Engineering Mathematics 2

Lecture 12

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IMPORTANT!

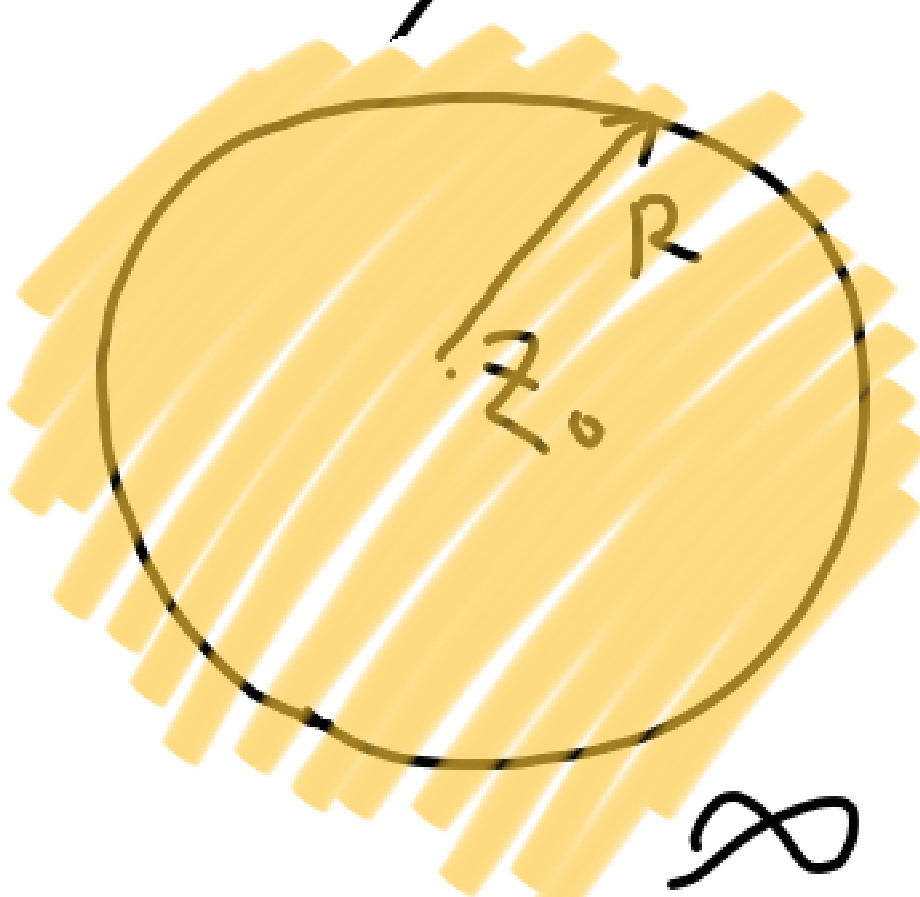
- Exam 2 on 2 Nov
- Ch 13–16 Complex analysis
- In-class exam!

