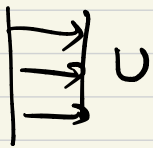
8.2 Elementary plane flow solutions

- Uniform stream : $u = U$, $v = 0$

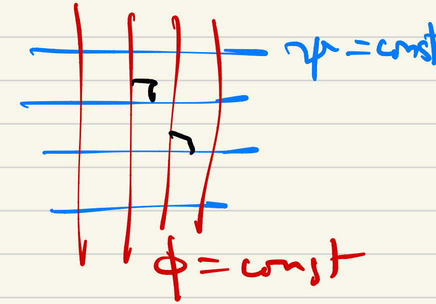


$$u = U = \frac{\partial\phi}{\partial x} = \frac{\partial\psi}{\partial y}$$

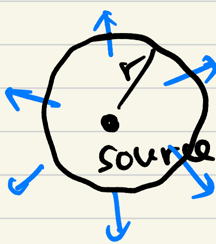
$$v = 0 = \frac{\partial\phi}{\partial y} = -\frac{\partial\psi}{\partial x}$$

$\phi = Ux$

$\psi = Uy$

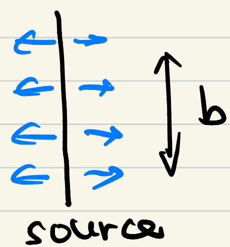


- Line source/sink @ the origin



$v_r \neq 0$

$v_\theta = 0$



$$Q = v_r\, 2\pi r\, b$$

$$v_r = \frac{Q}{2\pi r b} = \frac{m}{r} = \frac{1}{r}\frac{\partial\psi}{\partial\theta} = \frac{\partial\phi}{\partial r}$$

$m = Q/2\pi b$: source strength

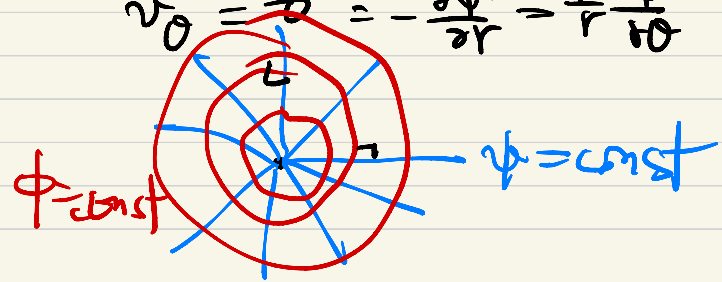
$$v_\theta = 0 = -\frac{\partial\psi}{\partial r} = \frac{1}{r}\frac{\partial\phi}{\partial\theta}$$

$m > 0$ source

$m < 0$ sink

$\rightarrow$ $\psi = m\theta$

$\phi = m \ln r$



- Line irrotational vortex (free vortex)

$\omega = 0$

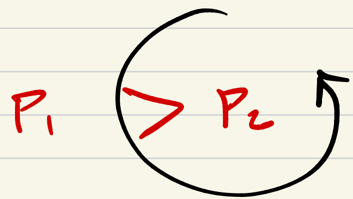
$v_r = 0$, $v_\theta \neq 0$ → $v_\theta = f(r)$

$\nabla \times \underline{v} = 0 \rightarrow \omega_z = \frac{1}{r}\frac{\partial}{\partial r}(r v_\theta) - \cancel{\frac{1}{r}\frac{\partial v_r}{\partial \theta}} = 0$

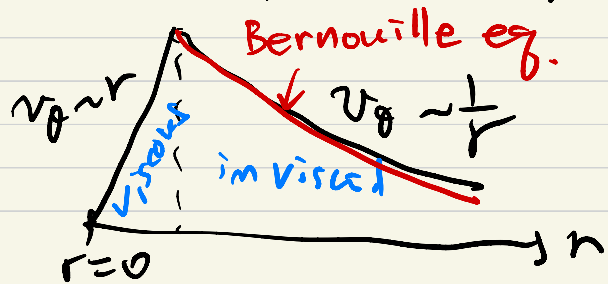
$r v_\theta = \text{const} = K$

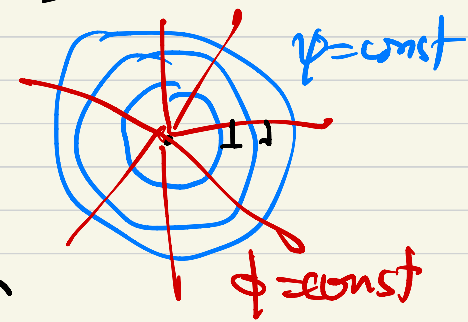
$\rightarrow v_\theta = \frac{K}{r}$ K: vortex strength

