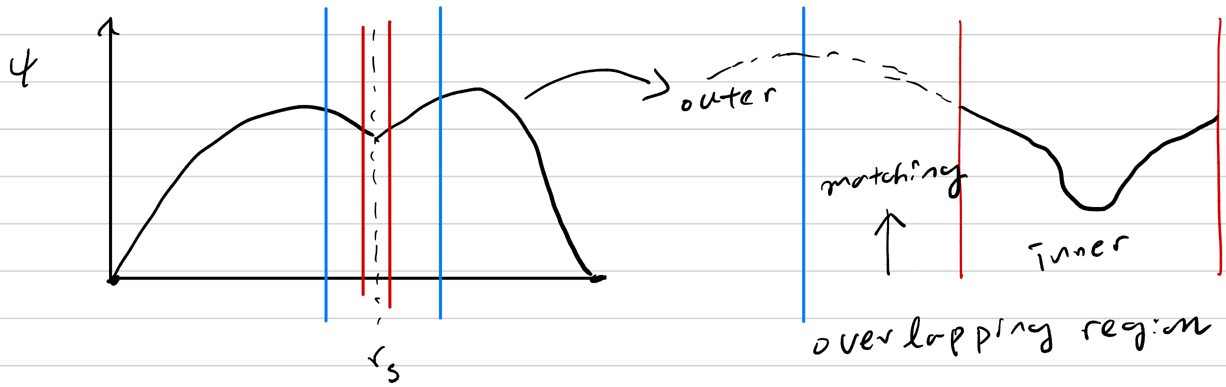
5/31 Resistive tearing instability – Continued

4. Boundary layer matching

2. outer ideal MHD
   2nd order ODE for ψ

3. inner (inertia, resistive) MHD
   4th order ODE for ψ



<u>Short cuts</u> are given when

(1) Internal linear problem

$$\left[\frac{1}{\psi}\frac{d\psi}{dr} = \frac{d}{dr}\ln\psi \right]$$ only important for matching

$$\boxed{\Delta'_{outer} = \frac{1}{\psi}\frac{d\psi}{dr}\Bigg|^{r = r_s + O(\delta)}_{r = r_s - O(\delta)}}$$ layer width

tearing mode index

$$\underline{\Delta'_{outer} \sim \Delta'_{inner}}$$

ideal MHD — non-ideal layer MHD

(2) constant-ψ approximation

take ψ as constant within layer
unless $\frac{d^n\psi}{dr^n}$

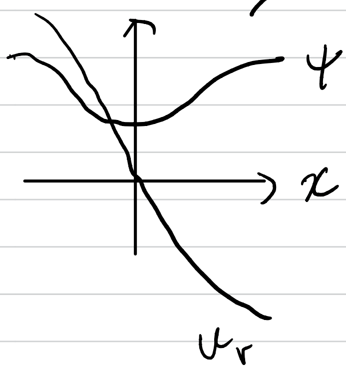
$$\Delta'_{inner} = \int_{r_s - O(\delta)}^{r_s + O(\delta)} dr \frac{d}{dr}\left(\frac{\psi'}{\psi}\right) \sim \int dr \frac{1}{\psi}\frac{d^2\psi}{dr^2}$$

$$= \frac{\mu_0 \gamma}{\eta}\int dr \left(1 + \frac{B_\theta}{\gamma}\left(1-\frac{nq}{m}\right)\frac{v_r}{\psi}\right)$$

expand $r = r_s + x$, $1-\frac{nq}{m} \approx -\frac{q'}{q}x$, $y = \frac{v_r}{\psi}$

$$\left[\begin{aligned} \frac{\gamma\rho r^2}{m^2}\frac{d^2 v_r}{dx^2} &= -\frac{B_\theta}{\mu}\frac{q'}{q}x\frac{d^2\psi}{dx^2} - \delta\phi'\psi \\ \frac{d^2\psi}{dx^2} &= \frac{\mu_0}{\eta}\left(\gamma\psi - B_\theta\frac{q'}{q}x\, v_r\right)\end{aligned}\right.$$

$$\rightarrow \frac{d^2 y}{dx^2} - \left(\frac{m^2 B_\theta^2 q'^2}{\rho\eta\gamma r^2 q^2}\right)xy = -\left[\frac{m^2 B_\theta q'}{\rho\eta r^2 q}\right]x - \frac{m^2}{\rho\gamma r^2}\delta\phi'$$



$$y = y_{odd} + y_{even}$$

$$\Delta'_{inner} = \frac{\mu_0\gamma}{\eta}\int_{-\theta(\delta)}^{\theta(\delta)} dx \left(1 - \frac{B_\theta q'}{\gamma q}x\left(y_{odd} + \cancel{y_{even}}\right)\right)$$