Previously, we discussed

- Taylor series of analytic functions

Laurent series

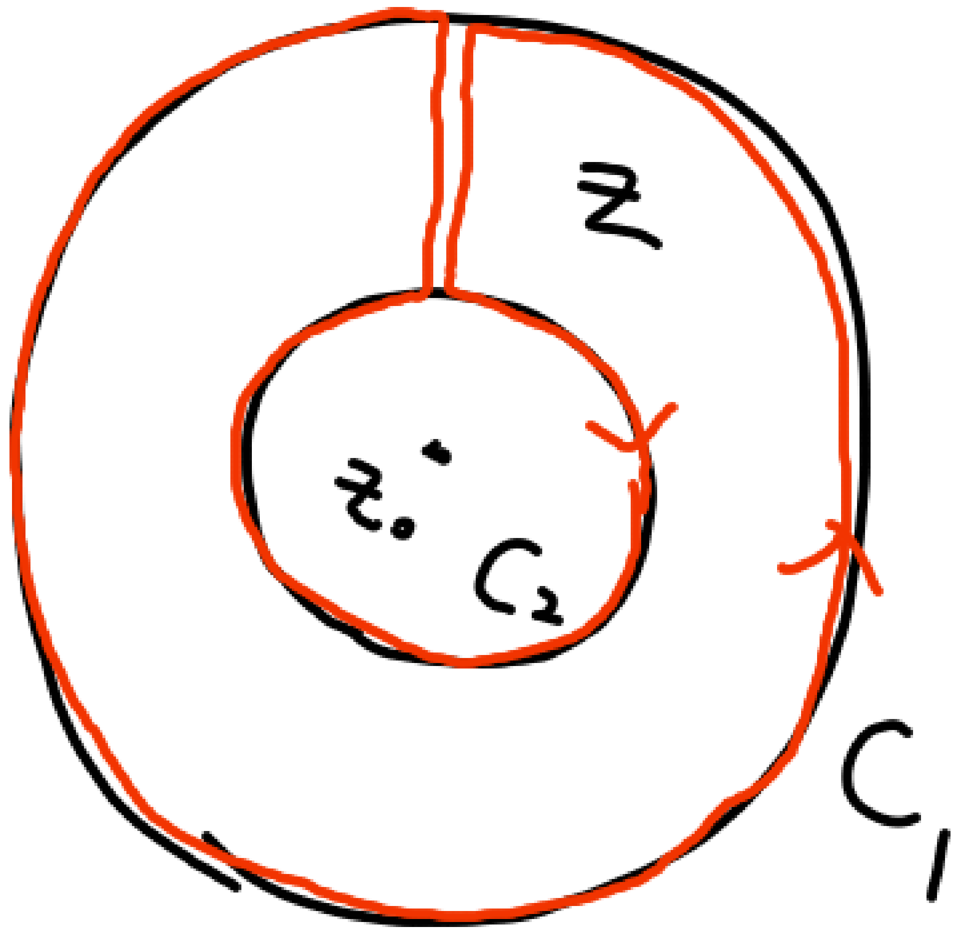
Taylor vs. Laurent



$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \qquad f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(t)\,dt}{(t - z_0)^{n+1}}$$

From Cauchy's integral formula



$$f(z) = \underbrace{\frac{1}{2\pi i}\oint_{C_1}\frac{f(t)\,dt}{t-z}}_{} - \frac{1}{2\pi i}\oint_{C_2}\frac{f(t)\,dt}{t-z}$$

On C_1 : $|t-z_0| > |z-z_0|$:

$$\frac{1}{t-z} = \frac{1}{t-z_0-(z-z_0)}$$

$$= \frac{1}{t-z_0}\,\frac{1}{1-\frac{z-z_0}{t-z_0}}$$

$$= \frac{1}{t-z_0}\left(1+\frac{z-z_0}{t-t_0}+\left(\frac{z-z_0}{t-t_0}\right)^2+\cdots\right)$$

On C_2 : $|t-z_0| < |z-z_0|$

Example : Find the Laurent series of

(a) $f(z) = z^{-5} \sin z$

$$= \frac{1}{z^5}\left(z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots \right)$$

$$= \frac{1}{z^4} - \frac{1}{3!}\frac{1}{z^2} + \frac{1}{5!} - \frac{z^2}{7!} + \cdots, \quad |z| > 0$$

(b) $f(z) = z^2 \exp \frac{1}{z}$

$$= z^2\left(1 + \frac{1}{z} + \frac{1}{2!z^2} + \frac{1}{3!z^3} + \cdots \right)$$

$$= z^2 + z + \frac{1}{2!} + \frac{1}{3!}\frac{1}{z} + \cdots$$

Example:

Develop $\dfrac{1}{1-z}$

(a) in non-negative powers of z , $|z|<1$

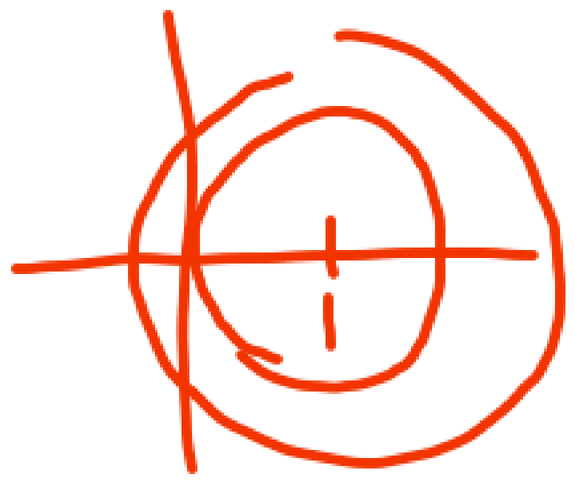
$$\frac{1}{1-z} = 1+z+z^2+\cdots$$

(b) in negative powers of z , $|z|>1$

$$\frac{1}{1-z} = \frac{\frac{1}{z}}{\frac{1}{z}-1} = -\frac{1}{z}\cdot\frac{1}{1-\frac{1}{z}} = -\frac{1}{z}\left(1+\frac{1}{z}+\frac{1}{z^2}+\cdots\right)$$

