Engineering Mathematics 2

Lecture 13

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Important Notice

- Exam 2 on next Monday, 2 Nov.
- You MUST come to 4th Floor Building No. 38 by 10:55 AM.

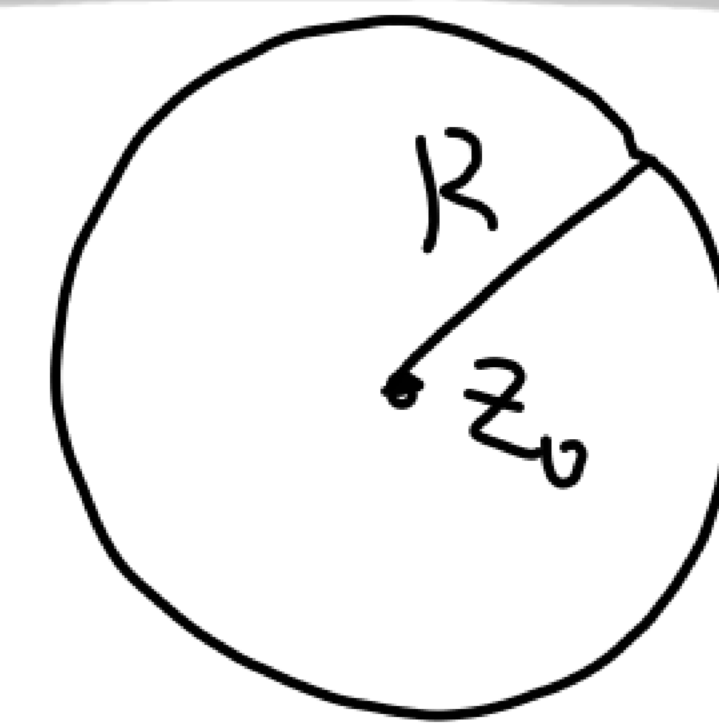
Previously,

- Laurent Series

- Poles and zeros

Residue integration method

Recall $\oint_C (z-z_0)^m \, dz$



$$z - z_0 = Re^{i\theta}, \quad dz = iRe^{i\theta} d\theta$$

$$\int_0^{2\pi} R^m e^{im\theta} \, iRe^{i\theta} d\theta = \int_0^{2\pi} iR^{m+1} e^{i(m+1)\theta} d\theta$$

$$= \begin{cases} 0, & m \neq -1 \\ 2\pi i, & m = -1 \end{cases}$$

Suppose function $f(z)$ analytic except at $z=z_0$, then

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z-z_0)^n , \quad 0<|z-z_0|<R$$

$$a_n = \frac{1}{2\pi i} \oint \frac{f(t)\,dt}{(t-z_0)^{n+1}}$$

$$\oint_c f(z)\,dz = 2\pi i \, \underline{a_{-1}}$$

$$a_{-1} = \operatorname*{Res}_{z=z_0} f(z)$$

Example) $\oint_C z^{-4} \sin z = 2\pi i \left(-\frac{1}{3!}\right) = -\frac{\pi i}{3}$, C : unit circle

$$z^{-4}\sin z = \frac{1}{z^4}\left(z - \frac{1}{3!}z^3 + \frac{1}{5!}z^5 + \cdots\right)$$

$$= \frac{1}{z^3} - \frac{1}{3!}\frac{1}{z} + \frac{1}{5!}z + \cdots$$

- If z_0 is a simple pole, $f(z) = \frac{a_{-1}}{z-z_0} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$