Now, Rescale by $X = x/\delta$ $Y = y/\beta$

$$\rightarrow \frac{\beta}{\delta^2}\frac{d^2Y}{dX^2} - \beta\delta^2(\ )X^2Y = -[\ ]\delta X$$

$$\frac{d^2Y}{dX^2} - \delta^4(\ )X^2Y = -[\ ]\frac{\delta^3}{\beta}X$$

$$\frac{d^2Y}{dX^2} - X^2Y = X$$

layer width

$$\left(\begin{aligned} \delta &\equiv \left(\frac{\rho\eta\gamma r^2 q^2}{B_\theta^2 m^2 q'^2}\right)^{\frac{1}{4}} \\ \beta &= \left(\frac{\rho\eta r^2 q}{m^2 B_\theta^2 q'}\right)^{-1}\delta^3 \end{aligned}\right.$$

$$\rightarrow Y = \frac{X}{2}\int_0^1 dt\, e^{-X^2 t/2}(1-t^2)^{-\frac{1}{4}}$$

$$\rightarrow \Delta'_{inner} = \frac{\mu_0\gamma\delta}{\eta}\int_{-\infty}^{\infty} dX(1-XY) \qquad \delta \ll x \ll 1$$

$$= \frac{\mu_0 \gamma \delta}{\eta}\left(\frac{\pi\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right) = \Delta'_{outer} \equiv \Delta'$$

$$\rightarrow \gamma^{\frac{5}{4}}\frac{\mu_0}{\eta}\left(\frac{\rho\eta r^2 q^2}{B_\theta^2 n^2 q'^2}\right)^{\frac{1}{4}}\left(\frac{\pi\Gamma(\frac{3}{4})}{\Gamma(\frac{1}{4})}\right) = \Delta'$$

$$\boxed{\gamma = \left(\frac{\Gamma(\frac{1}{4})\, r_s}{\pi\Gamma(3/4)}\right)^{4/5}\left(n\frac{r_s q'}{q}\right)^{2/5}(\Delta')^{4/5}\,\tau_R^{-3/5}\,\tau_A^{-2/5}}$$

$$\tau_A = \frac{r_s}{B_\phi/(\mu_0\rho)^{\frac{1}{2}}} \qquad \tau_R = \frac{\mu_0 r_s^2}{\eta}$$

5. physical implications for our γ

→ Surprisingly faster than τ_R

Furth/Killeen/Rosenbluth (FKR) '1962

ex) $B_\phi = 5T$, $n_e = 10^{20} m^{-3}$, $T_e = 5keV$, $r_s = 1m$

$\tau_A \sim 0.1\mu s$, $\tau_R \sim 10 min$, $\gamma \sim 70 ms$

→ Unstable if $\Delta' > 0$

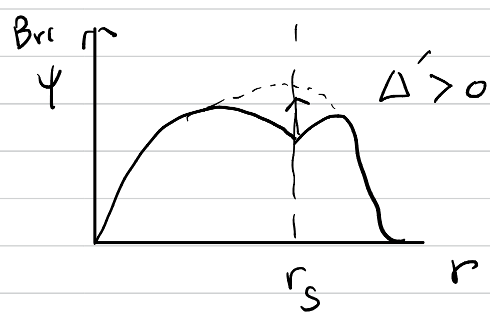
tearing mode (TM) stability

totally determined

by outer layer sol. Δ'

ex)