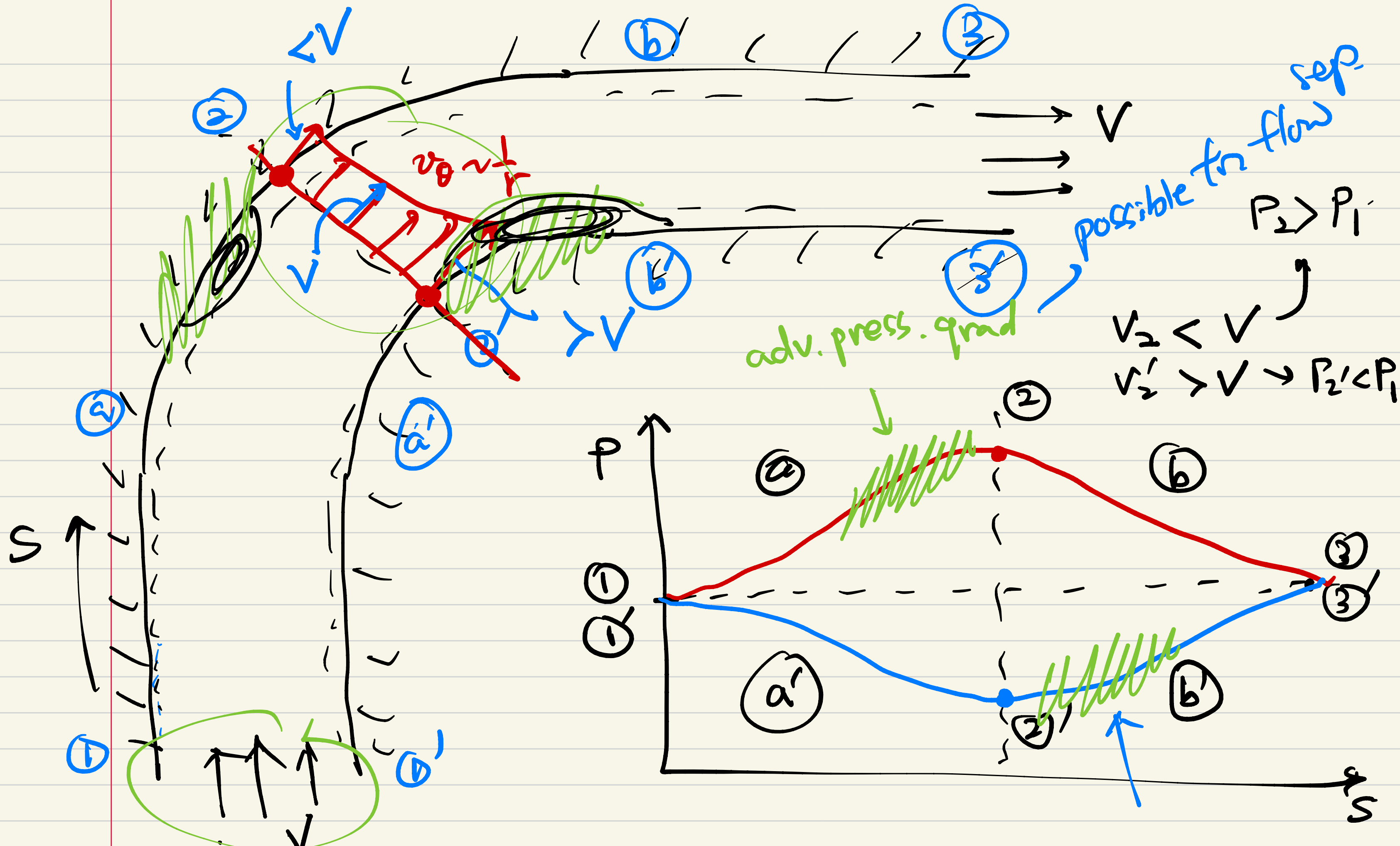
As $r \to 0$, $v_\theta \to \infty$ ⇒ pressure @ vortex center is lowest



