

• 2nd-order Adams-Basforth method (AB2) - widely used

노트 제목

2019-10-21

$$y_{n+1} = y_n + h y'_n + \frac{1}{2} h^2 y''_n + \frac{1}{6} h^3 y'''_n + \dots$$

$\underbrace{f(y_n, t_n)}_{\text{if}}$ $\underbrace{\frac{y_n - y_{n-1}}{h}}_{\text{if}} + \underbrace{\Theta(h)}$ $\therefore \text{neglect } \Theta(h) \text{ term}$

$$\Theta(h^3)$$

$$\rightarrow y_{n+1} = y_n + h y'_n + \frac{1}{2} h (y'_n - y'_{n-1}) + \Theta(h^3)$$

$$= y_n + h f(y_n, t_n) + \frac{1}{2} h [f(y_n, t_n) - f(y_{n-1}, t_{n-1})] + \Theta(h^3)$$

$$\rightarrow \boxed{y_{n+1} = y_n + \frac{1}{2} h [3f(y_n, t_n) - f(y_{n-1}, t_{n-1})]} + \Theta(h^3) \quad AB2$$

$$\text{or, } \boxed{\frac{y_{n+1} - y_n}{h} = \frac{1}{2} [3f(y_n, t_n) - f(y_{n-1}, t_{n-1})]} + \Theta(h^2)$$

globally 2nd-order accurate
explicit

multi-step method \rightarrow not self-starting

$$y' = \lambda y : y_{n+1} = y_n + \frac{1}{2}h[3\lambda y_n - \lambda y_{n-1}]$$

$$\rightarrow y_{n+1} - (1 + \frac{3}{2}\lambda h)y_n + \frac{1}{2}\lambda h y_{n-1} = 0$$

Assume $y_n = \sigma^n y_0 \rightarrow \sigma^2 - (1 + \frac{3}{2}\lambda h)\sigma + \frac{1}{2}\lambda h = 0$

$$\rightarrow \sigma = \frac{1}{2} \left[(1 + \frac{3}{2}\lambda h) \pm \sqrt{1 + \lambda h + \frac{9}{4}\lambda^2 h^2} \right]$$

$$\left(\sqrt{1+x} = 1 + \frac{1}{2}x - \frac{1}{8}x^2 + \frac{3}{48}x^3 + \dots \right)$$

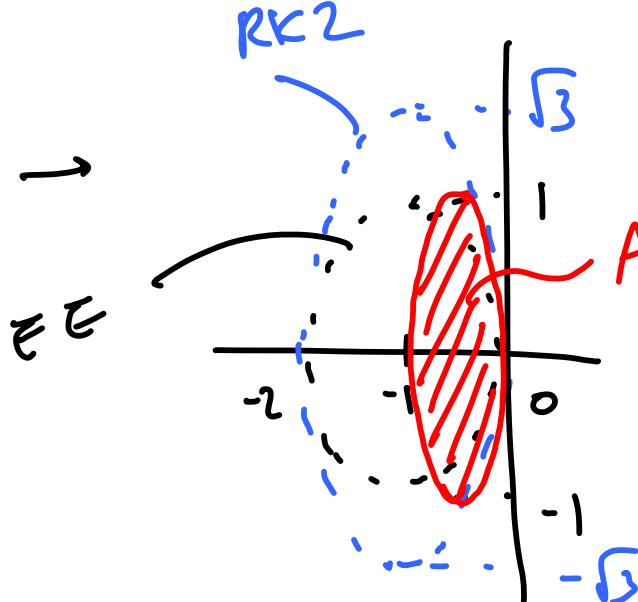
$$\rightarrow \sigma_1 = 1 + \lambda h + \frac{1}{2} \lambda^2 h^2 + \dots \text{ 2nd-order accurate}$$

$$\sigma_2 = \frac{1}{2} \lambda h - \frac{1}{2} \lambda^2 h^2 + \dots \text{ spurious root.}$$

$$q_n = c_1 \sigma_1^n + c_2 \sigma_2^n \quad \text{as } h \rightarrow 0, \sigma_2 \rightarrow 0$$