$$\operatorname*{Res}_{z=z_0} f(z) = \lim_{z\to z_0} (z-z_0) f(z)$$

- If z_0 is a simple zero of $g(z)$, $g(z_0) = 0$

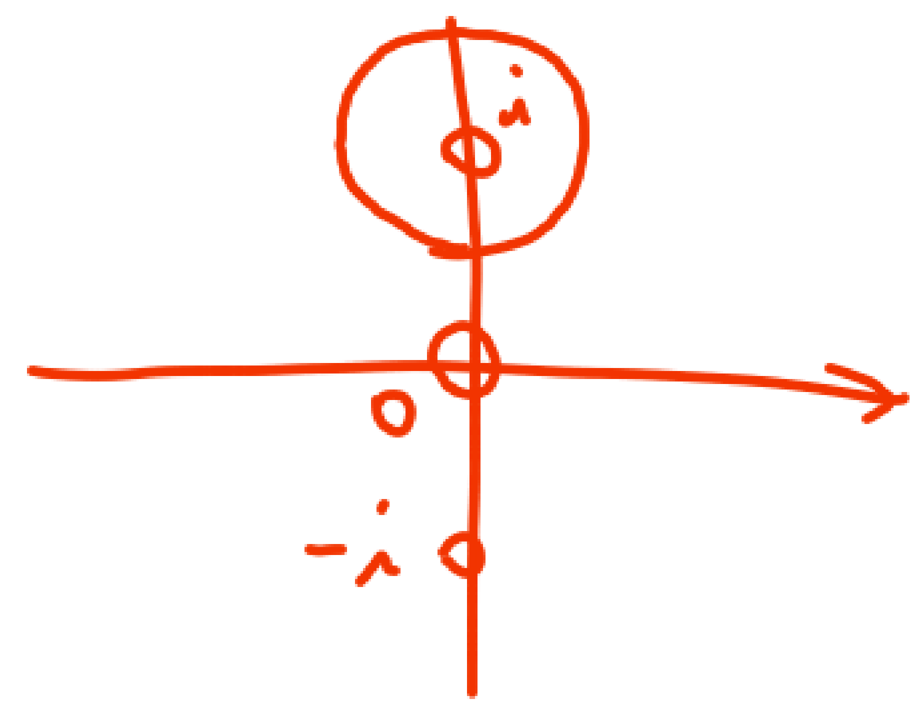
$$\operatorname*{Res}_{z=z_0} \frac{p(z)}{g(z)} = \lim_{z\to z_0} (z-z_0)\frac{p(z)}{g(z)} = \lim_{z\to z_0} \frac{p(z)}{\frac{g(z)-g(z_0)}{z-z_0}}$$

