5/24 Interchange - continued / screw-pinch

1. Gravitational Rayleigh-Taylor by $\rho\vec{g}$

one needs to add

$$\rho_0 \frac{\partial \vec{u}_1}{\partial t} = \vec{J}_0 \times \vec{B}_1 + \vec{J}_1 \times \vec{B}_0 - \nabla P_1 - \rho_1 \vec{g}$$

potential energy $\delta W_F \rightarrow \delta W_F - \frac{1}{2}\int d\vec{x}\, \vec{\xi}^* \cdot \vec{g}\,(\vec{\rho}(\vec{\nabla}\cdot\vec{\xi}) + \vec{\xi}\cdot\vec{\nabla}\rho)$

In a simple case w/o $\begin{cases}\text{bending} \\ \text{compression, current } J_\parallel\end{cases}$

$$\delta W_F \sim -\frac{1}{2}\int d\vec{x}\,(\vec{\xi}^*\cdot\vec{g})(\vec{\xi}\cdot\vec{\nabla}\rho) \sim -\frac{1}{2}\int d\vec{x}\,\vec{\xi}^*\cdot\rho\frac{\partial^2\vec{\xi}}{\partial t^2}$$

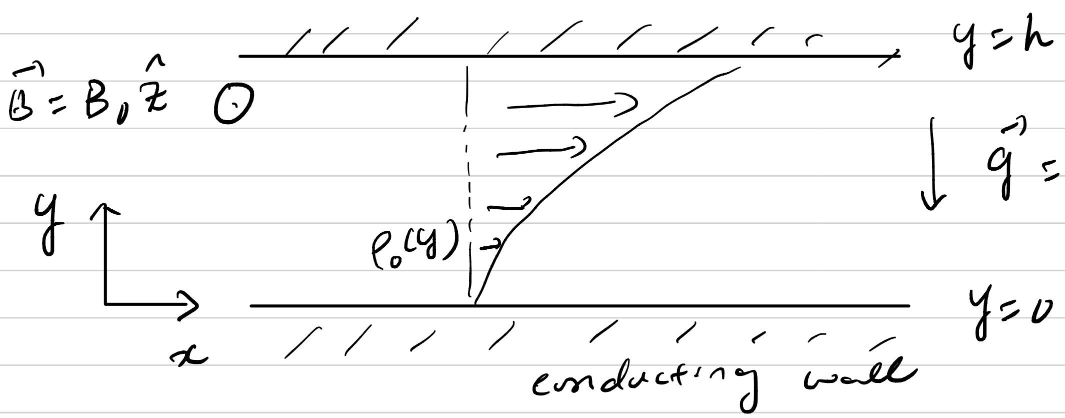
For 1-d case, i.e. $\vec{g} = -g\hat{y}$ $\rho = \rho(y)$

$$|\xi|^2 g\rho' \sim \rho\omega^2|\xi|^2 \qquad \omega \sim \left(\frac{g\rho'}{\rho}\right)^{\frac{1}{2}}$$

* For more general case, → H.W.
* simple case can be fully analyzed w/o using energy principle.

<u>Goldstn ch. 19</u>

Consider a plasma in equilibrium as



- Equil. $\frac{\partial}{\partial y}\left(P_0 + \frac{B_0^2}{2\mu_0}\right) + \rho_0 g = 0$
- Add a perturbation $f \to f_0 + f_1$ ($f_0 \gg f_1$)

  $f_1 = f_1(y) e^{i(kx-\omega t)}$, ignoring $\frac{\partial}{\partial z}$.

- $\frac{\partial \vec{B}_1}{\partial t} = \vec{\nabla}\times(\vec{u}_1\times\vec{B}_0) = \cancel{(\vec{B}_0\cdot\vec{\nabla})\vec{u}_1} - (\vec{u}_1\cdot\vec{\nabla})\vec{B}_0 + \cancel{\vec{u}_1(\vec{\nabla}\cdot\vec{B}_0)} - \vec{B}_0(\vec{\nabla}\cdot\vec{u}_1)$

  implies $B_{x1} = B_{y1} = 0$

- momentum conservation.

$$\hat{z}\cdot\vec{\nabla}\times\left\{\rho_0 \frac{\partial \vec{u}_1}{\partial t} = -\vec{\nabla}\left(P_1 + \frac{B_0 B_{z1}}{\mu_0}\right) + \rho_1 \vec{g}\right\}$$

$$-i\omega\left(ik\rho_0 u_y - \frac{\partial}{\partial y}(\rho_0 u_x)\right) = -ik\rho_1 g$$