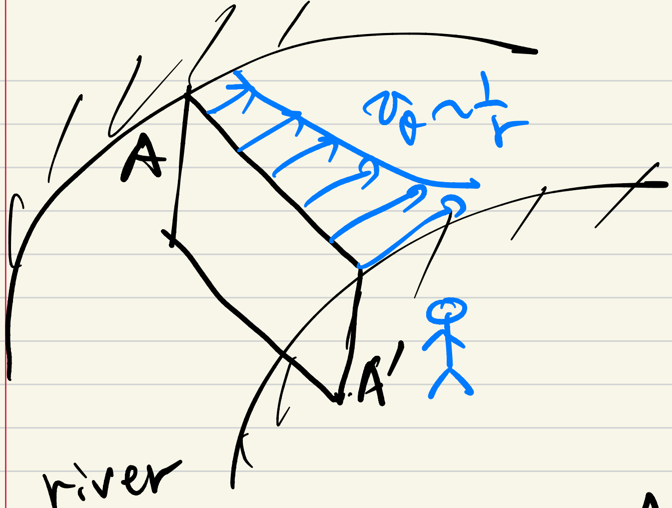
$$v_r = \frac{1}{r}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r} = 0$$
$$v_\theta = -\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta} = \frac{K}{r}$$
$\Bigg\}$ $\psi = -K \ln r$, $\phi = K\theta$