stable



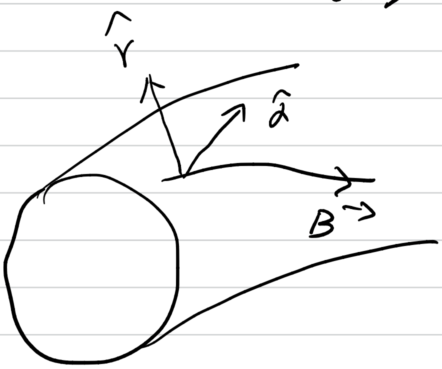
unstable

→ finite $B_{r1} e^{i(m\theta - n\phi)}$ at $q = \frac{m}{n}$

Let $B_{r1} \sin(m\theta - n\phi)$

$$= B_{r1} \sin\left(m\left(\theta - \frac{n}{m}\phi\right)\right) = B_{r1} \sin(m\alpha)$$

$$\alpha \equiv \theta - \frac{n}{m}\phi = \theta - \phi/q$$



binormal to $\hat{r}$, $\hat{b}$

$$\vec{B}\cdot\vec{\nabla}\alpha = \vec{B}\cdot\vec{\nabla}\theta - \vec{B}\cdot\vec{\nabla}\phi / q$$

$$= \frac{1}{r} B_\theta \left(1 - \frac{n}{m} q\right)$$

$$\simeq -\frac{1}{r_s} B_\theta \frac{q'}{q} x$$

$$\vec{B}\cdot\vec{\nabla} r = B_{r1} \sin(m\alpha)$$

$$\frac{dr}{d\alpha} = \frac{\vec{B}\cdot\vec{\nabla} r}{\vec{B}\cdot\vec{\nabla}\alpha} \qquad r = r_s + x$$

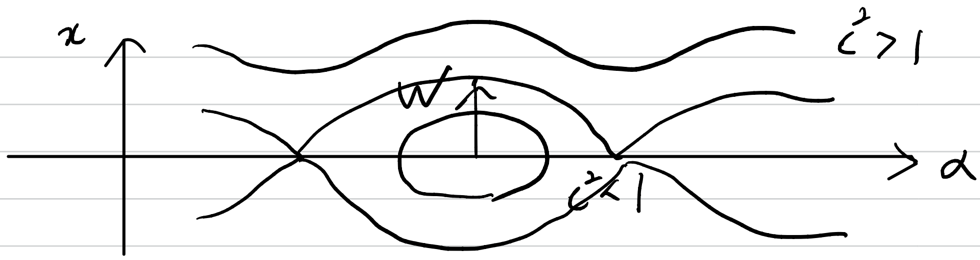
$$x \frac{dx}{d\alpha} = -\frac{r_s q B_{r1}}{B_\theta q'} \sin(m\alpha)$$

$$\frac{1}{2} x^2 = C + \frac{r_s q B_{r1}}{m B_\theta q'} \cos(m\alpha)$$

$$x = \pm W \left(C + \frac{\cos(m\alpha)}{2}\right)^{\frac{1}{2}}$$

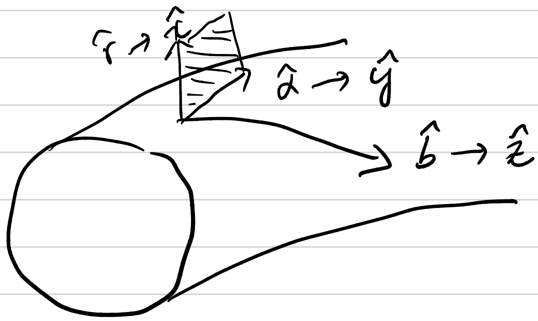
$$= \pm W \left(c^2 - \sin^2\left(\frac{m\alpha}{2}\right)\right)^{\frac{1}{2}} \qquad W = \left(\frac{4 r_s q}{m q' B_\theta} B_{r1}\right)^{\frac{1}{2}}$$

