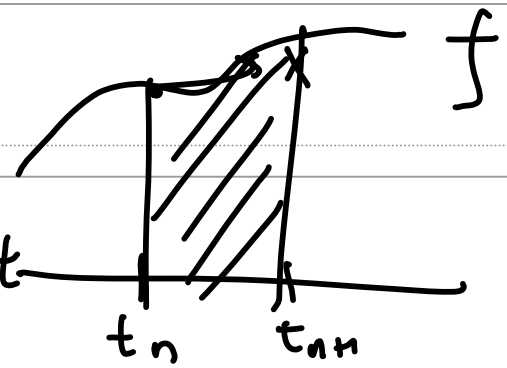
4.6 Trapezoidal method (TR)



$$y' = f(y,t) \rightarrow y_{n+1} = y_n + \int_{t_n}^{t_{n+1}} f(y,t)\,dt$$

$$\rightarrow \boxed{y_{n+1} = y_n + \frac{h}{2}\left[f(y_{n+1}, t_{n+1}) + f(y_n, t_n)\right]}$$ TR, implicit method

$$y' = f \rightarrow \frac{y_{n+1} - y_n}{h} = \frac{1}{2}\left[f_{n+1} + f_n\right]$$: TR, implicit

$$= f_n$$: EE, explicit

$$= f_{n+1}$$: IE, implicit

when TR is applied to PDE, it is called 'Crank-Nicolson' method.

Model problem : $y' = \lambda y$

TR : $y_{n+1} = y_n + \frac{h}{2}[\lambda y_{n+1} + \lambda y_n]$

$$\rightarrow \quad y_{n+1} = \frac{1+\lambda h/2}{1-\lambda h/2} y_n = \sigma y_n$$

$$\sigma = \frac{1+\lambda h/2}{1-\lambda h/2} = 1 + \lambda h + \frac{1}{2}\lambda^2 h^2 + \underbrace{\frac{1}{4}\lambda^3 h^3}_{\text{error}} + \cdots$$

Exact sol. : $e^{\lambda h} = 1 + \lambda h + \frac{1}{2}\lambda^2 h^2 + \frac{1}{6}\lambda^3 h^3 + \cdots$

$\therefore$ TR is 3rd-order accurate for one time step

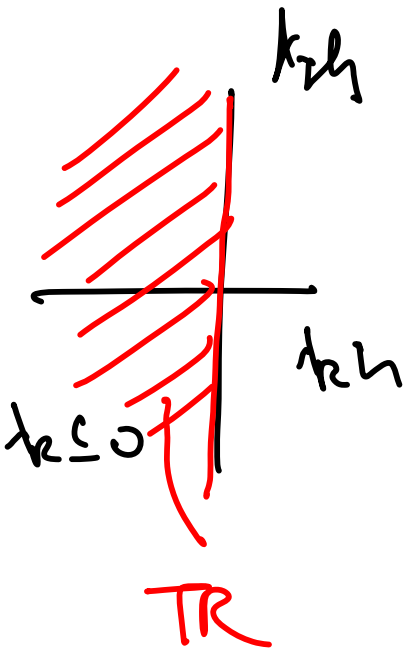
globally 2nd-order accurate.

stability : $\lambda = \lambda_R + i\lambda_I$

$$\sigma = \frac{1+\lambda h/2}{1-\lambda h/2} = \frac{1+\lambda_R h/2 + i\lambda_I h/2}{1-\lambda_R h/2 - i\lambda_I h/2} = \frac{Ae^{i\theta}}{Be^{i\alpha}} = \frac{A}{B}e^{i(\theta-\alpha)}$$

$$\theta = \tan^{-1}\frac{\frac{1}{2}\lambda_I h}{1+\frac{1}{2}\lambda_R h}, \quad \alpha = \tan^{-1}\frac{-\frac{1}{2}\lambda_I h}{1-\frac{1}{2}\lambda_R h}$$

$$|\sigma| = \frac{A}{B} = \frac{\sqrt{(1+\lambda_R h/2)^2+(\lambda_I h/2)^2}}{\sqrt{(1-\lambda_R h/2)^2+(\lambda_I h/2)^2}} \le 1 \quad \text{for } \lambda_R \le 0$$



$\therefore$ TR is unconditionally stable.