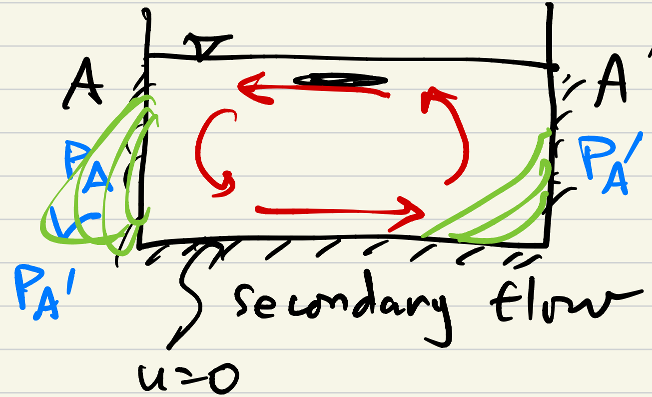


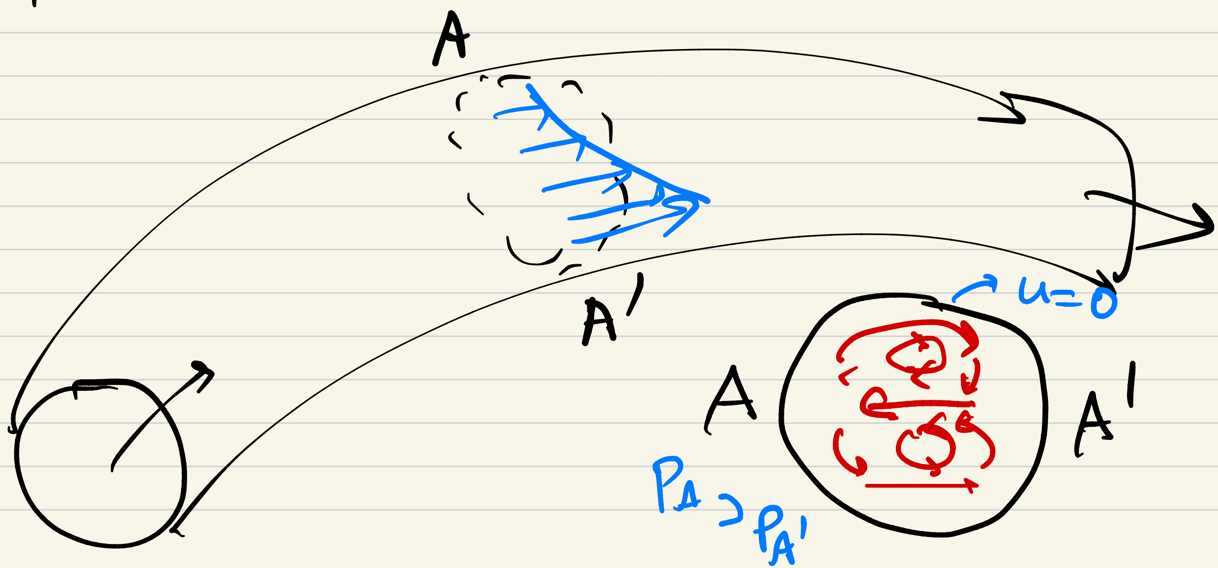




< V
②
$v_\theta \sim \frac{1}{r}$
V
> V
ⓑ
ⓑ'
③
③'
V
possible for flow sep.
$P_2 > P_1$
$V_2 < V$
$V_2' > V \rightarrow P_2' < P_1$
adv. press. grad
ⓐ
ⓐ'
S
①
①'
V
P
②
②'
ⓐ
ⓑ
ⓐ'
ⓑ'
①
①'
③
③'
S


A
A'
$v_\theta \sim \frac{1}{r}$
river


A
A'
P_A
P_A'
$P_{A'}$
secondary flow
$u=0$


A
A'
$u=0$
A
A'
$P_A > P_{A'}$

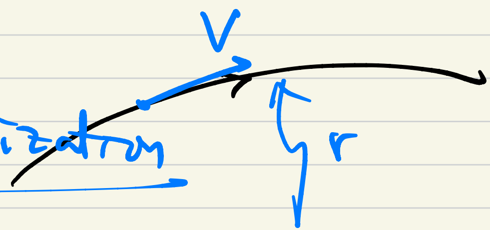
irrotational vortex ($v_r = 0,\ v_\theta \sim \frac{1}{r}$)

r-momentum eq

$$\overset{0}{\cancel{\frac{\partial v_r}{\partial t}}} + \overset{0}{\cancel{(\underline{v}\cdot\nabla) v_r}} - \frac{1}{r} v_\theta^2 = -\frac{1}{\rho}\frac{\partial p}{\partial r} + \underset{0}{\nu}\left(\overset{0}{\cancel{\nabla^2 v_r}} - \overset{0}{\cancel{\frac{v_r}{r^2}}} - \overset{0}{\cancel{\frac{2}{r^2}\frac{\partial v_\theta}{\partial \theta}}}\right)$$

