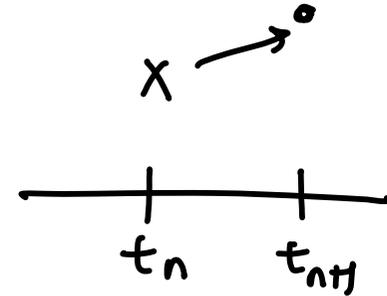


$$y' = f(y, t)$$

① Euler method (EE or forward Euler)

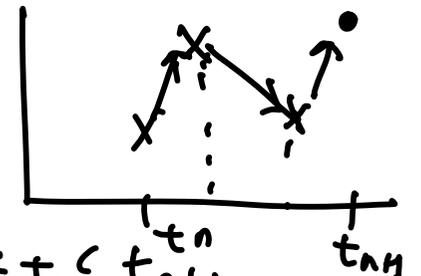
$$EE: \frac{y_{n+1} - y_n}{h} = f(y_n, t_n)$$



* Runge-Kutta method

y_{n+1} is obtained in terms of $y_n, f(y_n, t_n)$

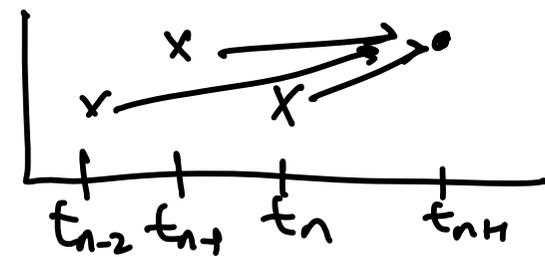
and values of f at intermediate times, $t_n \leq t \leq t_{n+1}$.



* Multi-step method ← not self-starting

uses information from $t \leq t_n$.

y_{n-1} @ t_{n-1} , y_{n-2} @ t_{n-2} , ...



* Explicit vs. implicit methods

↓
EE: $y_{n+1} = y_n + hf(y_n, t_n)$

→ involves $f(y_{n+1}, t_{n+1})$.

ex) implicit Euler method

$$y_{n+1} = y_n + hf(y_{n+1}, t_{n+1})$$

may involve solving a nonlinear algebraic eq. but provides better numerical stability.

4.2 Numerical stability

it is possible for numerical sol. of a diff'l eq to blow up (grow unbounded) even though the exact sol. is well behaved.

→ We seek parameters of the numerical method such as $h (= \Delta t)$ so that numerical sol. is well behaved.

- Stable numerical method (A-stable numerical method)
(absolutely stable " ")

Numerical sol. does not blow up w/ any choice of parameters.

- Unstable numerical method

Numerical sol. always blows up irrespective of the choice of parameters.

- Conditionally stable numerical method

Numerical sol. is well behaved w/ some choice of parameters.

$$\boxed{y' = f(y, t)}$$

Linear stability analysis λ

$$y' = f(y, t) = f(y_0, t_0) + \underbrace{(t-t_0)}_{\alpha_2} \frac{\partial f}{\partial t} \Big|_{y_0, t_0} + \underbrace{(y-y_0)}_{\lambda} \frac{\partial f}{\partial y} \Big|_{y_0, t_0} + \text{HOT}(y^2, t^2, \dots)$$

$$= \boxed{\lambda y} + \underbrace{\alpha_1 + \alpha_2 t}_{\text{a particular sol.}} + \text{HOT}$$

provides a decaying sol.
ex) $y' = \lambda y^2 \rightarrow y = \frac{1}{a - \lambda t}$

Model problem: $\boxed{y' = \lambda y}$ given $y(0) = y_0$.

$\lambda = \lambda_R + i\lambda_I$: complex number

exact sol. $y = y_0 e^{\lambda t} = y_0 e^{\lambda_R t} e^{i\lambda_I t}$

4.3 Stability analysis for Euler method $\lambda_r \leq 0$ for stability

$$y' = f(y, t) \xrightarrow{EE} \frac{y_{n+1} - y_n}{h} = f(y_n, t_n)$$

$$y_{n+1} = y_n + h f(y_n, t_n)$$

Apply EE to model problem, $y' = \lambda y = f$

$$\Rightarrow y_{n+1} = y_n + h \lambda y_n = (1 + \lambda h) y_n$$

$$\rightarrow y_n = (1 + \lambda h)^n y_0 \quad (t = nh)$$

$$= (1 + \lambda_r h + i \lambda_z h)^n y_0$$

Whether sol. remains bounded depends on $\lambda_R h$ & $\lambda_I h$.
 For exact sol. to be well behaved.