$$= \frac{p(z_0)}{g'(z_0)}$$

Example 3

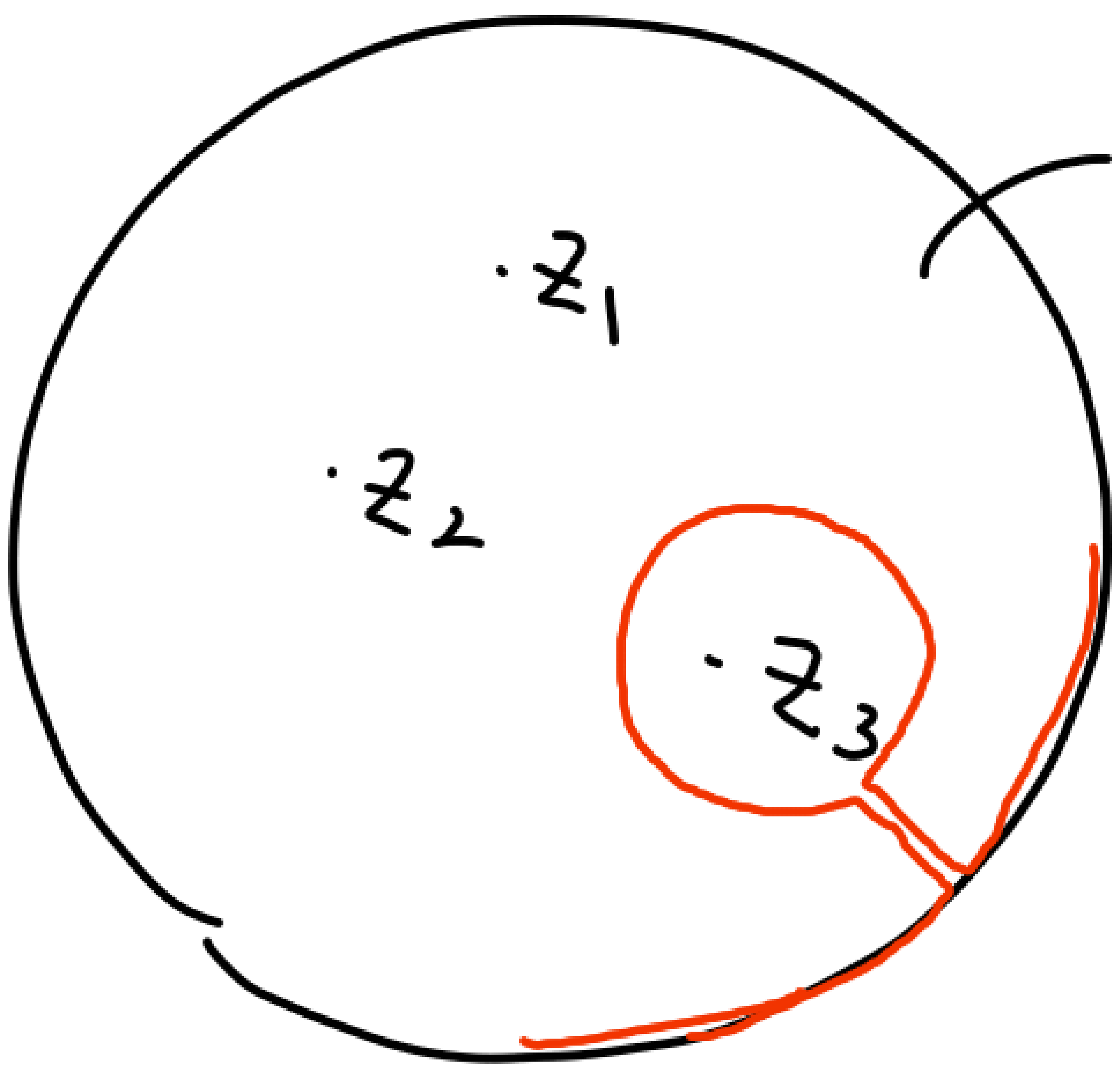
$$\oint_C \frac{9z+i}{z^3+z}\,dz \quad , \quad C: |z-i| = \frac{1}{2}$$

$$z^3+z = z(z+i)(z-i)$$



$$= 2\pi i \operatorname*{Res}_{z=i} f(z) = \cancel{2}\pi \cancel{i}\,\frac{10\cancel{i}}{\cancel{i}(\cancel{2}\cancel{i})} = 10\pi$$

