Engineering Mathematics II

Lecture 11

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Previously, we discussed

- Sequences, series and their convergence
- Power series and its radius of convergence

$$R = \lim_{n\to\infty} \left| \frac{a_n}{a_{n+1}} \right|$$

Functions represented by power series

$$f(z) = \sum_{n=0}^{\infty} a_n z^n \quad , \quad |z| < R_1$$

$$g(z) = \sum_{n=0}^{\infty} b_n z^n \quad , \quad |z| < R_2$$

$$f \pm g = \sum_{n=0}^{\infty} (a_n \pm b_n) z^n , \quad |z| < \min(R_1, R_2)$$

$$fg = a_0 b_0 + (a_1 b_0 + a_0 b_1) z + (a_2 b_0 + a_1 b_1 + a_0 b_2) z^2 + \cdots , \quad |z| < \min(R_1, R_2)$$

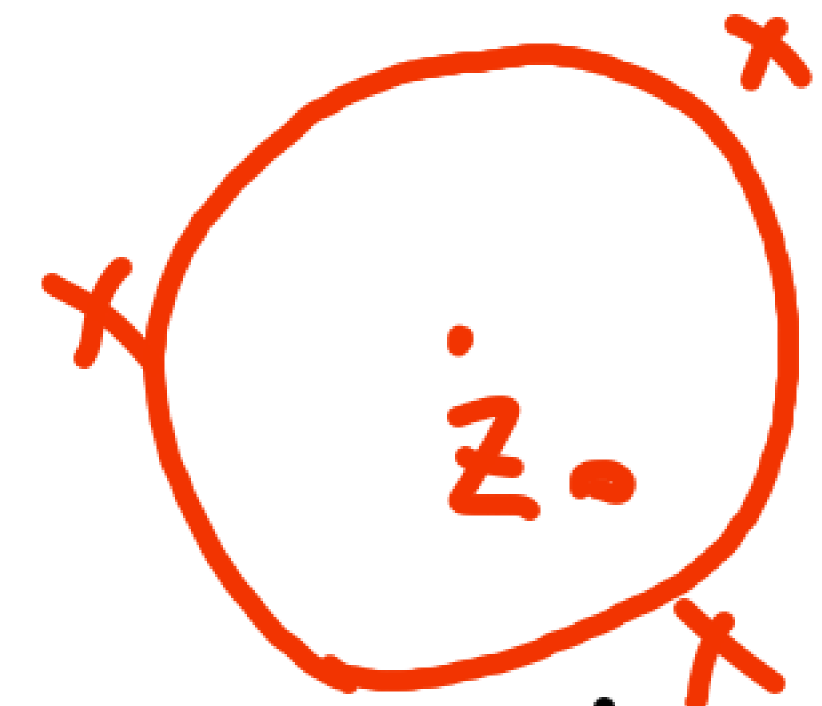
Taylor series

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n, \quad |z - z_0| < r$$

where, $a_n = \dfrac{f^{(n)}(z_0)}{n!} = \dfrac{1}{2\pi i} \oint_C \dfrac{f(t)\,dt}{(t - z_0)^{n+1}}$

For $z_0 = 0$, it is called Maclaurin series.

- Taylor series with the center z_0 is unique.



- The radius of convergence is equal to the distance from the center to the nearest singularity

- $|a_n| \le \dfrac{M}{r^n}$

Example: Show that a power series with a nonzero radius of convergence is Taylor series

$|z-z_0|<R$

$$f(z)=\sum_{n=0}^{\infty} a_n(z-z_0)^n = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

$$f(z_0)=a_0$$

$$f'(z_0)=a_1$$

$$f''(z_0)=2!\,a_2, \quad f^{(n)}(z_0)=n!\,a_n, \quad a_n=\frac{f^{(n)}(z_0)}{n!}$$

Example: Find the Maclaurin series

of $f(z) = \dfrac{1}{1-z} = 1 + z + z^2 + \cdots, \quad |z| < 1$

$$g(z) = \frac{1}{1-z^2} = \frac{1}{(1-z)(1+z)} = \frac{1}{2}\left(\frac{1}{1-z} + \frac{1}{1+z}\right)$$

$$\hookrightarrow 1 + z^2 + z^4 + \cdots, \quad |z|^2 < 1$$

Example: Find the Maclaurin series of

$$f(z) = e^z$$

$$= 1 + z + \frac{z^2}{2} + \frac{z^3}{3!} + \cdots$$

$$= \sum_{n=0}^{\infty} \frac{z^n}{n!}$$

$z = iy$

$$e^{iy} = 1 + iy + \frac{(-y^2)}{2} + \frac{-iy^3}{3!} + \cdots$$

$$= \left(1 - \frac{y^2}{2} + \frac{y^4}{4!} + \cdots\right) + i\left(y - \frac{y^3}{3!} + \cdots\right)$$

$$e^{iy} = \cos y + i \sin y$$