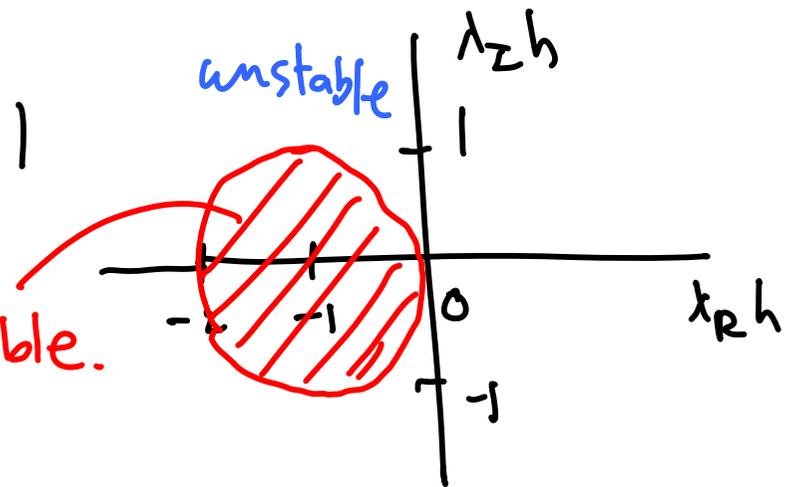
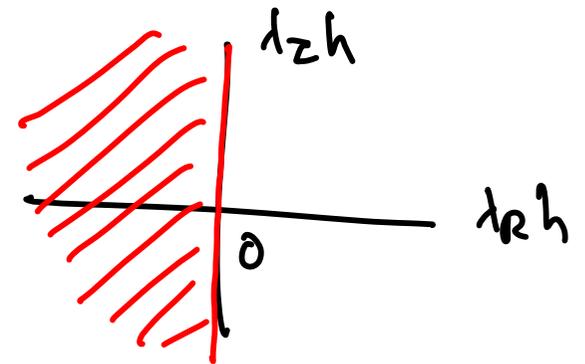
$$EE: y_n = (1 + \lambda_R h + i\lambda_I h)^n y_0 = \sigma^n y_0.$$

To be stable, $|\sigma| \leq 1$

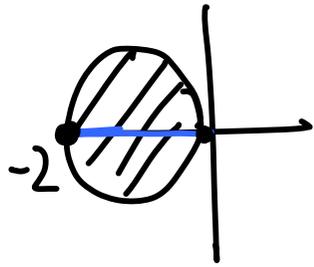
$$|\sigma|^2 = (1 + \lambda_R h)^2 + (\lambda_I h)^2 \leq 1$$

EE is stable.

\therefore EE is conditionally stable.



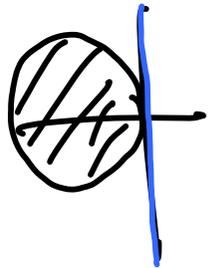
- When λ is real & negative ($\lambda_I = 0$)



$$|\lambda h| \leq 2 \rightarrow h_{\max} = \frac{2}{|\lambda|}$$

$$y' = -2y \xrightarrow{EE} h_{\max} = \frac{2}{|-2|} = 1$$

- When λ is purely imaginary ($\lambda_R = 0$) $\rightarrow y' = i\omega y$



EE is unstable for $\lambda = i\omega$. \rightarrow sol. blows up.

$$|\sigma|^2 = \underbrace{(1 + \lambda_R h)^2}_0 + (\lambda_I h)^2 > 1$$

- Accuracy of EE

$$y' = \lambda y \quad \text{exact sol.}$$

$$y = y_0 e^{\lambda t} = y_0 e^{\lambda n h} = y_0 \left(1 + \lambda h + \frac{1}{2} \lambda^2 h^2 + \dots \right)^n$$

$$EE: y_n = (1 + \lambda h)^n y_0$$

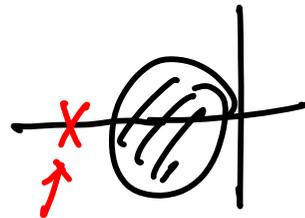
leading error

\therefore EE is 1st-order accurate.

