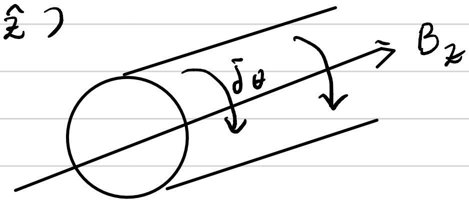
5/17 Stability assessment by energy principle

1. θ - pinch stability $(\hat{r}, \hat{\theta}, \hat{z})$



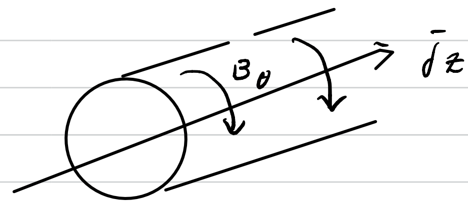
$\vec{j} = j_\theta \hat{\theta} \qquad \vec{B} = B_z \hat{z}$

$\rightarrow j_{\parallel} = 0. \qquad \vec{\kappa} = 0$

$\rightarrow \delta W_F$ is positive definite

$\rightarrow$ θ - pinch is always stable

2. Z - pinch stability



$\vec{j} = j_z \hat{z} \qquad \vec{B} = B_\theta \hat{\theta}$

$\rightarrow j_{\parallel} = 0$, but $\vec{\kappa} \neq 0 = \hat{\theta} \cdot \vec{\nabla} \hat{\theta} = -\frac{\hat{r}}{r}$

(0) Equilibrium $\quad \frac{d}{dr}\left(p + \frac{B_\theta^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$

perturbation $\quad \vec{x} \rightarrow \vec{x} + \vec{\xi}(\vec{x})$

Let $\vec{\xi}(\vec{x}) = \vec{\xi}_{mk}(r)\, e^{i(m\theta + kz)}$

$$= \left(\xi(r)\hat{r} + \xi_\theta(r)\hat{\theta} + \xi_z(r)\hat{z}\right) e^{i(m\theta + kz)}$$

$\frac{\partial}{\partial r} = \frac{d}{dr} = {}'$

(1) $\vec{\nabla} \cdot \vec{\xi} = \frac{1}{r}(r\xi)' + \frac{im}{r}\xi_\theta + ik\xi_z$

$\xi_{\parallel} = \xi_\theta = \frac{i}{m}\left[(r\xi)' + ikr\xi_z\right]$ for $m \neq 0$

can eliminate $\vec{\nabla} \cdot \vec{\xi} = 0$.

$\vec{\nabla} \cdot \vec{\xi} = \vec{\nabla} \cdot \vec{\xi}_\perp = \frac{1}{r}(r\xi)' + ik\xi_z \neq 0$ for $m = 0$

(2) $\vec{Q}_\perp = \left(\vec{\nabla} \times (\vec{\xi}_\perp \times B_\theta \hat{\theta})\right)_\perp$

$= \Big(\vec{\nabla} \times \underbrace{(\xi B_\theta \hat{z} - \xi_z B_\theta \hat{r})}_{\vec{A}}\Big)_\perp$

$$= \left(\frac{1}{r}\frac{\partial A_z}{\partial \theta} - \cancel{\frac{\partial A_\theta}{\partial z}},\ 0,\ \frac{1}{r}\frac{\partial}{\partial r}\cancel{(rA_\theta)} - \frac{1}{r}\frac{\partial A_r}{\partial \theta} \right)$$

$$= \frac{imB_\theta}{r}(\xi \hat{r} + \xi_z \hat{z})$$

(3) $\vec{\nabla}\cdot\vec{\xi}_\perp + 2\vec{\xi}_\perp\cdot\vec{\kappa} = \frac{1}{r}(r\xi)' + ik\xi_z - \frac{2}{r}\xi = r\left(\frac{\xi}{r}\right)' + ik\xi_z$