For $\lambda_R = 0$, $y' = i\omega y$ $(\lambda = i\omega)$

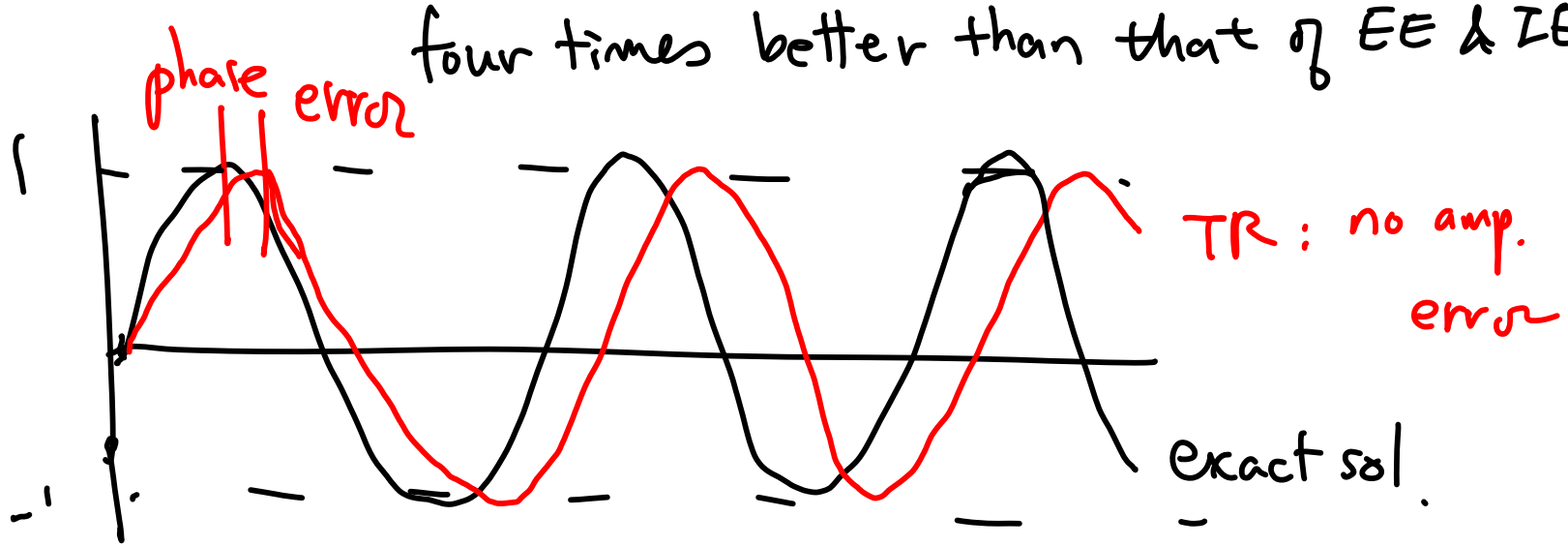
TR : $|\sigma| = 1$ $\therefore$ no amplitude error

phase : $\sigma = \dfrac{1 + i\frac{1}{2}\omega h}{1 - i\frac{1}{2}\omega h} = e^{i \cdot 2\theta}$ $\qquad \theta = \tan^{-1}\dfrac{\omega h}{2}$

$$\text{phase error} = \omega h - 2\tan^{-1}\frac{\omega h}{2}$$

$$= \omega h - 2\left[\frac{\omega h}{2} - \frac{1}{24}(\omega h)^3 + \cdots\right] = \frac{1}{12}(\omega h)^3 + \cdots$$

four times better than that of EE & IE.

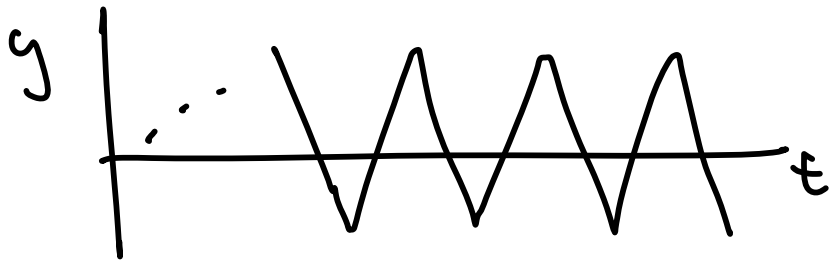


For $\lambda_I = 0$, λ is real & negative ($\lambda_R < 0$)

TR: $\sigma = \dfrac{1 + \frac{1}{2}\lambda_R h}{1 - \frac{1}{2}\lambda_R h} \longrightarrow q_n = q_0 \sigma^n$

for large h, $\sigma \to -1$

σ^n oscillates between -1 and 1, but will not blow up.



be careful!

