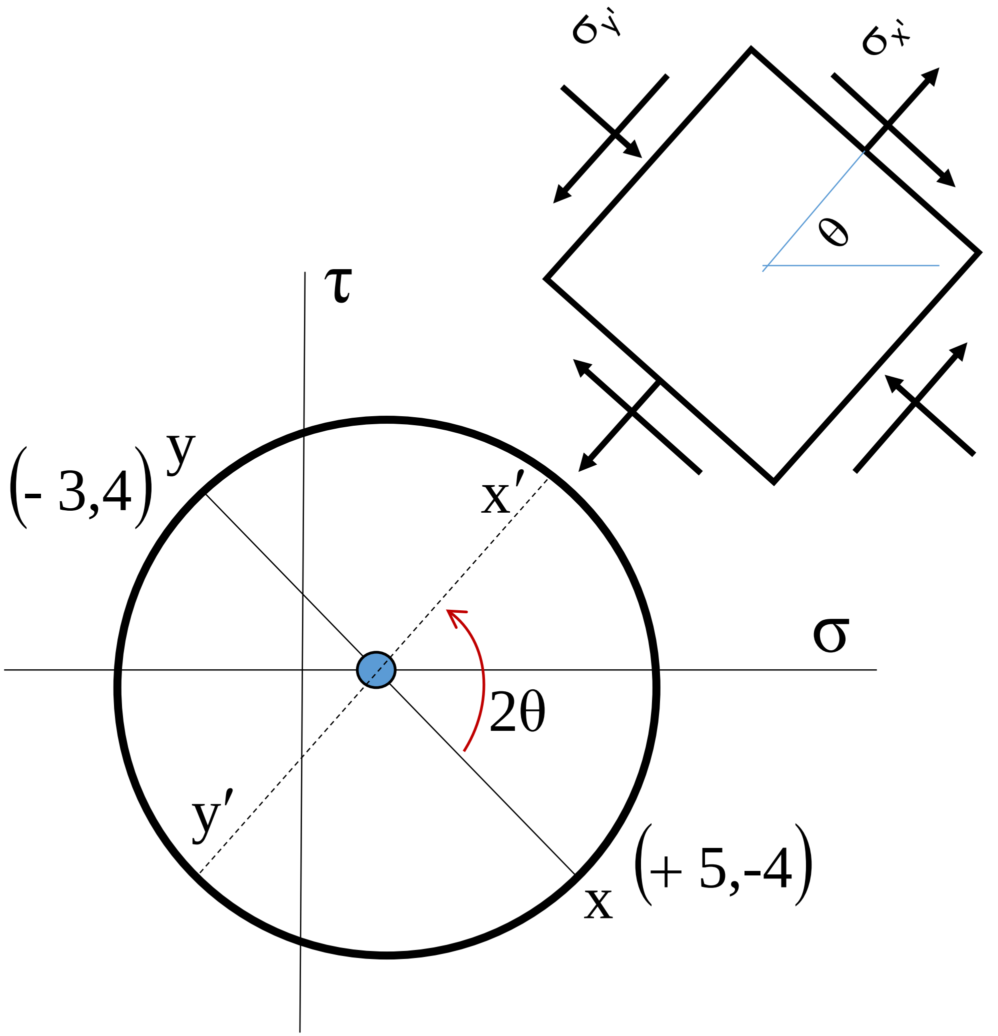


Transformation of Stresses: 2D, Mohr Circle

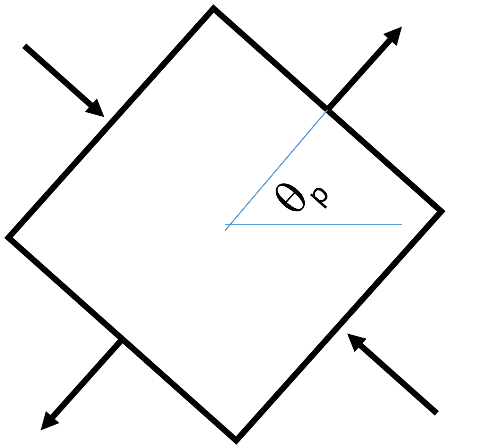
Steps to draw Mohr's circle: illustration

Step 4:

- Determine stresses on a square that has been rotated through an angle θ with respect to the original square
- Rotate the line in the same direction through 2θ. This new end points of the line are labeled as x' and y'.