(4) $-2\mu_0(\vec{\xi}_\perp\cdot\vec{\nabla}p)(\vec{\xi}_\perp^*\cdot\vec{\kappa}) = \frac{2\mu_0 p'}{r}|\xi|^2$

3. z-pinch $m \neq 0$ δW_F minimization and stability

$$\delta W_F = \frac{1}{2\mu_0}\int_0^{2\pi R_0} dz \int_0^{2\pi} d\theta \int_0^a r\,d\{\ \}\,dr = 2\pi R_0 \int_0^a rW(r)\,dr$$

$$W(r) = \underset{(2)}{\frac{m^2 B_\theta^2}{\mu_0 r^2}\left(|\xi|^2 + |\xi_z|^2\right)} + \underset{(3)}{\frac{B_0^2}{\mu_0}\left| r\left(\frac{\xi}{r}\right)' + ik\xi_z \right|^2} + \underset{(4)}{\frac{2p'}{r}|\xi|^2}$$

$\propto A\xi_z^2 + B\xi_z + C \quad \propto (\xi_z - D)^2 + E$

so, minimizing complete square for ξ_z by

$$\xi_z = \frac{ikr^3}{m^2 + k^2 r^2}\left(\frac{\xi}{r}\right)'$$

$$W(r) = \left(2rp' + \frac{m^2 B_\theta^2}{\mu_0}\right)\left|\frac{\xi}{r}\right|^2 + \frac{m^2 r^2 B_\theta^2}{\mu_0(m^2 + k^2 r^2)}\left|\left(\frac{\xi}{r}\right)'\right|^2$$

(can be minimized by variational principle)

$\longrightarrow \xi(r).\ \xi_z(r).\ \xi_\theta(r)$

Here, just look at the most unstable mode $k \to \infty$

$$W(r) \longrightarrow \left(2rp' + \frac{m^2 B_\theta^2}{\mu_0}\right)\left|\frac{\xi}{r}\right|^2$$

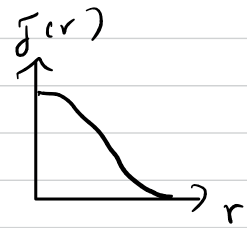
$\therefore$ z-pinch is stable for $m \neq 0$ iff

$$\boxed{rp' + \frac{m^2 B_\theta^2}{2\mu_0} > 0}$$ Kadomtsev (1966)

By eliminating $p' = -\frac{1}{2\mu_0}(B_\theta^2)' - \frac{B_\theta^2}{\mu_0 r}$

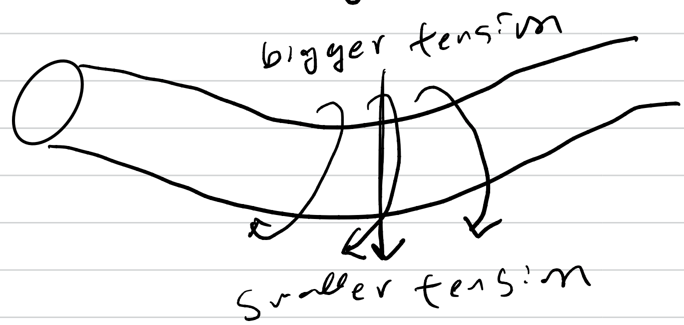
$$\frac{r^2}{B_\theta^2}\left(\frac{B_\theta}{r}\right)' < \frac{1}{2}(m^2-4) \qquad \text{(I)}$$

$$\frac{1}{B_\theta^2}(rB_\theta^2)' < m^2-1 \qquad \text{(II)}$$



- For $j(r)$ decreasing towards edge
- (I) $\left(\frac{B_\theta}{r}\right)' < 0 \rightarrow$ stable for $m \geq 2$ mode
- (II) $\begin{cases} B_\theta \propto \frac{1}{r} & \text{near the edge} \\ B_\theta \propto r & \text{near the core} \end{cases}$ $\quad j(r) = A(1-r^2/a^2)$