less dangerous

$$|\sigma| \leq 1 \rightarrow$$

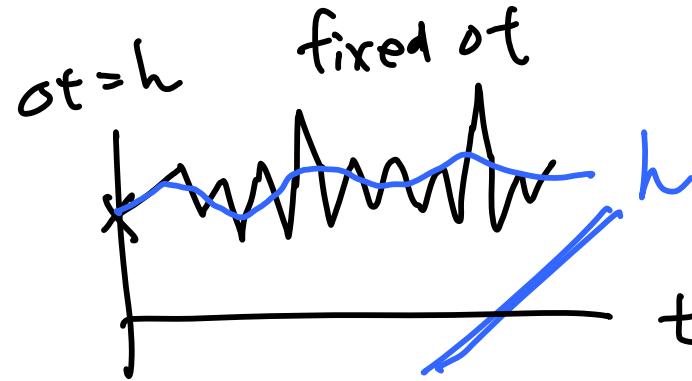
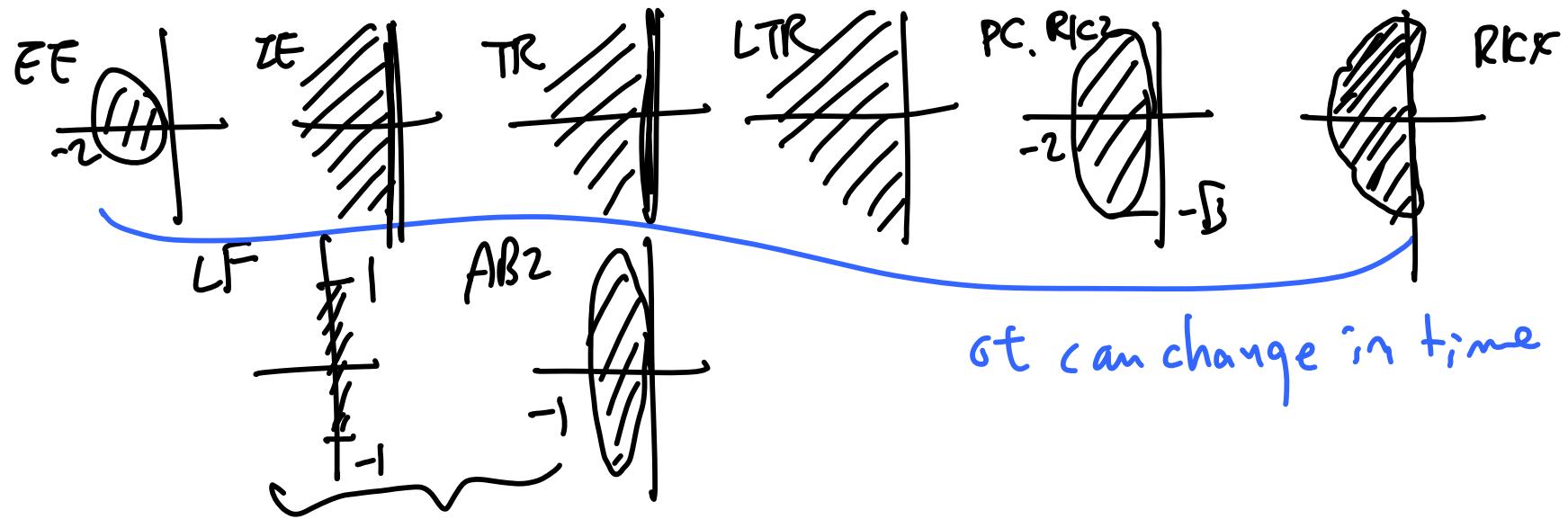