- Incompressibility

$$\vec{\nabla}\cdot\vec{u} = iku_x + \frac{\partial u_y}{\partial y} = 0$$

- mass conservation

$$\frac{\partial \rho_1}{\partial t} + \vec{u}_1\cdot\vec{\nabla}\rho_0 = 0 \qquad -i\omega\rho_1 = -u_y\frac{\partial \rho_0}{\partial y}$$

- Then,

$$\left\{\omega k \rho_0 u_y + \frac{i\omega}{-ik}\frac{\partial}{\partial y}\left(\rho_0 \frac{\partial u_y}{\partial y}\right) = -\frac{ik}{i\omega} g\rho_0' u_y\right\} \times \left(-\frac{k}{\rho_0\omega}\right)$$

$$\frac{1}{\rho_0}\frac{\partial}{\partial y}\left(\rho_0\frac{\partial u_y}{\partial y}\right) - k^2\left(1 + \frac{g\rho_0'}{\omega^2}\right)u_y = 0$$

- Let $\rho_0 \propto \exp(y/s)$, $u_y(0) = 0$ $u_y(h) = 0$

$\rightarrow$ $u_y = \sin\left(\frac{n\pi y}{h}\right)\exp\left(-\frac{y}{2s}\right)$

$\rightarrow$ $k^2\left(1 + \frac{g\rho_0'}{\omega^2}\right) = -\frac{n^2\pi^2}{h^2} - \frac{1}{4s^2}$

$\rightarrow \omega^2 = \left(-\frac{g}{s} \frac{h^2k^2}{h^2k^2 + n^2\pi^2 + h^2/4s^2} \right)^{\frac{1}{2}}$

$\rightarrow$ Unstable, always

$\rightarrow$ most unstable mode $k \to \infty$

$$\omega \sim \left(-\frac{g}{s}\right)^{\frac{1}{2}} = \left(-g\frac{\rho_0'}{\rho_0}\right)^{\frac{1}{2}} \qquad \rho_0/\rho_0' = s$$

Roughly, for any simple interchange instability

$$\omega \sim \left(-\frac{p'}{p\rho} f\right)^{\frac{1}{2}} \sim \left(-\frac{g\rho'}{\rho}\right)^{\frac{1}{2}} \quad \text{for } f = \rho g,\ T = \text{const}$$

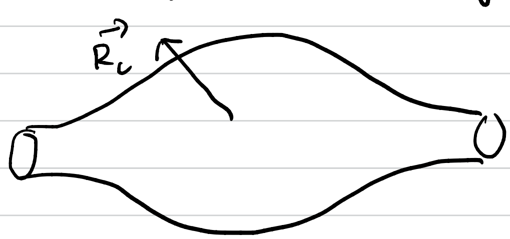
any force

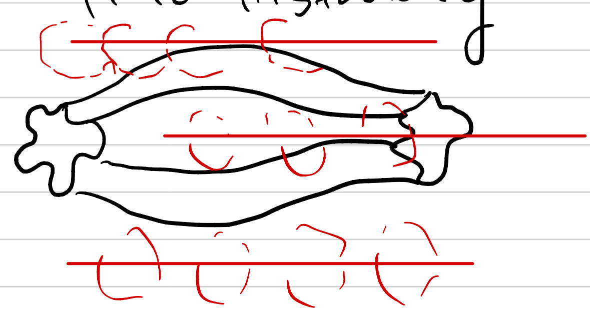
$$\sim \left(-\frac{2p'}{\rho R_c}\right)^{\frac{1}{2}} \quad \text{for } f = 2p/R_c \text{ by curvature}$$

Z-pinch

$$\frac{1}{2}\rho\omega^2\xi^2 \simeq -\frac{1}{2}(2rp')\left(\frac{\xi}{r}\right)^2 \rightsquigarrow \left(-\frac{2p'}{\rho r}\right)^{\frac{1}{2}} \qquad R_c \simeq r$$

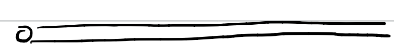
In mirror, interchange $\rightarrow$ "flute" instability

