Engineering Mathematics 2

Lecture 14

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Previously, we discussed

- Residue theorem and residue integration method

- Real integration

11.1 Fourier series

- $f(x+p) = f(x)$ for all x:

  $f(x)$ is a periodic function with the period p.

- Then, $f(x+np) = f(x)$ where n is an integer.

Recall

$$e^{i(\theta_1+\theta_2)} = e^{i\theta_1} e^{i\theta_2}$$

$$\cos(\theta_1+\theta_2) + i\sin(\theta_1+\theta_2)$$

$$= (\cos\theta_1 + i\sin\theta_1)(\cos\theta_2 + i\sin\theta_2)$$

$$= \cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2$$

$$+ i(\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2)$$

$$\int_{-\pi}^{\pi} f(x)\,dx = \int_{-\pi}^{\pi}\left[a_0 + \sum_n (a_n \cos nx + b_n \sin nx)\right]dx$$

$$\int_{-\pi}^{\pi} f(x)\,dx = 2\pi a_0$$

$$\int_{-\pi}^{\pi} \cos mx \cos nx \, dx$$

$$= \frac{1}{2}\int_{-\pi}^{\pi}\left[\cos(m+n)x + \cos(m-n)x\right]dx$$

$m=n$: $$= \frac{1}{2}\int_{-\pi}^{\pi}(\cos 2nx + 1)\,dx$$

$$= \pi$$

- Fourier series for a function with $p = 2\pi$:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

where Fourier coefficients are given as

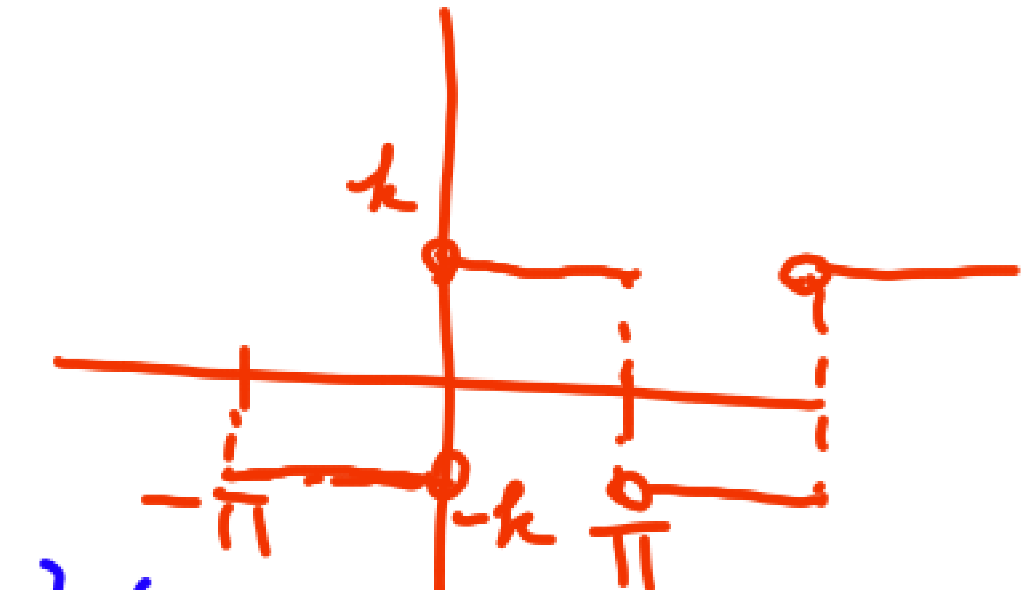
$$a_0 = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(x)dx$$

$$a_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \cos nx \, dx$$

$$b_n = \frac{1}{\pi}\int_{-\pi}^{\pi} f(x) \sin nx \, dx$$

Example Find the Fourier coefficients of

$$f(x)=\begin{cases} -k, & -\pi<x<0 \\ k, & 0<x<\pi \end{cases}$$



and $f(x+2\pi)=f(x)$.

$$f(x)=\frac{4k}{\pi}\left(\sin x+\frac{1}{3}\sin 3x+\cdots\right)$$

$$f(x)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos nx+\underline{b_n\sin nx}\right)$$

$$a_0=\frac{1}{2\pi}\int_{-\pi}^{\pi}f(x)\,dx=\frac{1}{2\pi}\left(-k\pi+k\pi\right)=0$$

$$a_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\cos nx\,dx=\frac{1}{\pi}\left[\int_{-\pi}^{0}(-k)\cos nx\,dx+\int_{0}^{\pi}k\cos nx\,dx\right]$$

$$=\frac{k}{n\pi}\left(-\sin nx\Big|_{-\pi}^{0}+\sin nx\Big|_{0}^{\pi}\right)=0$$