$$\frac{1}{2} \lambda^2 h^2 \cdot n = \frac{1}{2} \lambda^2 h^2 \frac{T}{h} \rightarrow O(h)$$

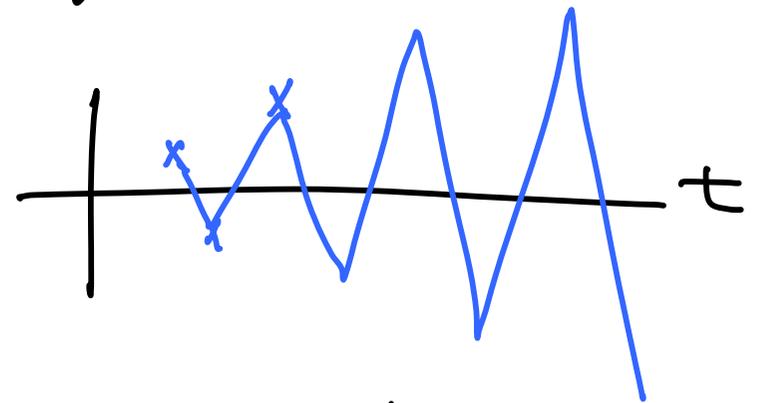
- Signal for instability (λ : real & negative)

$$\text{EE: } y_n = (1 + \lambda h)^n y_0$$



$$(1 + \lambda h) > 1$$

$$1 + \lambda h < 0$$



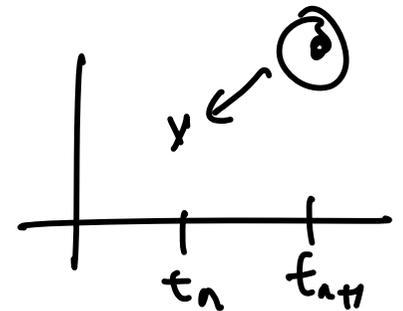
Ex. 4 Implicit Euler or backward Euler method (IE)

$$y' = f(y, t)$$

IE \rightarrow

$$\frac{y_{n+1} - y_n}{h} = f(y_{n+1}, t_{n+1})$$

$$y_{n+1} = y_n + h f(y_{n+1}, t_{n+1})$$



cost/timestep is higher than explicit Euler.

model prob. $y' = \lambda y$

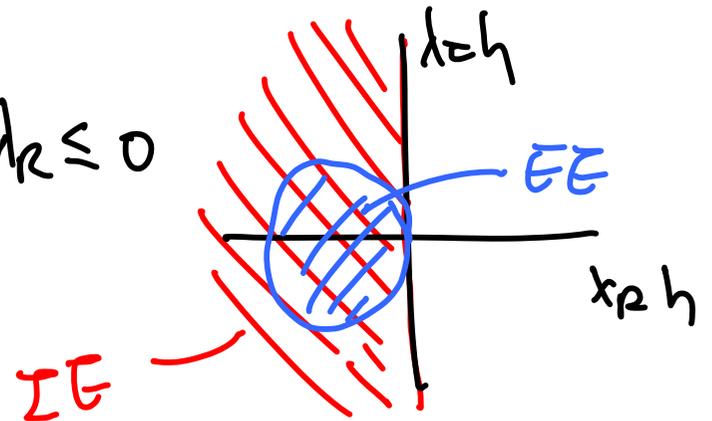
$$\text{IE: } y_{n+1} = y_n + h \cdot \lambda y_{n+1} \rightarrow y_{n+1} = \frac{1}{1-\lambda h} y_n \rightarrow y_n = \left(\frac{1}{1-\lambda h}\right)^n y_0$$

$$\sigma = \frac{1}{1-\lambda h} = \frac{1}{1-\lambda_r h - i\lambda_i h}$$

$$= \sigma^n y_0$$

$$|\sigma|^2 = \frac{1}{(1-\lambda_r h)^2 + (\lambda_i h)^2} \leq 1 \text{ for } \lambda_r \leq 0$$