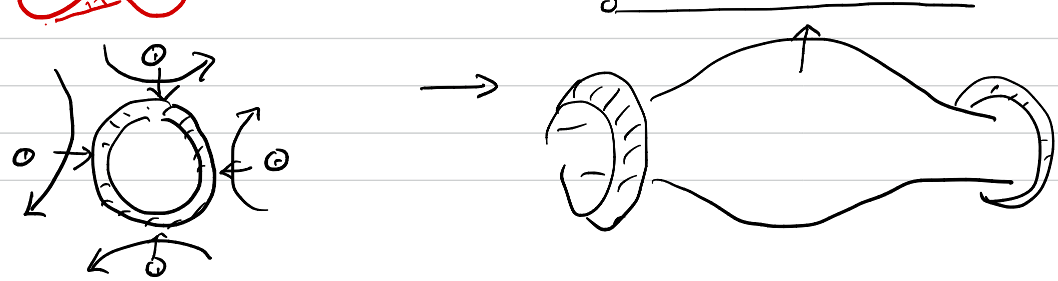




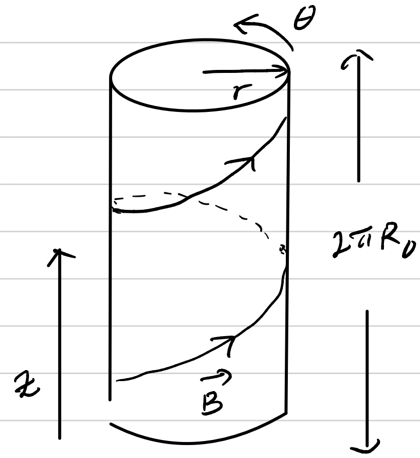
additional wire to stabilize flute

or introduce magnetic shear





2. General Screw pinch



(r, θ, z)

(1) Equilibrium

$$\frac{d}{dr}\left(p + \frac{B_\theta^2 + B_z^2}{2\mu_0}\right) + \frac{B_\theta^2}{\mu_0 r} = 0$$

(2) perturbation

$$\vec{\xi} = \xi_r \hat{r} + \xi_\theta \hat{\theta} + \xi_z \hat{z}$$
$$= \xi \hat{r} + \eta \hat{e}_\eta + \xi_\parallel \hat{b}$$

where, $\hat{b} = \dfrac{B_\theta \hat{\theta} + B_z \hat{z}}{B}$, $\hat{e}_\eta = \dfrac{B_z \hat{\theta} - B_\theta \hat{z}}{B}$

$$\xi_\parallel = \frac{\xi_\theta B_\theta + \xi_z B_z}{B}, \quad \eta = \frac{\xi_\theta B_z - \xi_z B_\theta}{B}$$

$$\vec{\xi} = \vec{\xi}(r) e^{i(m\theta + kz)} = \vec{\xi}(r) e^{i(m\theta - \frac{n}{R_0} z)} = \vec{\xi}(r) e^{i(m\theta - n\phi)}$$

(3) Energy principle.

(i) $\vec{\nabla} \cdot \vec{\xi} = 0$ determine $\xi_\parallel$

(ii) η apperas as algebraically,

(iii) $\delta W_F = \dfrac{2\pi^2 R_0}{\mu_0} \displaystyle\int_0^a dr \left[f\left(\frac{d\xi}{dr}\right)^2 + g\xi^2 \right]$

$$f = \frac{r B_\theta^2 (m - nq)^2}{k_0^2} \qquad \text{where, } q = \frac{r B_z}{R_0 B_\theta}, \; k_0^2 = m^2 + k^2 r^2$$

$$g = \underbrace{\frac{2\mu_0 k^2 r^2}{k_0^2} p'}_{\text{Interchange or pressure-driven}} + \underbrace{\frac{B_\theta^2}{r}(m - nq)^2 \frac{k^2 r^2 + m^2 - 1}{k_0^2}}_{\text{Bending always possitive } m \neq 0} + \underbrace{(n^2 q^2 - m^2)\frac{2k^2 r^2 B_\theta^2}{k_0^4}}_{\text{current-driven positive when } q > \frac{m}{n}}$$

(4) Eigenfcn for ξ

variational principle $\xi \to \xi + \delta\xi$

$$\delta(\delta W_F) \propto \int_0^a dr \; 2f \frac{d\delta\xi}{dr}\frac{d\xi}{dr} + 2g\xi\delta\xi$$

$$= 2\left\{ \cancel{f \frac{d\xi}{dr}\delta\xi}\Big|_0^a - \int_0^a dr \left[\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi\right]\delta\xi \right\}$$

internal mode

$$\therefore \boxed{\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0 \quad : \text{Newcomb equation}}$$

Generalized for toroidal / stellarator / kinetic by Glasser, Park.