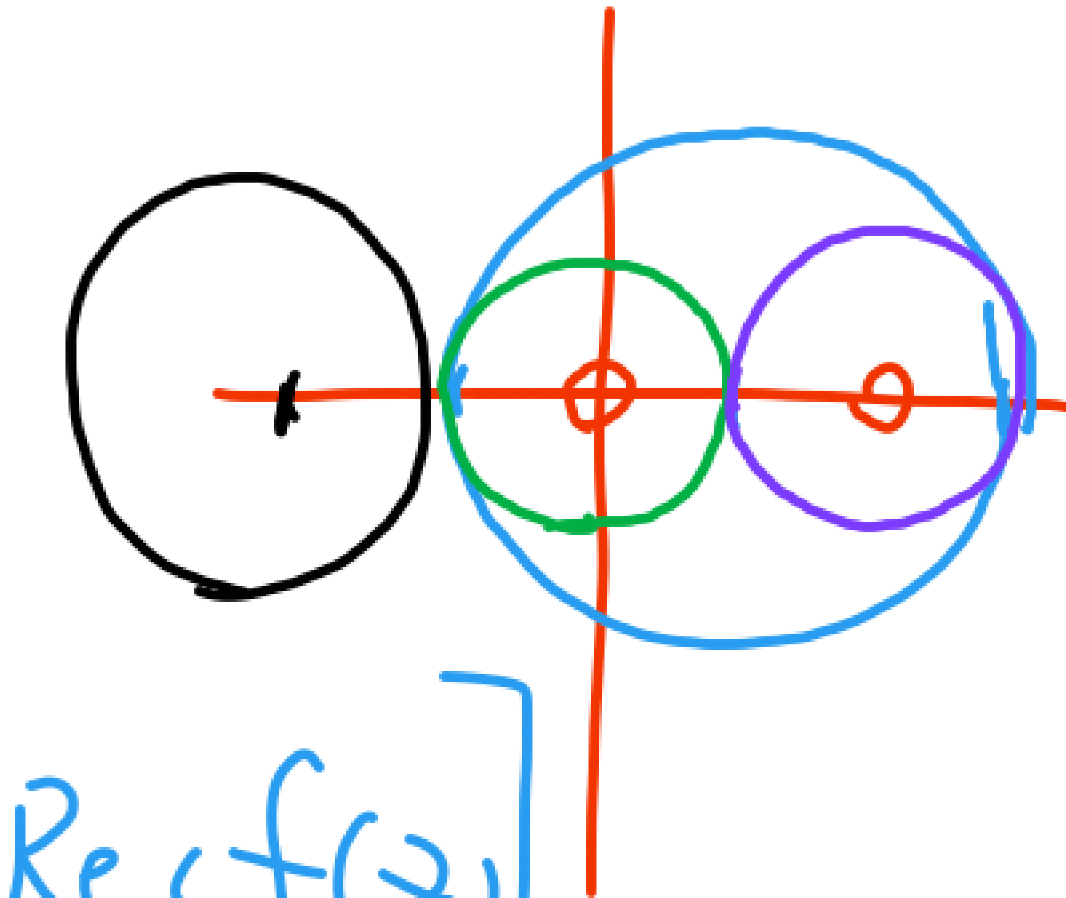
Residue theorem



$f(z)$ analytic except at $z_1, z_2, \cdots, z_n$

$$\oint_C f(z)\,dz = 2\pi i \sum_{k=1}^{n} \operatorname*{Res}_{z=z_k} f(z)$$

Example 5 $I = \oint_C \frac{4-3z}{\underbrace{z^2-z}_{=z(z-1)}}\,dz$



(a) $C: \left|z-\frac{1}{2}\right| = 1$, $I = 2\pi i\left[\operatorname*{Res}_{z=0} f(z) + \operatorname*{Res}_{z=1} f(z)\right]$

$= 2\pi i(-4+1) = -6\pi i$

(b) $C: |z| = \frac{1}{2}$, $I = 2\pi i \operatorname*{Res}_{z=0} f(z) = -8\pi i$

(c) $C: |z-1| = \frac{1}{2}$, $I = 2\pi i \operatorname*{Res}_{z=1} f(z) = 2\pi i$

(d) $C: |z+1| = \frac{1}{2}$, $I = 0$

Residue integration of REAL integrals

$$\int_0^{2\pi} F(\cos\theta, \sin\theta)\, d\theta = \oint_C f(z)\frac{dz}{iz},$$