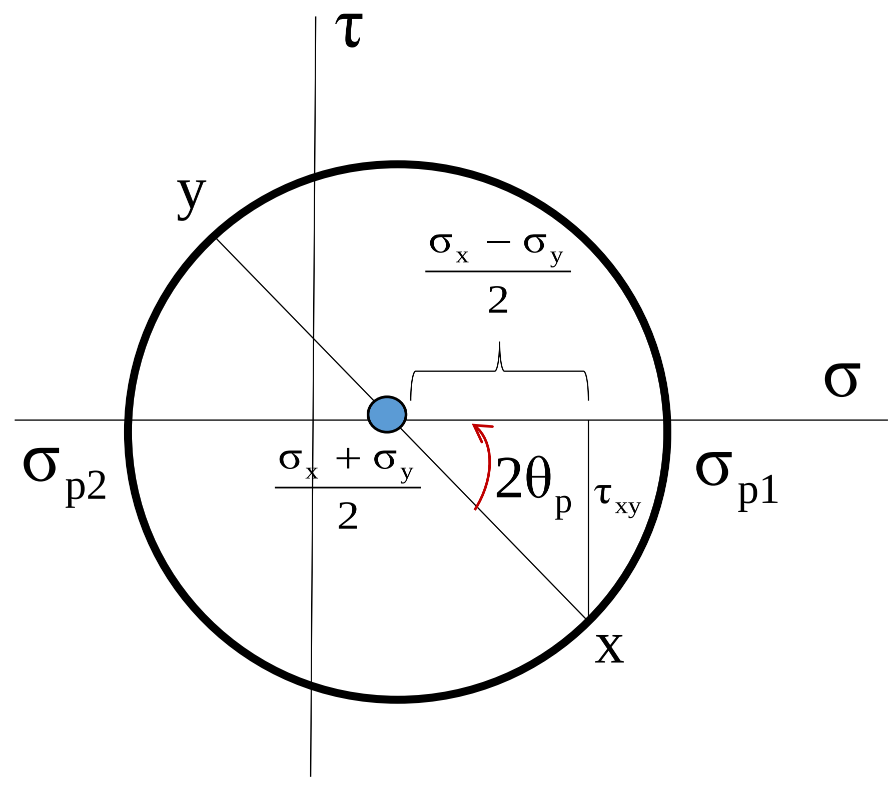


Transformation of Stresses: Principal stress



- Normal stresses become maximum values and the shear stresses are zero

- These normal stresses are called “principal” stresses, σ_{p1} and σ_{p2}



Transformation of Stresses: Principal stress

$$\tan 2\theta_p = \frac{\tau_{xy}}{\left(\sigma_x - \sigma_y\right)/2}$$

By Pythagorean construction

$$\boxed{\sigma_{p1,p2} = \frac{\left(\sigma_x + \sigma_y\right)}{2} \pm \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}}$$

Principal stresses

$$\boxed{\tau_{max} = \frac{\sigma_{p1} - \sigma_{p2}}{2} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}}$$

Maximum shear stress

"The maximum shear are 90 º away from the principal stress points on the Mohr's circle"

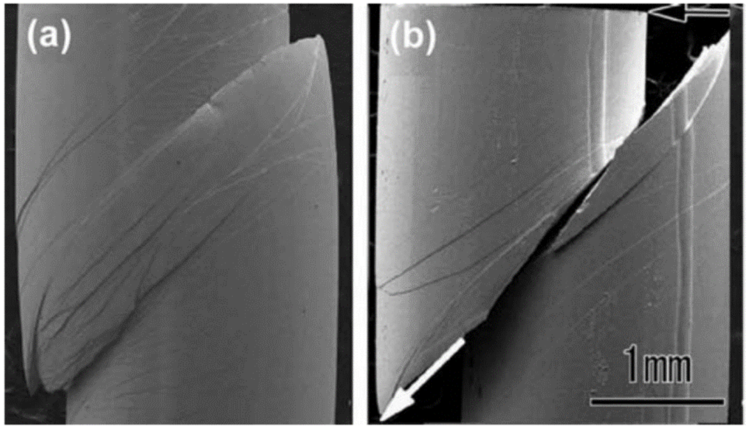
Transformation of Stresses: 2D, Mohr Circle

Maximum shear stress and its plane

Maximum shear stress

$$\tau_{max} = \frac{\sigma_{p1} - \sigma_{p2}}{2} = \sqrt{\left(\frac{\sigma_x - \sigma_y}{2}\right)^2 + \left(\tau_{xy}\right)^2}$$

“ In the tensile specimen, the maximum shear are 45 ^{o} away from the loading direction which is the direction of principal stress”



Transformation of Stresses: Invariant

$$\sigma_{x'} = \frac{\sigma_x + \sigma_y}{2} + \frac{\sigma_x - \sigma_y}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'} = \frac{\sigma_x + \sigma_y}{2} - \frac{\sigma_x - \sigma_y}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_x - \sigma_y}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$\Downarrow$$