(5) Local interchange instability

near the rational surface $m - nq \simeq 0$

why? — no-line bending, no δW

$f \to 0$ Newcomb eq. $\to$ singular

$$q(r_s) = \frac{n}{m}, \quad r = r_s + x \quad (r_s \gg x)$$

$$m - nq(r) = (m - nq(r_s)) + \frac{d}{dr}(-nq)\Big|_{r=r_s} x + \sim$$

$$\sim -nq'(r_s)x$$

$$f \sim \frac{rB_\theta^2}{K_0^2} n^2 (q')^2 x^2$$

$$g \sim \frac{2\mu_0 k^2 r^2}{K_0^2} p'$$

$$\delta W_F \sim \frac{2\pi^2 R_0}{\mu_0} \cdot \frac{rB_\theta^2 n^2 (q')^2}{K_0^2} \int_{r_s-\epsilon}^{r_s+\epsilon} x^2 \left(\frac{d\xi}{dx}\right)^2 - D_s \xi^2 \, dx$$

$$D_s = -\frac{2\mu_0 k^2 r^2 p'}{r B_\theta^2 n^2 (q')^2} = -\left(\frac{2\mu_0 p' q^2}{r B_z^2 (q')^2}\right)$$

$$\because \quad k = -\frac{n}{R_0} = -\left(\frac{m B_\theta}{r B_z}\right)$$

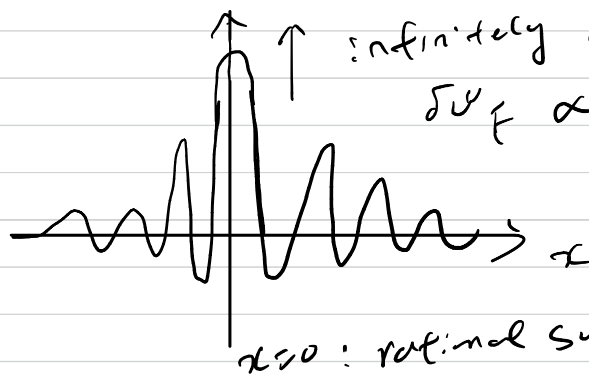
Newcomb eq

$$\frac{d}{dx}\left(x^2 \frac{d\xi}{dx}\right) + D_s \xi = 0$$

$$\xi = c_1 |x|^{p_1} + c_2 |x|^{p_2} \qquad p_{1,2} = -\frac{1}{2} \pm \frac{1}{2}(1-4D_s)^{\frac{1}{2}}$$

If $1-4D_s < 0$, $D_s > \frac{1}{4}$ sol → oscillatory

→ possible to have ξ such that



infinitely large

$$\delta W_F \propto \int -D_s \xi^2 \, dx \to -\infty$$

Therefore, $D_s < \frac{1}{4}$ necessary / sufficient condition for stability

$$\boxed{r B_z^2 \left(\frac{q'}{q}\right)^2 + 8\mu_0 p' > 0}$$

"Suydam" criterion

p' : destabilizing

q' : stabilizing

In fact, toroidal effects stabilizing

$$\boxed{r B_z^2 \left(\frac{q'}{q}\right)^2 + 8\mu_0 p' (1 - q^2) > 0}$$

"Mercier" stable $q > 1$

* Newcomb proved it's unstable iff the solution of

$$\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0$$

cross "zero".

(6) External mode needs

$$\delta W = \delta W_F + \delta W_V$$

(7) Large-aspect-ratio exp. for straight tokamak

Read Freidberg ch. 11 Wesson ch 6.9

5/26 Resistive tearing mode

1. Singular (resonant) response in ideal MHD

With a perturbation, ideal MHD forms displacement profiles by δW E-L (Newcomb) eq.

i.e) $\frac{d}{dr}\left(f\frac{d\xi}{dr}\right) - g\xi = 0$

near the resonant surface $m - nq(r_s) = 0$

$r = r_s + x$

$\rightarrow \xi_{mn} \sim c_1|x|^{p_1} + c_2|x|^{p_2} \quad p_{1,2} = -\frac{1}{2} \pm \frac{1}{2}(1-4D_s)^{\frac{1}{2}}$