$$* \quad y'' + \omega^2 y = 0 \qquad y(0) = y_0, \; y'(0) = 0$$

$$y_1 = y$$

$$y_2 = y_1' \rightarrow y_2' = y_1'' = -\omega^2 y_1$$

$$\Rightarrow \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \overbrace{\begin{pmatrix} 0 & 1 \\ -\omega^2 & 0 \end{pmatrix}}^{A} \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

$$= S^{-1} \Lambda S \begin{pmatrix} y_1 \\ y_2 \end{pmatrix}$$

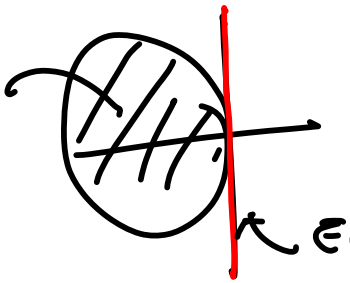
$$\det(A - \lambda I) = \begin{vmatrix} -\lambda & 1 \\ -\omega^2 & -\lambda \end{vmatrix} = 0 \rightarrow \lambda^2 = -\omega^2$$

$$\rightarrow \lambda = \pm i\omega \quad \text{pure imaginary eigenvalues}$$

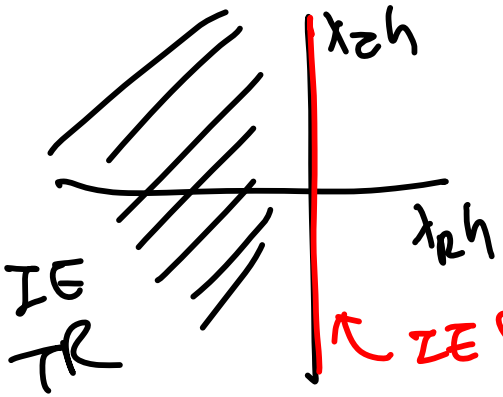
$$S \begin{pmatrix} y_1' \\ y_2' \end{pmatrix} = \Lambda S \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rightarrow \begin{pmatrix} z_1' \\ z_2' \end{pmatrix} = \Lambda \begin{pmatrix} z_1 \\ z_2 \end{pmatrix} \Rightarrow \begin{array}{l} z_1' = \lambda_1 z_1 \\ z_2' = \lambda_2 z_2 \end{array}$$

model prob.

EE



EE is unstable.



IE
TR

IE & TR are stable.

4.7 Linearization for implicit methods

TR : $y' = f(y,t) \longrightarrow \dfrac{y_{n+1}-y_n}{h} = \dfrac{1}{2}\left[f(y_{n+1},t_{n+1}) + f(y_n,t_n)\right] + O(h^2)$

!

→ solve nonlinear algebraic eq.

→ require iterative solution procedure.

⇒ can be avoided by linearization technique.

$$f(y_{n+1}, t_{n+1}) = f(y_n, t_{n+1}) + (y_{n+1} - y_n)\left.\frac{\partial f}{\partial y}\right|_{y_n, t_{n+1}} + \frac{1}{2}(y_{n+1} - y_n)^2 \left.\frac{\partial^2 f}{\partial y^2}\right|_{y_n, t_{n+1}} + \cdots$$

$$h \left.\frac{\partial y}{\partial t}\right|_n + \frac{1}{2} h^2 \left.\frac{\partial^2 y}{\partial t^2}\right|_n + \cdots$$

$O(h^2)$

$O(h^3)$

$$\text{TR}: \; y_{n+1} = y_n + \frac{h}{2}\left[f(y_{n+1}, t_{n+1}) + f(y_n, t_n)\right] + O(h^3)$$

$\therefore$ neglect this term w/o losing any accuracy.

$$\Rightarrow \; y_{n+1} = y_n + \frac{h}{2}\left[f(y_n, t_{n+1}) + (y_{n+1} - y_n)\left.\frac{\partial f}{\partial y}\right|_{y_n, t_{n+1}} + f(y_n, t_n)\right] + O(h^3)$$

$$\rightarrow \boxed{y_{n+1} = y_n + \frac{h}{2} \frac{f(y_n, t_{n+1}) + f(y_n, t_n)}{1 - \frac{h}{2} \frac{\partial f}{\partial y}\Big|_{y_n, t_{n+1}}}} + O(h^3)$$

linearized TR (LTR)

This formula does not require iteration while retaining global second-order accuracy.

Linear stability analysis $\rightarrow$ unconditionally stable ($y' = \lambda y$)

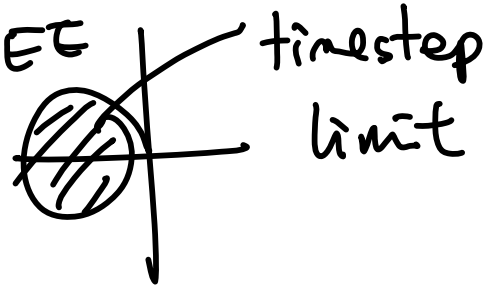
Linearization may lead to some loss of total stability for nonlinear f.