AB2, oval shape

conditionally stable

unstable for $\lambda = i\omega$

mild instability



$$h_{\text{old}} + h_{\text{new}}(1-\alpha) \quad 0 < \alpha < 1$$

λ real & negative

stability $\rightarrow |\lambda| h \leq *$

$$\rightarrow h \leq * / |\lambda| \quad \xrightarrow{\text{def}} x(t)$$

$$h_{\max} = * / |\lambda| \rightarrow h'_{\max} = 0.9 * / |\lambda|$$

4.10 System of first-order ODEs

ex) chemical reactions

$$\left\{ \begin{array}{l} \frac{dy_1}{dt} = \alpha_{11} y_1^2 + \alpha_{12} y_1 y_2 + \dots \\ \frac{dy_2}{dt} = \alpha_{21} y_1 y_2 + \alpha_{22} y_2^2 + \dots \\ \dots \end{array} \right.$$

higher order
ODE

→ system of 1st-order
ODEs,

ex) Blasius eq im fluid mechanics

$$f''' + f f'' = 0$$

$$y_1 = f$$

$$y_2 = y_1' = f'$$

$$y_3 = y_2' = f''$$

$$y_3' = -y_1 y_3$$

↓

$$\left\{ \begin{array}{l} y_1' = y_2 \\ y_2' = y_3 \\ y_3' = -y_1 y_3 \end{array} \right.$$

From the conceptual point of view, there is only one fundamental difference between numerical sol. of one ODE and that of a system of ODEs → stiffness

single ODE vs. system of ODE's

$$\frac{dy}{dt} = f(y, t)$$

$$\frac{dy}{dt} = \lambda y$$

Model prob.

$$\frac{dy_i}{dt} = f_i(t, y_1, y_2, \dots, y_m) \quad i=1, 2, \dots, m$$

$$\frac{dy}{dt} = Ay$$

Assume that A has a complete set of eigenvalues.

$$\rightarrow A = S \Lambda S^{-1} \quad \Lambda: \text{eigenvalue matrix}$$

$$\rightarrow \frac{dy}{dt} = S^T \Lambda S y \rightarrow \frac{d(Sy)}{dt} = \Lambda(Sy) \quad u = Sy$$

$$\Lambda = \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix}$$

$$\rightarrow \frac{du}{dt} = \Lambda u \rightarrow \frac{du_i}{dt} = \lambda_i u_i$$

$\text{if } u_i^* = (1 + \lambda_i h)^n u_i^0$

for stability, $|1 + \lambda_i h| \leq 1 \quad i=1, 2, \dots, n$

largest eigenvalue \rightarrow smallest h

$$\frac{dy}{dt} = Ay \rightarrow E\Lambda E^{-1} y_{n+1} = y_n + hA y_n = (I + hA) y_n$$

$$\rightarrow y_n = (I + hA)^n y_0 \equiv B^n y_0$$

for stability $|\alpha_i| \leq 1$ α_i : eigenvalues of B

$$\alpha_i = 1 + h\lambda_i \quad \lambda_i : \text{ " } \quad \text{of } A$$

$$\hookrightarrow |\alpha_i| = |1 + h\lambda_i| \leq 1$$

$$\text{for real } \lambda_i, -1 \leq 1 + h\lambda_i \leq 1 \rightarrow h \leq \frac{2}{|\lambda_{\max}|}$$

$$\text{stiffness} \equiv \frac{|\lambda_{\max}|}{|\lambda_{\min}|} \gg 1 \rightarrow \text{system is stiff.}$$

Stiffness can arise in physical systems with several degrees of freedom, but with widely different response times.

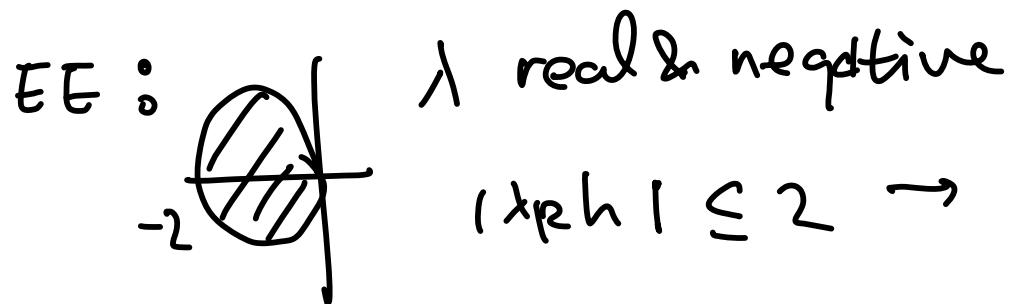
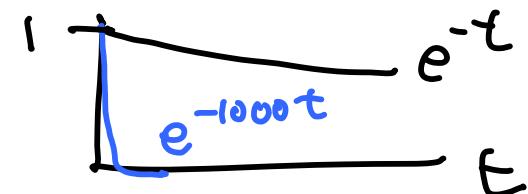
ex) a system composed of two springs,
one very stiff and the other very flexible