Even for $p' = 0$, $D_s = 0$, then $P_{1,2} = 0, 1$

$\rightarrow \xi_{mn} \sim |x|^0, |x|^{-1}$

Ideal sol. — prohibited in ideal MHD but allowed by non-ideal effects

Note $B_{r1} = \hat{r}\cdot\vec{\nabla}\times(\vec{\xi}\times\vec{B}_0) = \vec{\nabla}\cdot((\vec{\xi}\times\vec{B}_1)\times\vec{r})$

$= \vec{\nabla}\cdot(\vec{B}_0\xi) = (\vec{B}_0\cdot\vec{\nabla})\xi = (B_\theta\hat{\theta} + B_\phi\hat{\phi})\cdot\vec{\nabla}\xi$

$B_{r1,mn} \propto e^{i(m\theta - n\phi)}$

$B_{r1,mn} = \oint i\left(\frac{m}{r}B_\theta - \frac{n}{R}B_\phi\right)\xi_{mn} = \frac{iB_\theta}{r}(m - nq)\xi_{mn}$

$\rightarrow B_{r1mn} \sim x\xi_{mn} \sim 0$, or finite value

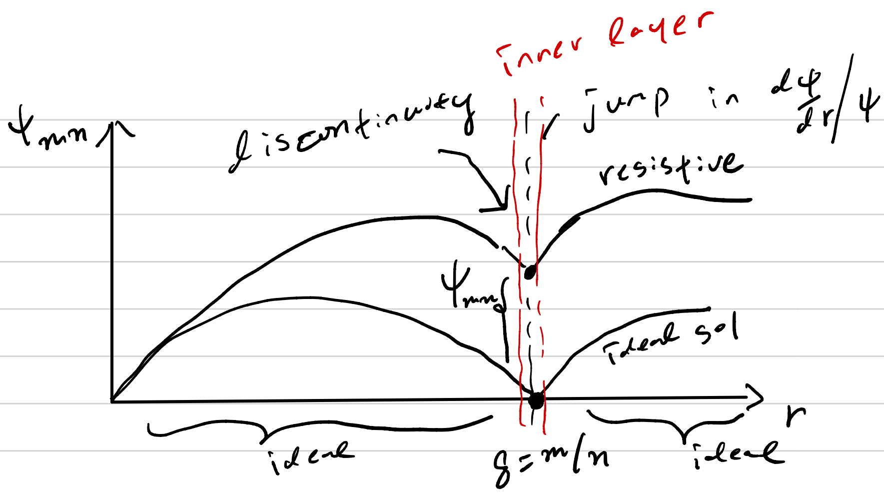
ideal sol — non-ideal.

many literature use flux "ψ" instead of B_{r1}.

$B_{r1} = -\frac{1}{r}\frac{\partial\psi}{\partial\theta}$, $B_{\theta 1} = \frac{\partial\psi}{\partial r}$ $(\vec{\nabla}\cdot\vec{B}_1 = 0,\ B_{\phi 1} \sim O(\epsilon))$

For (m,n) mode $B_{r1mn} = -\frac{im}{r}\psi_{mn}$

$\psi_{mn} = -B_\theta(m - nq)\xi_{mn}$



2. Reduced MHD for ideal (outer) layer

Full equation $\to$ $\vec{j}_1 \times \vec{B}_0 + \vec{j}_0 \times \vec{B}_1 = \vec{\nabla} P_1$

Reduced $\to$ $\underline{\vec{B}_0 \cdot \vec{\nabla} j_{\phi 1} = 0}$

$\hat{\phi} \cdot \left(\vec{\nabla} \times (\vec{j} \times \vec{B}) = \vec{\nabla} \times \vec{\nabla} p = 0 \right) \to \vec{B} \cdot \vec{\nabla} j_\phi + \cancel{\vec{j} \cdot \vec{\nabla} B_\phi} = 0$