half width



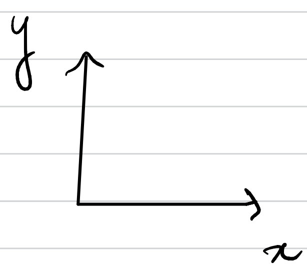
→ finite B_{r1} → magnetic island

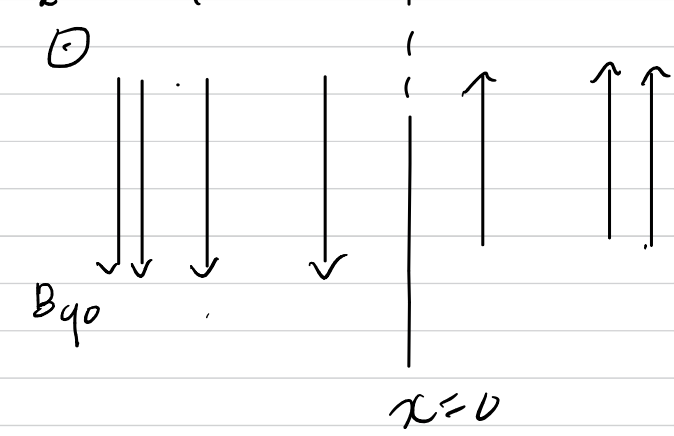
6. Reduced to a slab geometry



$$\vec{B} = B_\phi \hat{\phi} + B_\theta \hat{\theta}$$
$$\to B_z \hat{z} + B_\theta \left(1 - \frac{n}{m} q\right) \hat{y}$$
$$\approx B_z \hat{z} - B_\theta \frac{q'}{q} x \hat{y}$$
$$\to B_z \hat{z} + \frac{B_{y0}}{a} x \hat{y}$$

$B_\theta \to B_{y0}$, $\frac{q'}{q} \to \frac{1}{a}$, $\frac{m}{r} \to k$

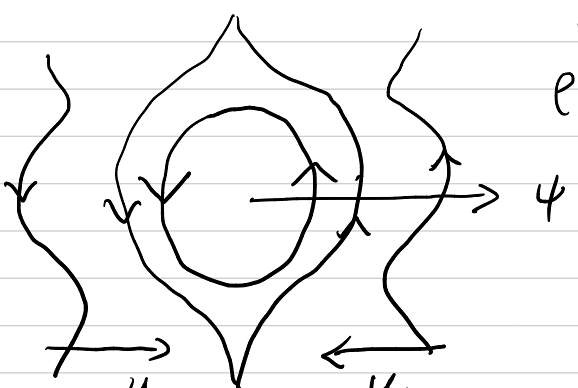




$$\star \begin{cases} \vec{B} = \vec{\nabla}\psi \times \hat{z} + B_\phi \hat{z} \\ \vec{u} = \vec{\nabla}\phi \times \hat{z} + \cancel{u_z \hat{z}} \end{cases}$$

$$\frac{\partial \psi}{\partial t} + \vec{u_\perp} \cdot \vec{\nabla}\psi = -\eta j$$

$$\rho\left(\frac{\partial \omega}{\partial t} + \vec{u_\perp} \cdot \vec{\nabla}\omega\right) = (\vec{B} \cdot \vec{\nabla}) j$$



6/2 non linear and drift instabilities

★ notes for non-linear tearing mode

Linear theory : assume islands are small

width $\delta \propto (\eta\gamma)^{\frac{1}{4}} \propto (\eta \cdot \eta^{\frac{3}{5}})^{\frac{1}{4}} \propto \eta^{\frac{2}{5}} \gg w$

Non-linear theory for big enough island

$$\frac{\partial \psi}{\partial t} + \vec{u}_E \cdot \vec{\nabla}\psi = -\eta j_z \qquad \vec{u}_E = \vec{\nabla}\phi \times \hat{z}$$

$$(\vec{\nabla}\phi \times \hat{z}) \cdot \vec{\nabla}\psi$$

Ohm's law in a slab

$$\frac{\partial \psi}{\partial t} - \vec{B}\cdot\vec{\nabla}\phi = -\eta j_z = \frac{\eta}{\mu_0}\nabla^2\psi$$

$$= -(\vec{\nabla}\psi \times \hat{z}) \cdot \vec{\nabla}\phi$$

note $\psi = \psi_0 + \psi_1(x,t)\, e^{iky}$ "Fully" non-linear

Integral over magnetic islands $\langle \vec{B}\cdot\vec{\nabla}\phi \rangle = 0$

$$w\frac{\partial \psi}{\partial t} \approx \frac{\eta}{\mu_0}\frac{\partial \psi}{\partial x}\Big|_{-w}^{w} \approx \frac{\eta}{\mu_0}\psi\Delta'$$