\therefore IE is unconditionally stable.
A-stable



4.5 Numerical accuracy

$$\text{IE} : \sigma = \frac{1}{1-\lambda h} = \underline{1 + \lambda h} + (\lambda h)^2 + (\lambda h)^3 + \dots$$

$$\text{exact sol.} : e^{\lambda h} = \underline{1 + \lambda h} + \frac{1}{2}(\lambda h)^2 + \frac{1}{6}(\lambda h)^3 + \dots$$

$$\text{IE} : y_n = \sigma^n y_0 = (1 + \lambda h + (\lambda h)^2 + (\lambda h)^3 + \dots)^n y_0$$

error

$$\lambda^2 h^2 \cdot n = \lambda^2 h^2 \cdot \frac{T}{h} \rightarrow O(h)$$

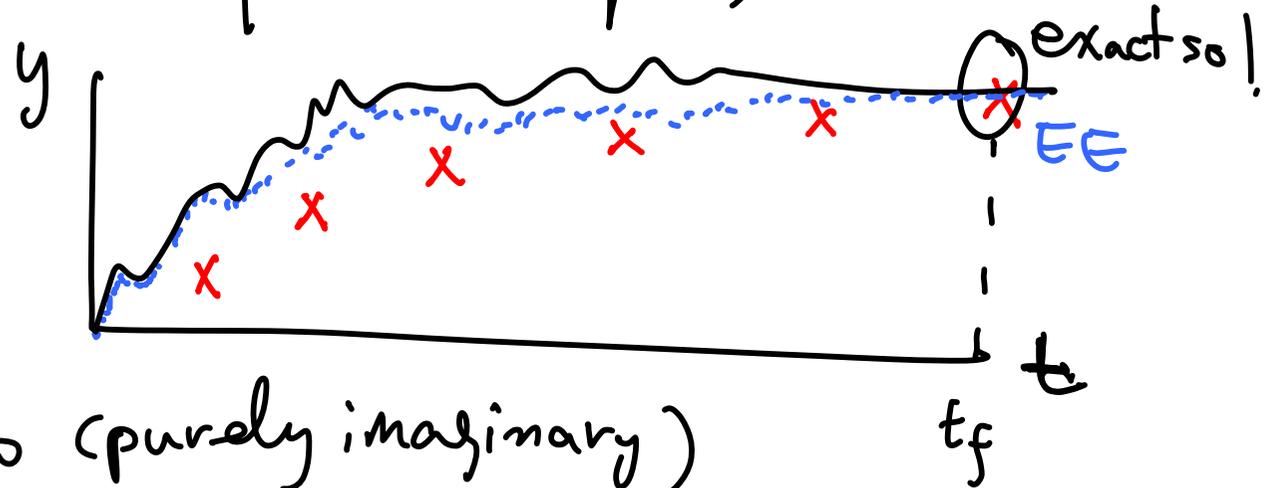
\therefore IE is 1st-order accurate.

$$\text{IE} : e^{\lambda h} - \sigma = e^{\lambda h} - \frac{1}{1-\lambda h} = -\frac{1}{2} \lambda^2 h^2 + \dots$$

$$\text{EE} : e^{\lambda h} - \sigma = e^{\lambda h} - (1 + \lambda h) = \frac{1}{2} \lambda^2 h^2 + \dots$$

\Rightarrow stability is nothing to do with accuracy.

From stability point of view,
 our objective ^{is} to take largest time step h ,



- Accuracy for $\lambda = i\omega$ (purely imaginary)

$$y' = \lambda y \quad (\lambda_r = 0, \lambda_z = \omega)$$

$$\text{exact sol. } y = y_0 e^{i\omega t} = y_0 (\cos \omega t + i \sin \omega t)$$

$$\text{EE: } y_{n+1} = y_n + \lambda h y_n = (1 + \lambda h) y_n$$

$$\rightarrow y_n = (1 + i\omega h)^n y_0 = \sigma^n y_0, \quad \sigma = 1 + i\omega h$$

exact sol.: amplitude $|e^{i\omega t}| = 1$

EE : " $|\sigma| = \sqrt{1 + \omega^2 h^2} > 1$ unstable

$$\sigma = |\sigma| e^{i\theta} \quad \theta = \tan^{-1} \frac{\text{Im}(\sigma)}{\text{Re}(\sigma)} = \tan^{-1} \frac{\omega h}{1} = \tan^{-1} \omega h$$