$$= -\frac{1}{z}-\frac{1}{z^2}-\frac{1}{z^3}-\cdots$$

Example: Find the Laurent series of



$f(z) = \frac{1}{z^2}$ with the centre $z_0 = 1$

$$|z-1|<1$$

$$\frac{1}{z^2} = -\left(\frac{1}{z}\right)'$$

$$\frac{1}{z} = \frac{1}{1+z-1} = \frac{1}{1-[-(z-1)]} = 1-(z-1)+(z-1)^2-(z-1)^3$$

$$\left(\frac{1}{z}\right)' = -1+2(z-1)-3(z-1)^2+4(z-1)^3-\cdots$$

$$f(z) = 1-2(z-1)+3(z-1)^2-4(z-1)^3-\cdots$$

Singularities and zeroes

- $f(z)$ is singular at $z = z_0$ if
  (i) $f(z)$ is not analytic at $z = z_0$, and
  (ii) every neighbourhood of z_0 contains a point that $f(z)$ is analytic.
- z_0 is singular point.

Laurent series

- $f(z) = \underbrace{\sum_{n=0}^{\infty} a_n (z-z_0)^n}_{\text{analytic part}} + \underbrace{\sum_{n=1}^{\infty} a_{-n} (z-z_0)^{-n}}_{\text{principal part}}$

- m-th order singular point = m-th order pole

- $m=1$: simple pole

- $m \to \infty$: isolated essential singularity

Classify the singular points

(a) $f(z) = \dfrac{1}{z(z-2i)^4}$

(b) $f(z) = \exp\left(\dfrac{1}{z}\right) = 1 + \dfrac{1}{z} + \dfrac{1}{2!\,z^2} + \dfrac{1}{3!\,z^3} + \cdots$

Behaviour near essential singularity of $f(z) = \exp\frac{1}{z}$

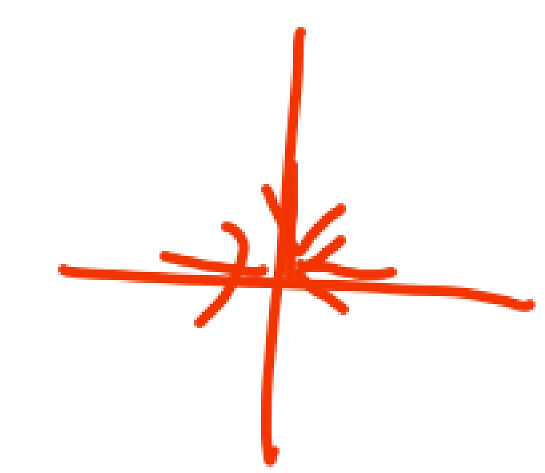
(a) Along imaginary axis $z = iy$, $f(iy) = \exp\frac{1}{iy}$

$$= \exp\left(-\frac{i}{y}\right)$$

$$= \cos\frac{1}{y} - i\sin\frac{1}{y}$$

(b) Along positive real axis $z = x$, $x \to +0$

$$\exp\frac{1}{x} \to \infty$$



(c) Along negative real axis $z = x$, $x \to -0$

$$\exp\frac{1}{x} \to 0$$

Removable singularity

$$\lim_{z \to z_0} |z - z_0| f(z) = 0$$

$$f(z) = \frac{\sin z}{z}$$

$$f(0) = 1$$

Zeros

$f(z_0) = 0 \rightarrow z_0$ is called zero

m-th order zero $\quad (z-z_0)^m g(z)$

Pole or zero at infinity : use $z = \frac{1}{w}$

$$\lim_{z \to \infty} f(z) = \lim_{w \to 0} \underline{f\left(\frac{1}{w}\right)}$$

Example : $f(z) = \frac{1}{z^2}$ is analytic at infinity.

$f\left(\frac{1}{w}\right) = w^2$ is analytic at $w=0$.