So, $(rB_\theta^2)' > 0$ near the core,
making $m=1$ unstable.



unfavorable curvature (∇p-driven) vs. line bending

4. Z-pinch $m=0$ δW_F minimization and stability

$Q_\perp = 0 \quad \vec{\nabla}\cdot\vec{\xi} = \vec{\nabla}\cdot\vec{\xi}_\perp \neq 0$

$$W(r) = \underbrace{\gamma p\left|\frac{(r\xi)'}{r} + ik\xi_z\right|^2}_{(1)} + \underbrace{\frac{B_\theta^2}{\mu_0}\left|r\left(\frac{\xi}{r}\right)' + ik\xi_z\right|^2}_{(3)} + \underbrace{\frac{2p'}{r}|\xi|^2}_{(4)}$$

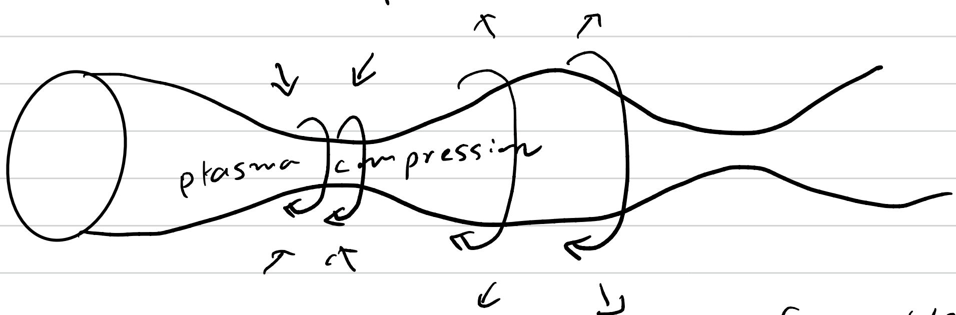
Again minimizing w.r.t ξ_z.

$$= \left[\left(\frac{4\gamma B_\theta^2}{B_\theta^2 + \mu_0 \gamma p}\right)p + 2rp'\right]\left(\frac{\xi}{r}\right)^2$$

∴ z-pinch is stable for $m=0$ iff

$$\boxed{-\frac{rp'}{p} < \frac{2\gamma B_\theta^2}{B_\theta^2 + \mu_0 \gamma p}}$$, Kadomtsev (1966)

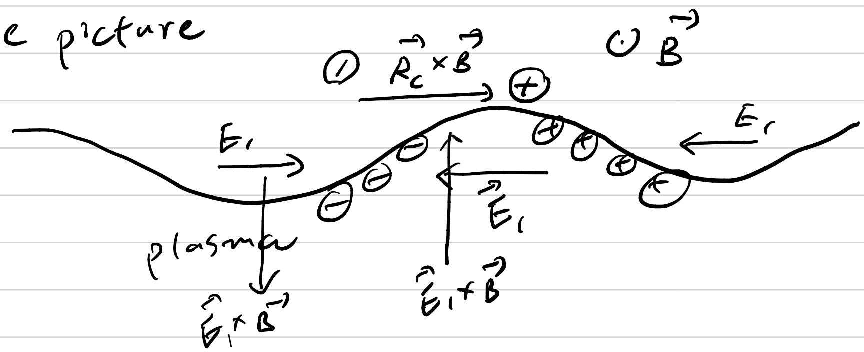
if $p \ll B_\theta^2/2\mu_0$ $\quad -\frac{rp'}{p} < 2\gamma = \frac{10}{3}$



"sausage" instability,

Unfavorable curvature vs. plasma compression

In particle picture



"Interchange" instability (no line bending, $\vec{Q}_\perp = 0$, $j_\parallel = 0$)

- Sausage - Z-pinch
- Flute - Mirror
- Localized interchange - Tokamak
- Rayleigh-Taylor - Fluids
- Gravitational R-T - Astrophysics