4.8 Runge-Kutta methods (RK)

① Predictor-corrector method (PC)

explicit method

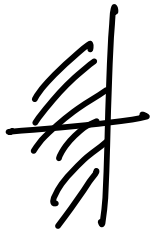
EE



implicit method

IE

TR



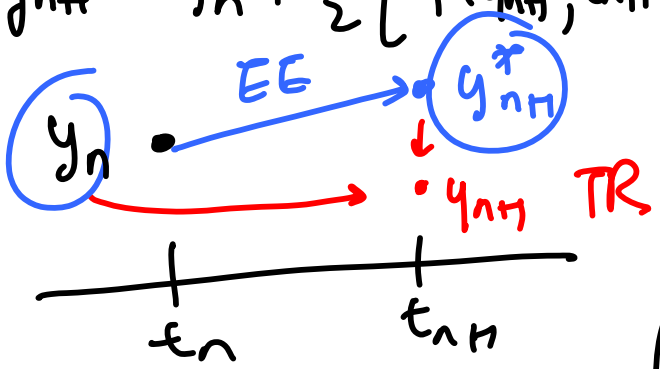
iterative sol. → LTR

predictor-corrector method

PC & RK methods provide better stability than implicit method explicit method like EE but less work/timestep than implicit method.

PC : $y' = f(y,t)$

$$\begin{cases} y_{n+1}^{*} = y_n + h f(y_n, t_n) & : \text{EE as predictor} \\ y_{n+1} = y_n + \frac{h}{2}\left[f(y_{n+1}^{*}, t_{n+1}) + f(y_n, t_n) \right] & : \text{TR as corrector} \end{cases}$$



model prob : $y' = \lambda y$

$$\begin{cases} y_{n+1}^{*} = y_n + h\lambda y_n = (1+\lambda h) y_n \\ y_{n+1} = y_n + \frac{h}{2}\left[\lambda(1+\lambda h) y_n + \lambda y_n \right] \end{cases}$$

$$= y_n \left(\underline{1 + \lambda h + \tfrac{1}{2}\lambda^2 h^2} \right) = \sigma y_n$$

exact sol. $e^{\lambda h} = \underline{1 + \lambda h + \frac{1}{2}\lambda^2 h^2} + \frac{1}{6}\lambda^3 h^3 + \cdots$

$\therefore$ PC is second-order accurate.

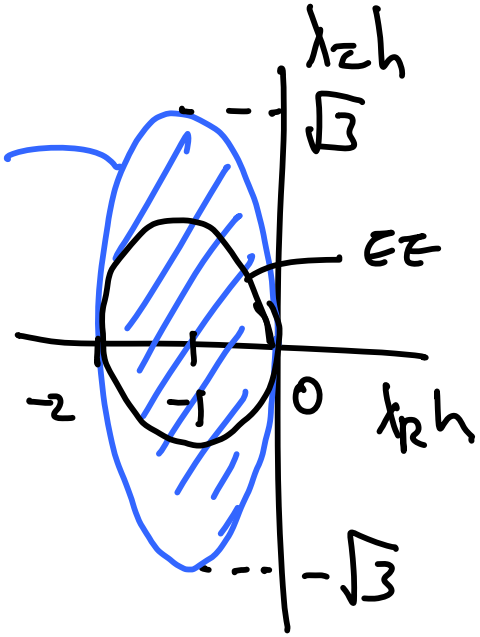
stability: $y_n = \sigma^n y_0$, $\sigma = 1 + \lambda h + \frac{1}{2}\lambda^2 h^2$

$\Rightarrow |\sigma| \le 1$ to be stable

$\rightarrow \left|\underbrace{1 + \lambda h + \frac{1}{2}\lambda^2 h^2}\right| \le 1$

$1 + \lambda h + \frac{1}{2}\lambda^2 h^2 = e^{i\theta}$

find λh for different trials for θ



For $\lambda = i\omega$, $\sigma = 1 + \lambda h + \frac{1}{2}\lambda^2 h^2 = 1 + i\omega h - \frac{1}{2}\omega^2 h^2$

$|\sigma|^2 = (1 - \frac{1}{2}\omega^2 h^2)^2 + (\omega h)^2 = 1 + \frac{1}{4}\omega^4 h^4 > 1$

unstable for purely imaginary λ.