(here w: full width X half width)

$$\frac{1}{\psi}\frac{\partial \psi}{\partial t} \approx \frac{1}{\psi}\frac{d\psi}{dw}\frac{dw}{dt} \qquad \psi \propto w^2$$

$$\approx \frac{2}{w}\frac{dw}{dt}$$

$$\boxed{\frac{dw}{dt} = \frac{\eta}{2\mu_0}\Delta'(w)}$$

Island grows when $\Delta' > 0$, again [Rutherford '73]

Reality, Kinetic/FLR stabilizing

Bootstrap currents destabilizing

$$\frac{\partial \psi}{\partial t} + \vec{u}_E \cdot \vec{\nabla}\psi = -\eta(j_z + j_{bs}) \qquad j_{bs} \sim \frac{\epsilon^{1/2}}{B_\theta}\frac{dp}{dx}$$

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neoclassical tearing mode.

P

$$\frac{dw}{dt} = \frac{\eta}{2\mu_0}\Delta'(w) - \frac{\eta}{2}\frac{\epsilon^{1/2}}{B_\theta}\frac{dp}{dx}\left(\frac{w}{\psi}\right) \propto \frac{1}{w}$$

"Neoclassical tearing mode"

< 0

so, can be unstable $\left(\frac{dw}{dt} > 0\right)$ even when $\Delta' < 0$

Island saturates when $w_{sat} \sim \mu_0 \frac{\epsilon^{1/2}\, dp/dx}{B_\theta\, \Delta'(w)}$

★ Resistive drift wave instabilities

- "Universal" instability even with
  a straight magnetic field (like θ-pinch)
- driven by fluid drift, density & temperature gradient
- critical to add the "Hall" term in ohm's law

Ideal MHD	Resistive MHD	Drift MHD
$\vec{E} = \vec{u} \times \vec{B}$	$\vec{E} = \vec{u} \times \vec{B} + \eta \vec{j}$	$\vec{E} = \vec{u} \times \vec{B} + \eta \vec{j} + \left(\dfrac{\vec{j} \times \vec{B} - \vec{\nabla} P_e}{en} \right)$

1. Slab model for Resistive drift waves (x, y, z)
   - $\vec{B}_0 = B_0 \hat{z}$ (Don't need to consider shear (rq'))
     $\vec{j}_0 = 0$ unlike resistive MHD
   - but similar to resistive MHD

     $\vec{u}_0 = 0$ no flow in eq.

     $\vec{\nabla} \cdot \vec{u} \approx \vec{\nabla} \cdot \vec{u}_{\perp 1} = 0$ Incompressibility ($\vec{u}_{\perp 1} = \vec{\nabla} \phi \times \hat{z}$)
   - $T_i \ll T_e$ cold ion assumption
   - Consider variation in equil. along with x

   → perturbation should be

   $$\psi_1 = \psi_1(x) \exp(i k_y y + i k_z z - i \omega t)$$

   but let's assume

   $$\sim \exp(i k_x x + i k_y y + i k_z z - i \omega t)$$

   by $k_x \gg L$ (L: Scale length of equilibrium gradient)

So, its just a wave problem in extended MHD.

2. Derivation of dispersion relation

Vorticity eq. (perp. momentum)

(1) $\hat{z}\cdot\vec{\nabla}\times\left(\rho_0\frac{\partial\vec{u}_1}{\partial t}\right) = \hat{z}\cdot\vec{\nabla}\times(\vec{j}\times\vec{B})_1 = \vec{B}_0\cdot\vec{\nabla} j_z$

$\underline{\text{Same as resistive MHD}}$

note $j_z = \frac{1}{\mu_0}\left(\frac{\partial B_y}{\partial x} - ik_yB_x\right) = \frac{i}{\mu_0 k_y}\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)B_x$

$\uparrow$ $\vec{\nabla}\cdot\vec{B}=0$ $\simeq -\frac{ik_\perp^2}{\mu_0 k_y}B_x$

$\hat{z}\cdot\vec{\nabla}\times\left(\rho_0\frac{\partial\vec{u}_1}{\partial t}\right) = -i\omega\left(\frac{\partial}{\partial x}(\rho_0 u_y) - ik_y\rho_0 u_x\right)$

$\vec{\nabla}\cdot\vec{u}_\perp = 0 \rightarrow \approx \frac{-i\omega\rho_0}{-ik_y}\left(\frac{\partial^2}{\partial x^2} - k_y^2\right)u_x$

$\rho_0 \simeq$ const.