$$\rightarrow \quad \frac{\partial p}{\partial r} = \rho\frac{v_\theta^2}{r}$$

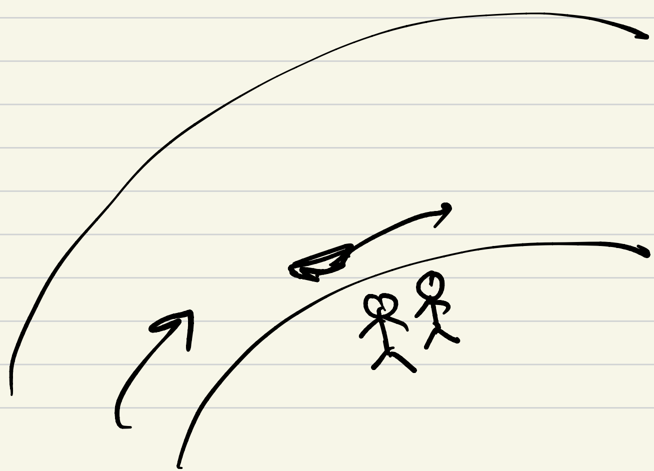
flow visualization

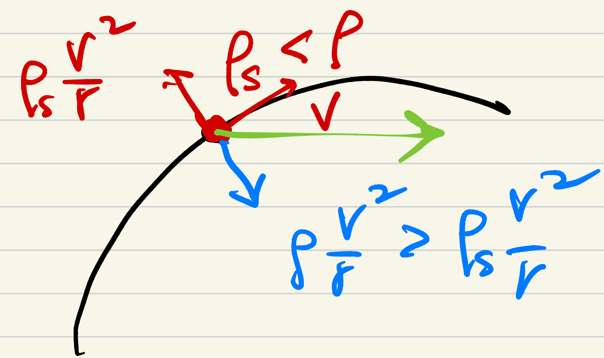


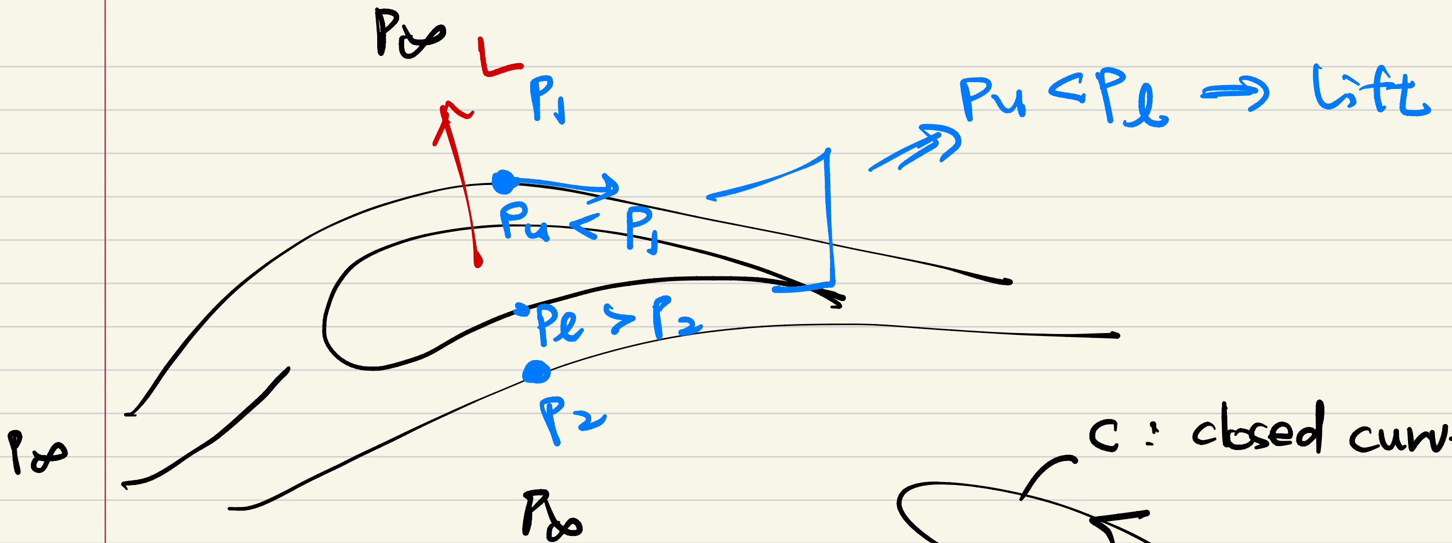
$$\frac{\partial p}{\partial r} = \frac{\rho v_\theta^2}{r} = \frac{\rho V^2}{r} > 0$$

$$\boxed{\frac{\partial p}{\partial n} = \frac{\rho V^2}{r} > 0}$$ Euler-n eq.

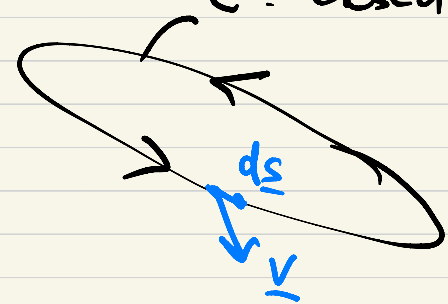
normal







C : closed curve



- Circulation Γ (순환)

$$\boxed{\Gamma = \oint_C \underline{V}\cdot d\underline{s}}$$

inviscid
viscous

$$= \int_C (u\,dx + v\,dy)$$

irrotational vortex $(\underline{\omega} = 0)$ $\Gamma = \oint_C \nabla\phi \cdot d\underline{s} = \oint_C d\phi$

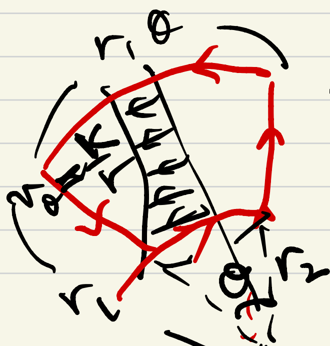
$\Gamma = 0$ for irrotational flow on the whole domain.