ex) a mixture of chemical species w/ very different
reaction times.

$$\begin{cases} u' = 998u - 1998v & u(0) = v(0) = 1 \\ v' = -999u - 1999v \end{cases}$$
$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \underbrace{\begin{pmatrix} 998 & -1998 \\ -999 & -1999 \end{pmatrix}}_A \begin{pmatrix} u \\ v \end{pmatrix}$$
$$\det(A - \lambda I) = 0$$
$$\rightarrow \lambda_1 = -1 \quad \lambda_2 = -1000 \quad) \text{ stiff}$$

exact sol.

$$\begin{cases} u = 4e^{-t} - 3e^{-1000t} \\ v = -2e^{-t} + 3e^{-1000t} \end{cases}$$

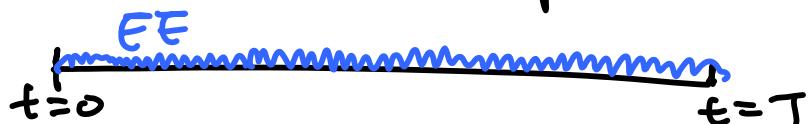


$$|\lambda_2 h| \leq 2 \rightarrow h \leq \frac{2}{|\lambda_1|} \text{ or } \frac{2}{|\lambda_2|} \Rightarrow h_{\max} = \frac{2}{|\lambda_{\max}|} = \frac{1}{500}$$

Advance 5 timesteps, $t = 5 \cdot \frac{1}{500} = 0.01$

$$\Rightarrow e^{-1000t} = 4.5 \times 10^{-5}, e^{-t} = 0.99$$

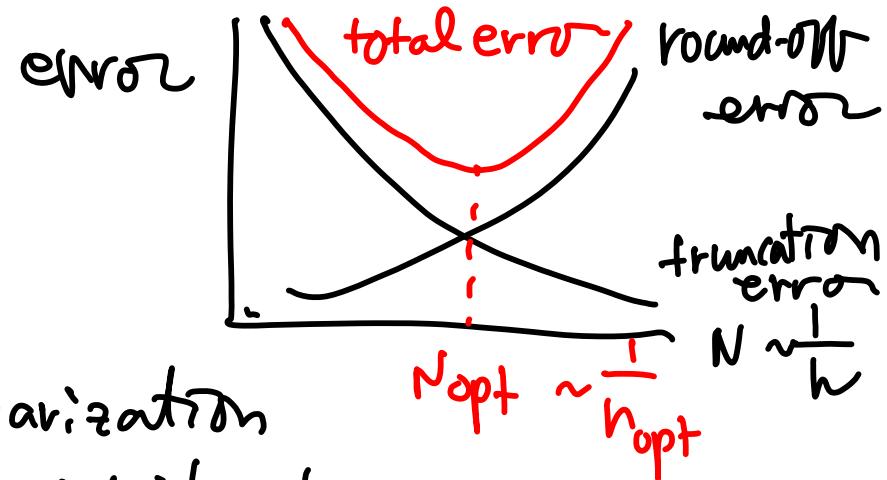
take 3500 timesteps to drive e^{-t} to 10^{-3} .



↓
use implicit methods !

non linear algebraic egs.

↳ iteration \leftarrow linearization
to avoid iteration



Midterm — Nov. 6 (Wed) 6:30 PM