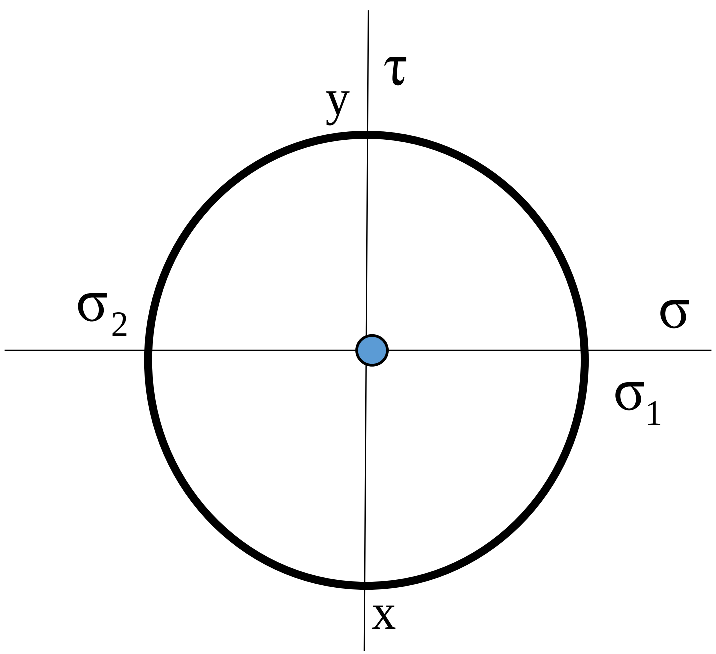
$$\boxed{\sigma_{x'} + \sigma_{y'} = \sigma_x + \sigma_y}$$

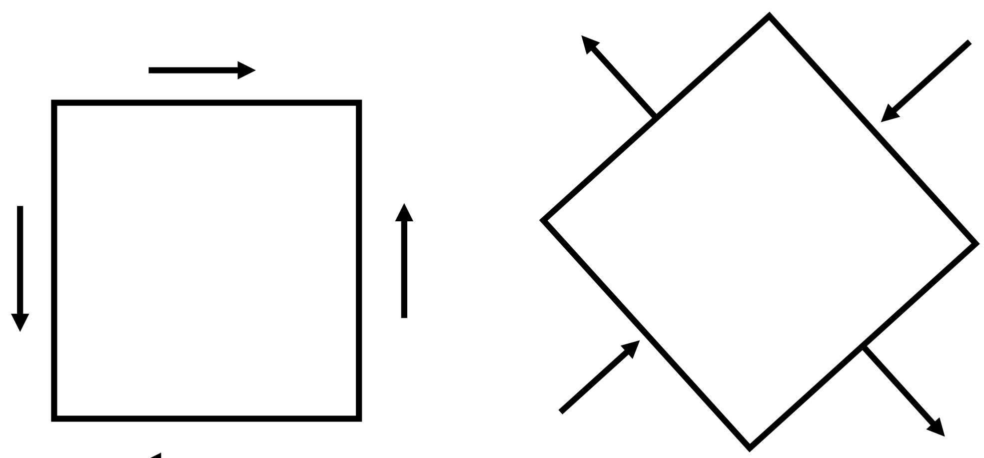
Transformation of Stresses: 2D, Mohr Circle

Pure shear

$$\frac{\sigma_x + \sigma_y}{2} = 0$$

When normal stresses vanish on the plane of maximum shear
Example: the stress state by the simple torsion





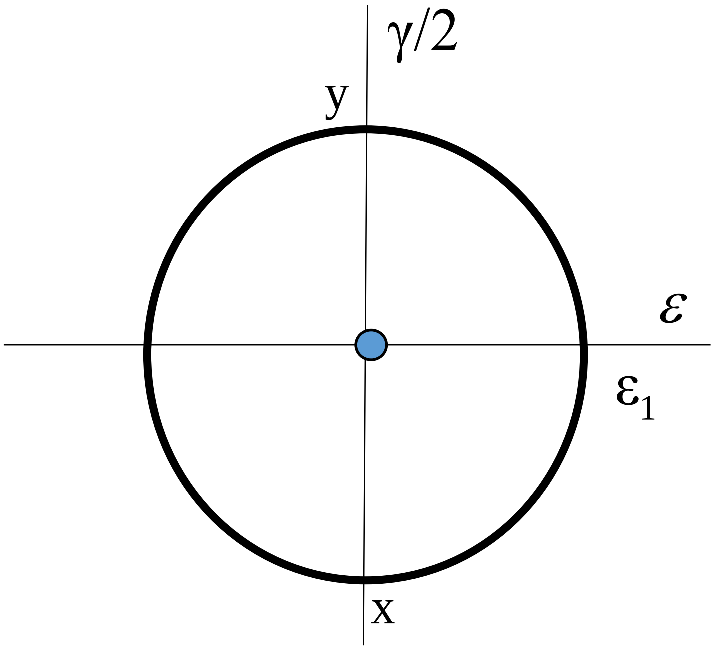
These two stresses are equivalent

Transformation of Stresses: 2D, Mohr Circle

Under Pure shear

$$\tau = G\gamma$$ Hooke's law for shear

$$\varepsilon_1 = \frac{\tau}{2G}$$ From the Mohr circle



Principal strain is related to the principal stress

$$\varepsilon_1 = \frac{1}{E}\left(\sigma_1 - \nu\sigma_2\right)$$

From the Mohr's circle

$$\frac{\tau}{2G} = \frac{1}{E}\left(\tau - \nu\left(-\tau\right)\right)$$

$$\boxed{G = \frac{E}{2\left(1+\nu\right)}}$$