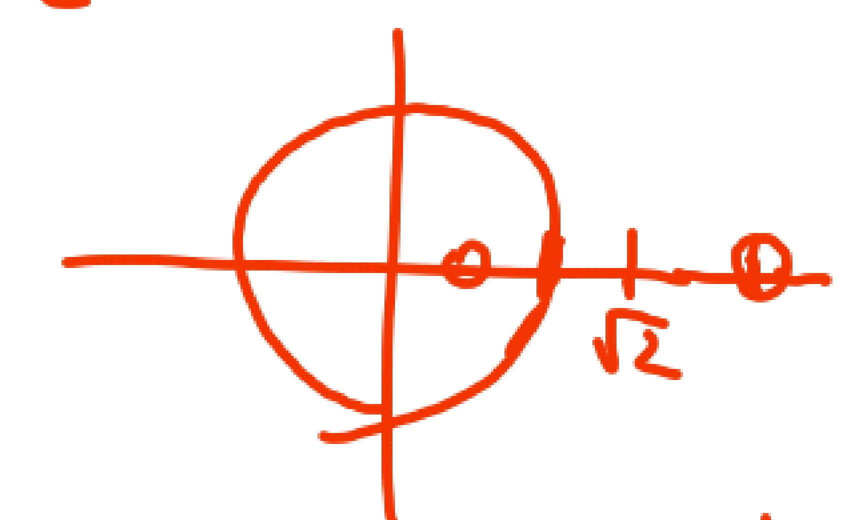
$z = e^{i\theta}$

$dz = ie^{i\theta}d\theta$

$d\theta = \frac{dz}{iz}$

$\cos\theta = \frac{1}{2}\left(z + \frac{1}{z}\right)$

Example: $\displaystyle\int_0^{2\pi} \frac{1}{\sqrt{2}-\cos\theta}\, d\theta = \oint_C \frac{dz/iz}{\sqrt{2}-\frac{1}{2}\left(z+\frac{1}{z}\right)}$



$$= 2i\oint_C \frac{dz}{z^2 - 2\sqrt{2}z + 1} = 2i \oint \frac{dz}{\left[z-(\sqrt{2}-1)\right]\left[z-(\sqrt{2}+1)\right]} = 2i(2\pi i)\left(-\frac{1}{2}\right)$$