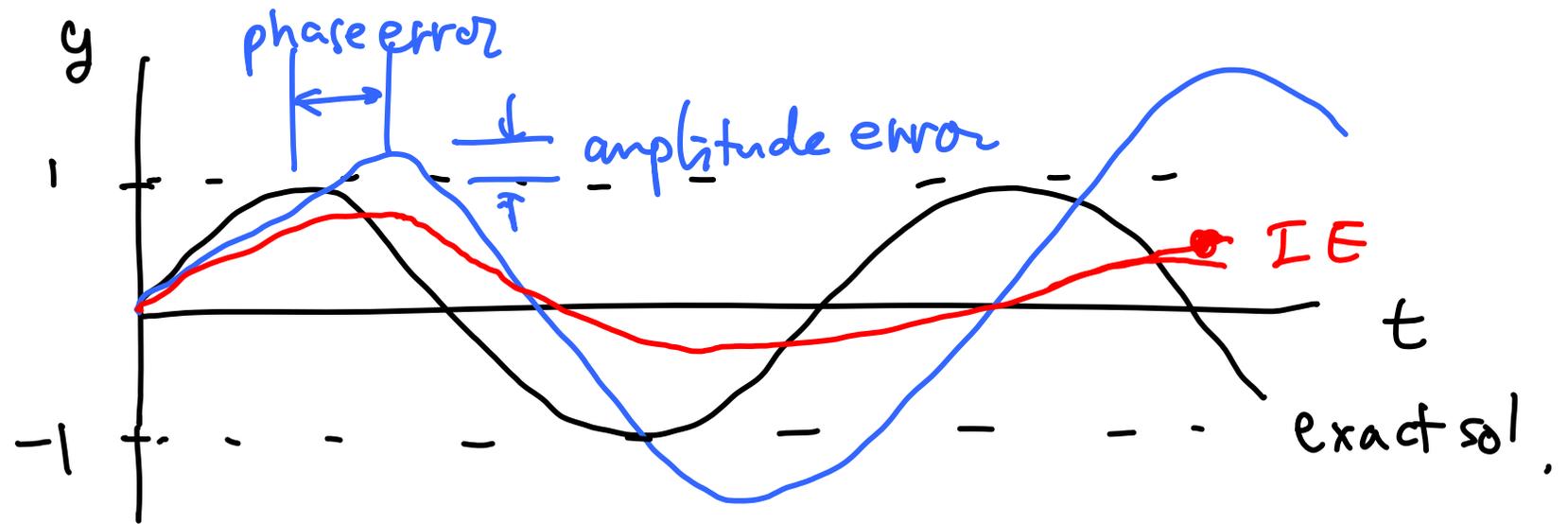
phase

$$\text{exact sol. : } y = y_0 e^{i\omega t} = y_0 e^{i\omega h n} = y_0 \cdot 1 \cdot e^{i\omega h n}$$

$$\text{EE : } y_n = \sigma^n y_0 = y_0 |\sigma|^n e^{i\theta n}$$

$$\text{phase error} = \omega h - \theta = \omega h - \tan^{-1} \omega h = \frac{1}{3} (\omega h)^3 + \dots$$

($\tan^{-1} \omega h = \omega h - \frac{1}{3} (\omega h)^3 + \frac{1}{5} (\omega h)^5 + \dots$)



$$IE: y_{n+1} = \frac{1}{1 - i\omega h} y_n = \frac{1}{1 - i\omega h} y_n \rightarrow y_n = \left(\frac{1}{1 - i\omega h} \right)^n y_0$$

$$\sigma = \frac{1}{1 - i\omega h} = |\sigma| e^{i\theta} = \sigma^n y_0$$

$$\frac{1 + i\omega h}{1 + \omega^2 h^2}$$

$$|\sigma|^2 = \frac{1}{1 + \omega^2 h^2} < 1 \quad \text{stable!} \rightarrow \text{but decaying sol.}$$

$$\theta = \tan^{-1} \frac{\text{Im}(\sigma)}{\text{Re}(\sigma)} = \tan^{-1} \omega h : \text{same as that of IE.}$$

$$\text{phase error} = \omega h - \theta = \frac{1}{3}(\omega h)^3 + \dots$$