For large aspect ratio $\epsilon = \frac{a}{R} \ll 1$

$$\left\{ \begin{array}{l} B_\theta \sim \epsilon B_\phi \ (\vec{\nabla} \cdot \vec{B} = 0), \ \xrightarrow{\mu_0 \vec{j} = \vec{\nabla} \times \vec{B}} \ , \ j_\theta \sim \epsilon j_\phi \\ B_{\phi 1} \sim \epsilon B_{r1} \sim \epsilon B_{\theta 1} \qquad , \ j_{r1} \sim j_{\theta 1} \sim \epsilon j_{\phi 1} \end{array} \right.$$

linearizing $\vec{B} \cdot \vec{\nabla} j_\phi = 0$

$$\underline{(\vec{B}_0 \cdot \vec{\nabla}) j_{\phi 1} + \vec{B}_1 \cdot \vec{\nabla} j_\phi = 0} \qquad j_\phi = j_\phi(r)$$

$$\mu_0 j_{\phi 1} = \frac{1}{r} \frac{\partial}{\partial r}(r B_{\theta 1}) - \frac{1}{r} \frac{\partial}{\partial \theta} B_{r1} = \nabla_\perp^2 \psi$$

For (m, n) mode,

$$\frac{i}{\mu_0} \left(\frac{m B_\theta}{r} - \frac{n B_\phi}{R} \right) \nabla_\perp^2 \psi - i \frac{m}{r} \psi j_\phi' = 0 \qquad ' = \frac{d}{dr}$$

$$q = r B_\phi / R B_\theta$$

$$\underline{\frac{1}{r} \frac{d}{dr} \left(r \frac{d\psi}{dr} \right) - \frac{m^2}{r^2} \psi - \frac{\mu_0 j_\phi'}{B_\theta (1 - \frac{nq}{m})} \psi = 0}$$

Reduced ideal outer layer equation

→ becomes singular near $m-nq \simeq 0$

→ need more physics.

3. Reduced MHD for inner (non-ideal) layer

near $r = r_s + x \qquad x \ll r_s$

(1) Inertial effect

$$\hat{\phi}\cdot(\vec{\nabla}\times(\vec{j}\times\vec{B})) = \hat{\phi}\cdot\vec{\nabla}\times(\rho\frac{\partial\vec{u}}{\partial t})$$

$$\rho = \text{const}, \quad \vec{u} = \cancel{\vec{u}_0} + \vec{u}_1$$

Let $\vec{u}, \psi \sim \exp(im\theta - in\phi + \gamma t)$

$$\text{R.H.S} = \gamma\rho\left[\frac{1}{r}\frac{\partial}{\partial r}(r u_\theta) - \frac{1}{r}\frac{\partial u_r}{\partial\theta}\right] \sim \gamma\rho\frac{\partial u_\theta}{\partial r}$$

due to narrowness of layer

$$\vec{\nabla}\cdot\vec{u} = 0 \qquad \sim \gamma\rho\left(\frac{ir}{m}\right)\frac{\partial^2 u_r}{\partial r^2}$$

$$\text{L.H.S} = \frac{i}{\mu_0}\left(\frac{mB_\theta}{r} - \frac{nB_\phi}{R}\right)\nabla_\perp^2\psi - \frac{im}{r}\psi\, \delta j_\phi'$$

$$\sim \frac{i}{\mu_0}\left(\frac{mB_\theta}{r} - \frac{nB_\phi}{R}\right)\frac{\partial^2\psi}{\partial r^2} - \frac{im}{r}\delta j_\phi'\psi$$

$$\boxed{\frac{\gamma\rho r^2}{m^2}\frac{d^2 u_r}{dr^2} = \frac{B_\theta}{\mu_0}\left(1 - \frac{nq}{m}\right)\frac{d^2\psi}{dr^2} - \delta j_\phi'\psi}$$

momentum equation for layer

(2) Resistive effect

$$\vec{E}_1 + \vec{u}_1\times\vec{B}_0 = \eta\vec{j}_1$$

$$\hat{r}\cdot\left(\frac{\partial\vec{B}_1}{\partial t} = \vec{\nabla}\times(\vec{u}_1\times\vec{B}_0) + \frac{\eta}{\mu_0}\nabla^2\vec{B}_1\right)$$

$$\frac{\partial B_{r1}}{\partial t} = (\vec{B}_0 \cdot \vec{\nabla}) u_r + \frac{\eta}{\mu_0} \nabla^2 B_{r1}$$

$$\gamma B_{r1} \sim \frac{i B_\theta}{r}(m - nq) u_r + \frac{\eta}{\mu_0} \frac{\partial^2 B_{r1}}{\partial r^2}$$

$$\boxed{\frac{d^2 \psi}{dr^2} = \frac{\mu_0}{\eta}\left(\gamma \psi + B_\theta \left(1 - \frac{nq}{m}\right) u_r\right)}$$

Resistive ohm's law for layer