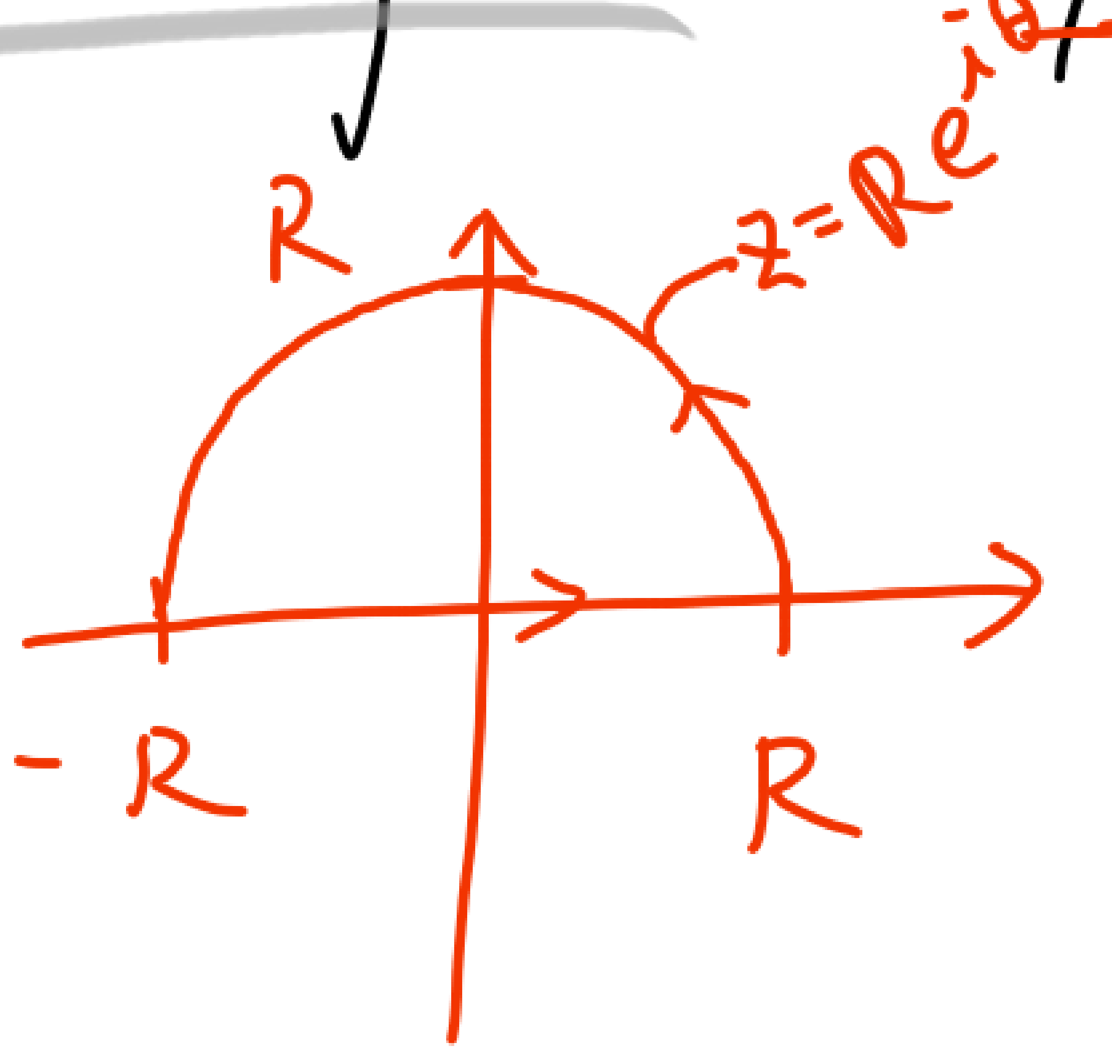
$$= 2\pi$$

$$\text{pr.v.}\int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx \qquad |f(z)| < \left|\frac{M}{z^2}\right|$$

ik

Example $\quad \text{pr.v.}\int_{-\infty}^{\infty}\frac{e^{isx}}{k^2+x^2}\,dx, \quad s>0,\ k>0$

$k^2+x^2 = (ik+x)(-ik+x)$



$$\oint f(z)\,dz = \int_{-R}^{R} f(x)\,dx + \int_0^{\pi} f(Re^{i\theta})\,iRe^{i\theta}\,d\theta$$