Now, $\frac{\partial^2}{\partial x^2} \sim -k_x^2 \qquad k_\perp^2 = k_x^2 + k_y^2$

$$\underline{\omega u_x = -k_z V_A^2 B_x/B_0}$$

(2) Ohm's law.

$\vec{E}_1 + \vec{u}_1\times\vec{B}_0 = \eta\vec{j}_1 + \frac{1}{en}(\vec{j}\times\vec{B} - \vec{\nabla}p_e)_1$

$\perp$: $\vec{E}_{\perp 1} + \vec{u}_1\times\vec{B}_0 = 0$, but

$\parallel$: $E_z = \eta j_\parallel - \frac{1}{en}(\nabla_\parallel p_e)_1$

$= \eta j_z - \frac{1}{en}(\hat{b}_0\cdot\vec{\nabla}p_{e1} + \hat{b}_1\cdot\vec{\nabla}p_{e0})$

$= \underline{\eta j_z - \frac{1}{en}\left(ik_z p_{e1} + \frac{B_x}{B_0}p_{e0}'\right)}$

$\left(\frac{\partial\vec{B}_1}{\partial t} = -\vec{\nabla}\times(-\vec{u}_1\times\vec{B}_0 + E_z\hat{z})\right)\cdot\hat{x}$

$$\rightarrow \omega B_x + k_z B_0 u_x = -\frac{i\eta}{\mu_0} k_\perp^2 B_x - \frac{k_y}{en}\left(ik_z p_{e1} + \frac{B_x}{B_0} p_{e0}'\right)$$

(3) continuity eq

$$ikp_{e1} + \frac{B_x}{B_0} p_{e0}' \approx T_{e0}\left(ikn_{e1} + \frac{B_x}{B_0} n_{e0}'\right)$$

(constant T_e or $\vec{B}\cdot\vec{\nabla} T_e = 0$)

with $n_e \approx n_i$

$$\rightarrow \frac{\partial n_{e1}}{\partial t} + \vec{u}_\perp \cdot \vec{\nabla} n_{e0} + \nabla_\parallel (n_{e0} u_\parallel) = 0$$

$\because u_e$

$$\rightarrow i\omega n_{e1} + u_x n_{e0}' + ik_z n_{e0} u_z = 0$$

{ * one can include $\vec{u}_\perp = \vec{u}_E + \vec{u}_{pol}$ }

(4) parallel momentum eq. (for u_z)

$$\rho_0 \frac{\partial u_\parallel}{\partial t} = -(\nabla_\parallel p_e)_1 \quad (\because p_i \ll p_e)$$

$$-i\omega\rho_0 u_z = -T_{e0}\left(ik_z n_{e1} + \frac{B_x}{B_0} n_{e0}'\right)$$

(4) $\rightarrow$ (3) $\rightarrow$ (2) $\rightarrow$ (1)

(u_z. n_{e1} B_x. u_x)

$$\rightarrow (\omega^2 - k_z^2 V_A^2)(\omega^2 - \omega_{*e}\omega - k_z^2 c_s^2) = -\frac{i\eta}{\mu_0} k_\perp^2 \omega(\omega^2 - k_z^2 c_s^2)$$

$V_A = \dfrac{B_0}{(\rho_0\mu_0)^{\frac{1}{2}}}$ $\quad C_s = (T_e/m)^{\frac{1}{2}}$, $\quad V_{*e} = -\dfrac{T_{e0}}{en_{e0}B_0}\dfrac{dn_{e0}}{dx}$, $\quad \omega_{*e} = k_y V_{*e}$