Free vortex $\phi = k\theta \rightarrow \Gamma = \oint_c d\phi = 2\pi k \neq 0$

Γ denotes the net algebraic strength of all the vortex filaments contained within the closed curve.

$$\Gamma \neq 0 \quad \omega_z = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} \neq 0 \quad \text{rotational flow}$$

$$= 0 \quad \text{irrotational "}$$



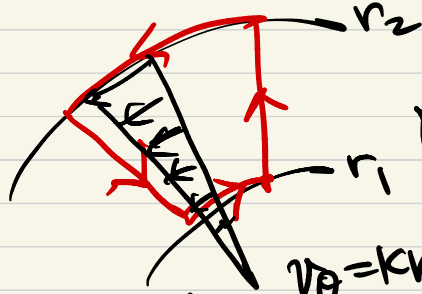
$$\oint \underline{v} \cdot d\underline{s}$$

$$= +\frac{k}{r_1} \cdot r_1\theta - \frac{k}{r_2} \cdot r_2\theta$$

$$= 0$$

irrotational flow

Viscous flow



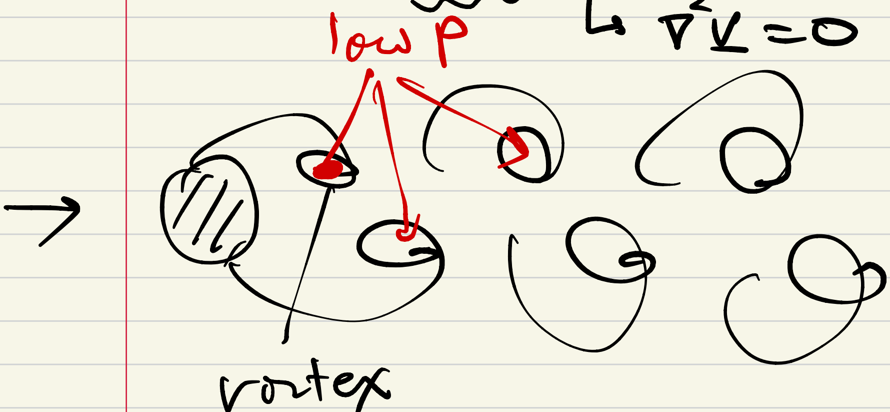
$$\Gamma = \oint \underline{v}\cdot d\underline{s} = k r_2 \cdot r_2\theta - k r_1 \cdot r_1 \theta$$
$$= k\theta(r_2^2 - r_1^2) \neq 0$$

$v_\theta = kr$

$v_\theta \sim r$ (solid-body rotation) ↳ $\nabla^2 \underline{v} = 0$

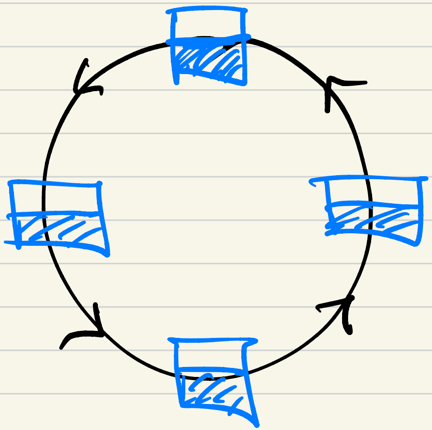
$$\nabla p = \rho(\cancel{\underline{g}} - \underline{a}) = -\rho \underline{a}$$
$$\underline{v} = \underline{\Omega} \times \underline{r}$$
$$\underline{a} = \underline{\Omega} \times (\underline{\Omega} \times \underline{r}) = -\rho r \Omega^2 \hat{r}$$
$$\rightarrow \frac{\partial p}{\partial r} = \rho r \Omega^2 = \rho \frac{v^2}{r} > 0$$

same as that of irrotational vortex



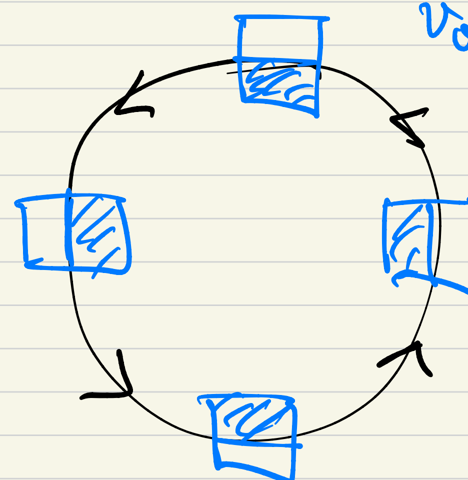
irrotational circular motion

$v_\theta \sim \frac{1}{r}$



rotational circular motion

$v_\theta \sim r$



vortex, $N-S \Rightarrow u_i^{(x)} , P(x)$

vorticity, loop, $\underline{v}$ $\nabla \underline{v}$ λ_2

$\underline{R}$ $\underline{S}$