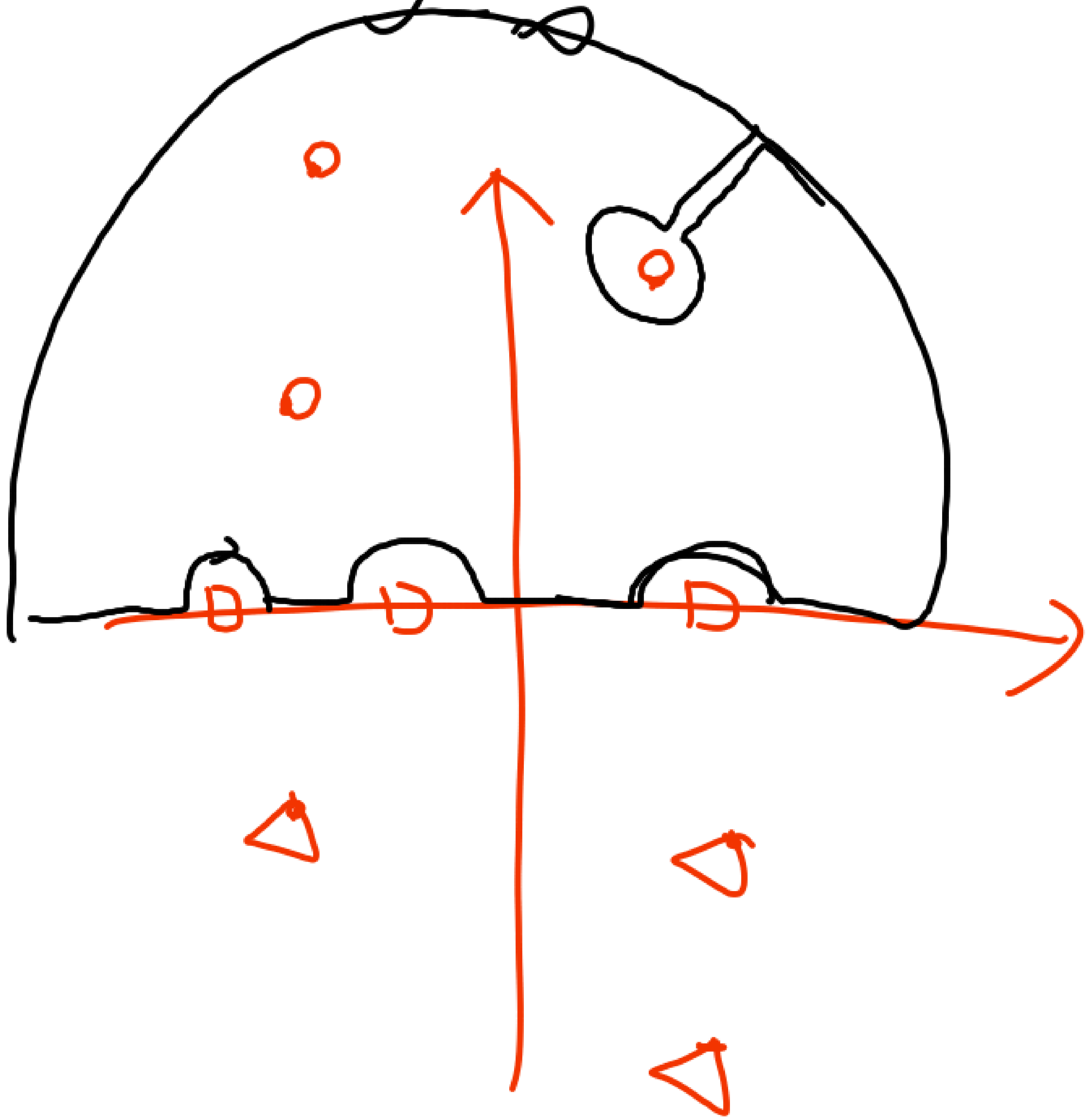
$$\left|\int_0^{\pi} f(Re^{i\theta})\,iRe^{i\theta}\,d\theta\right| < \frac{M}{R^{\cancel{2}}}\cancel{R}\cdot\pi \to 0$$

$$= \cancel{2}\pi\cancel{i}\,\frac{e^{is(ik)}}{\cancel{2}\cancel{i}k} = \frac{\pi e^{-sk}}{k}$$