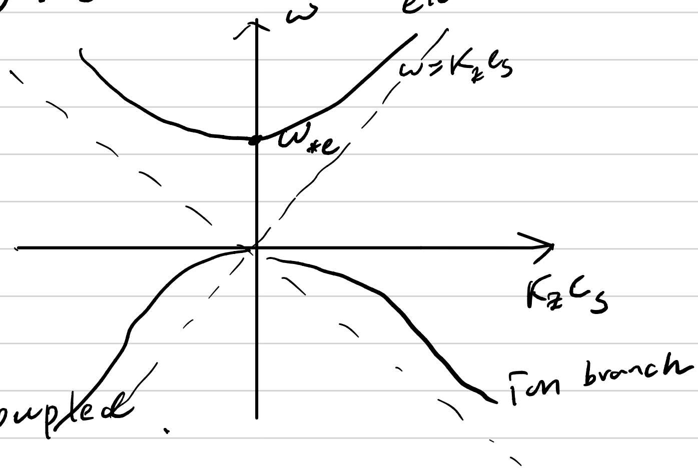
electron diamagnetic velocity with const T_e

3. Characteristics of wave instability

(1) for $\eta = 0$

$$(\omega^2 - k_z^2 V_A^2)(\underline{\omega^2 - \omega_{*e}\omega - k_z^2 C_s^2}) = 0$$

$\downarrow$ shear Alfven $\quad$ $\downarrow$ drift wave



— ω is all real w/o damping or growth.

(2) for $\eta \neq 0$, waves are coupled.

for $\omega_{*e} \sim k_z C_s \ll k_z V_A$

(i) High frequency branch

$$\omega^2 - k_z^2 V_A^2 \approx -\frac{i\eta}{\mu_0} k_\perp^2 \omega \qquad (\text{HW \#4.5})$$

(ii) Low frequency branch

$$\omega^2 - \omega_{*e}\omega - k_z^2 C_s^2 = \frac{i\eta k_\perp^2}{\mu_0} \frac{1}{k_z^2 V_A^2} \omega(\omega^2 - k_z^2 C_s^2)$$

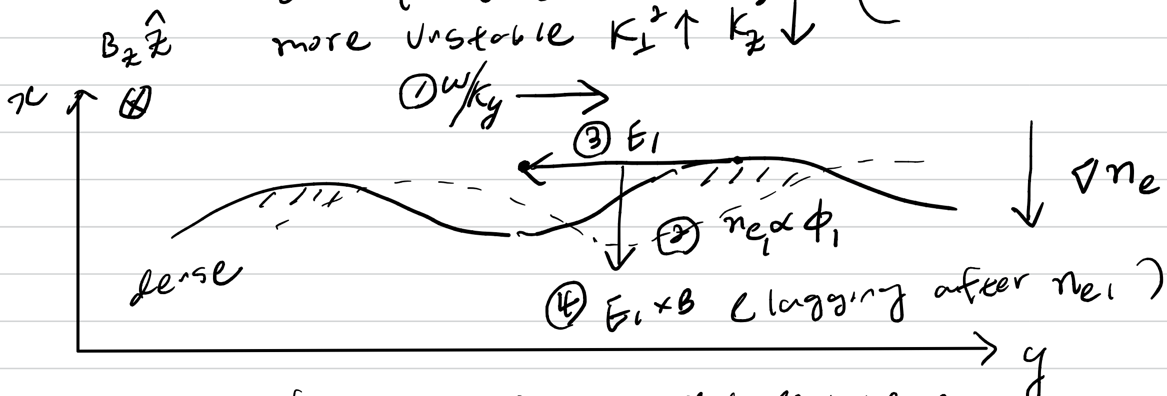
Let $\omega \to \omega + i\gamma$ $\quad (\gamma \ll \omega)$

(i) $\gamma \approx -\eta k_\perp^2 / 2\mu_0$ $\quad$ Shear Alfven damping

(ii) $\gamma \approx \dfrac{\eta k_\perp^2}{\mu_0} \dfrac{\omega^2(\omega^2 - k_z^2 C_s^2)}{k_z^2 V_A^2 (\omega^2 + k_z^2 C_s^2)}$ $\quad$ drift wave growth (instability)

always unstable $\quad$ ($\because \omega > k_z^2 C_s^2$.)

more unstable $k_\perp^2 \uparrow$ $k_z^2 \downarrow$



source of micro instability - small scale turbulence

$$(\omega^2 - k_z^2 V_A^2)(\omega^2 - \omega_{*e}\omega - k_z^2 C_s^2) = i(\qquad)$$

collisionless drift kinetic effects (e.g. ITG, ETG, TEM)