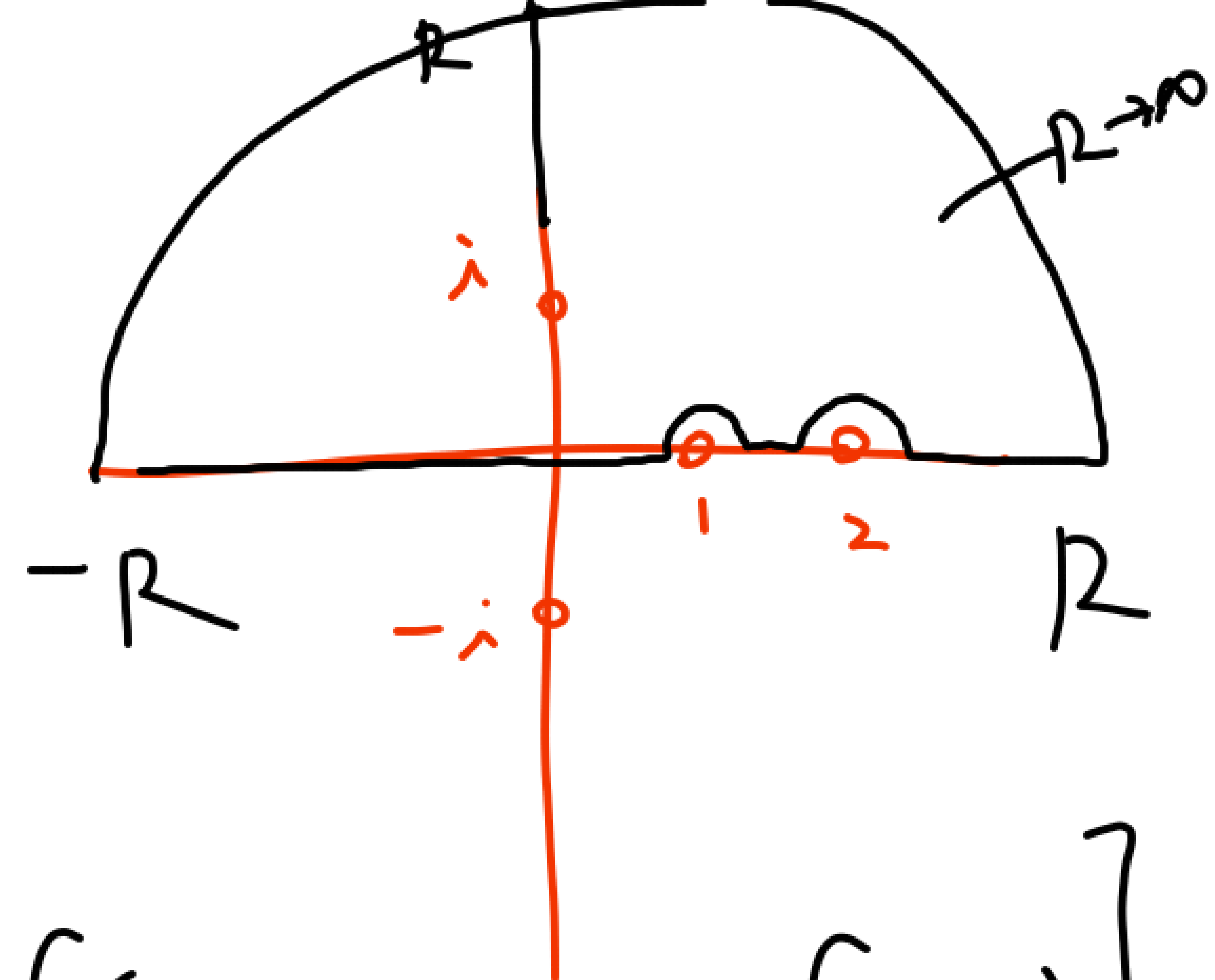
For $f(x)$, $\lim_{x \to a} |f(x)| = \infty$, $A < a < B$

$$\text{pr. v.} \int_A^B f(x)\,dx = \lim_{\epsilon \to 0} \left[\int_A^{a-\epsilon} f(x)\,dx + \int_{a+\epsilon}^B f(x)\,dx \right]$$

$$\text{pr.v.}\int_{-\infty}^{\infty} f(x)\,dx = 2\pi i \sum_{\circ} \operatorname{Res} f(z) + \pi i \sum_{\square} \operatorname{Res} f(z)$$



$$\text{pr. v.} \int_{-\infty}^{\infty} \frac{1}{\underbrace{(x^2-3x+2)}_{(x-1)(x-2)}\underbrace{(x^2+1)}_{(x-i)(x+i)}}\,dx$$



$$= 2\pi i \operatorname*{Res}_{z=i} f(z) + \pi i \left[\operatorname*{Res}_{z=1} f(z) + \operatorname*{Res}_{z=2} f(z) \right]$$

$$= \cancel{2}\pi \cancel{i} \left(\frac{1}{(i-1)(i-2)\cancel{2i}} \right) + \pi i \left(\frac{1}{-1\cdot 2} + \frac{1}{5} \right)$$

$$= \frac{\pi(1+3i)}{(1-3i)(1+3i)} + \frac{-3\pi i}{10} = \frac{\pi}{10}$$