$$b_n=\frac{1}{\pi}\int_{-\pi}^{\pi}f(x)\sin nx\,dx=-\frac{k}{n\pi}\left(-\cos nx\Big|_{-\pi}^{0}+\cos nx\Big|_{0}^{\pi}\right)$$

$$=-\frac{2k}{n\pi}\left(-1+(-1)^n\right)$$

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lecture14.m

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% Example 1 of Section 11.1
% for K = pi/4

clear all, close all

x = linspace(-pi, pi, 101);
f = zeros(size(x));

clist = 'bgrcmykw';
for n = 1:2:7
    f = f + 1/n*sin(n*x);
    figure(1), plot(x, f, clist(n)), hold on
end
xlabel('x'), grid on
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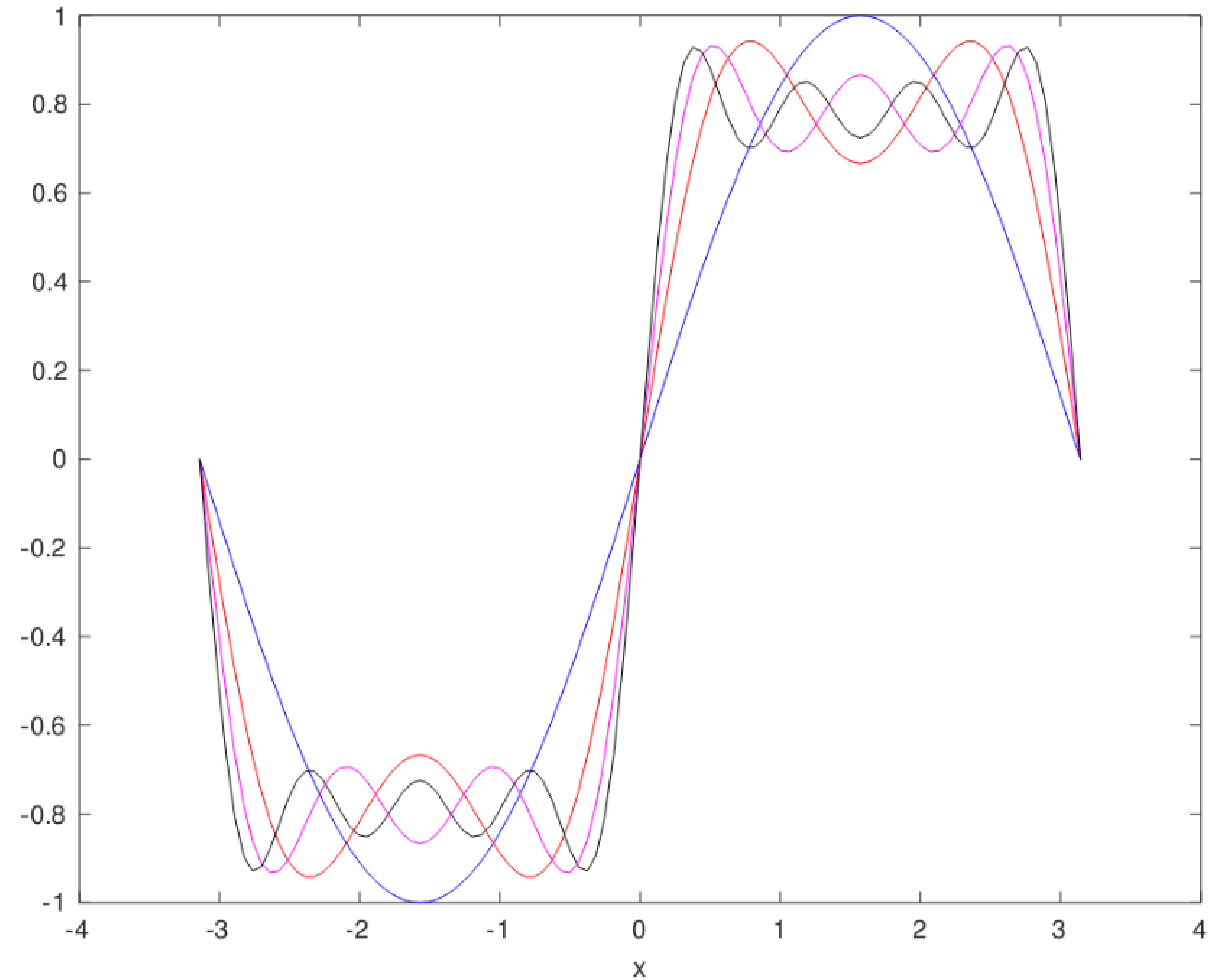


11.2 Arbitrary period. Even and odd functions. Half-range expansions

- Fourier series for a function with $p = 2L$:

$$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} x + b_n \sin \frac{n\pi}{L} x \right)$$

where Fourier coefficients are given as

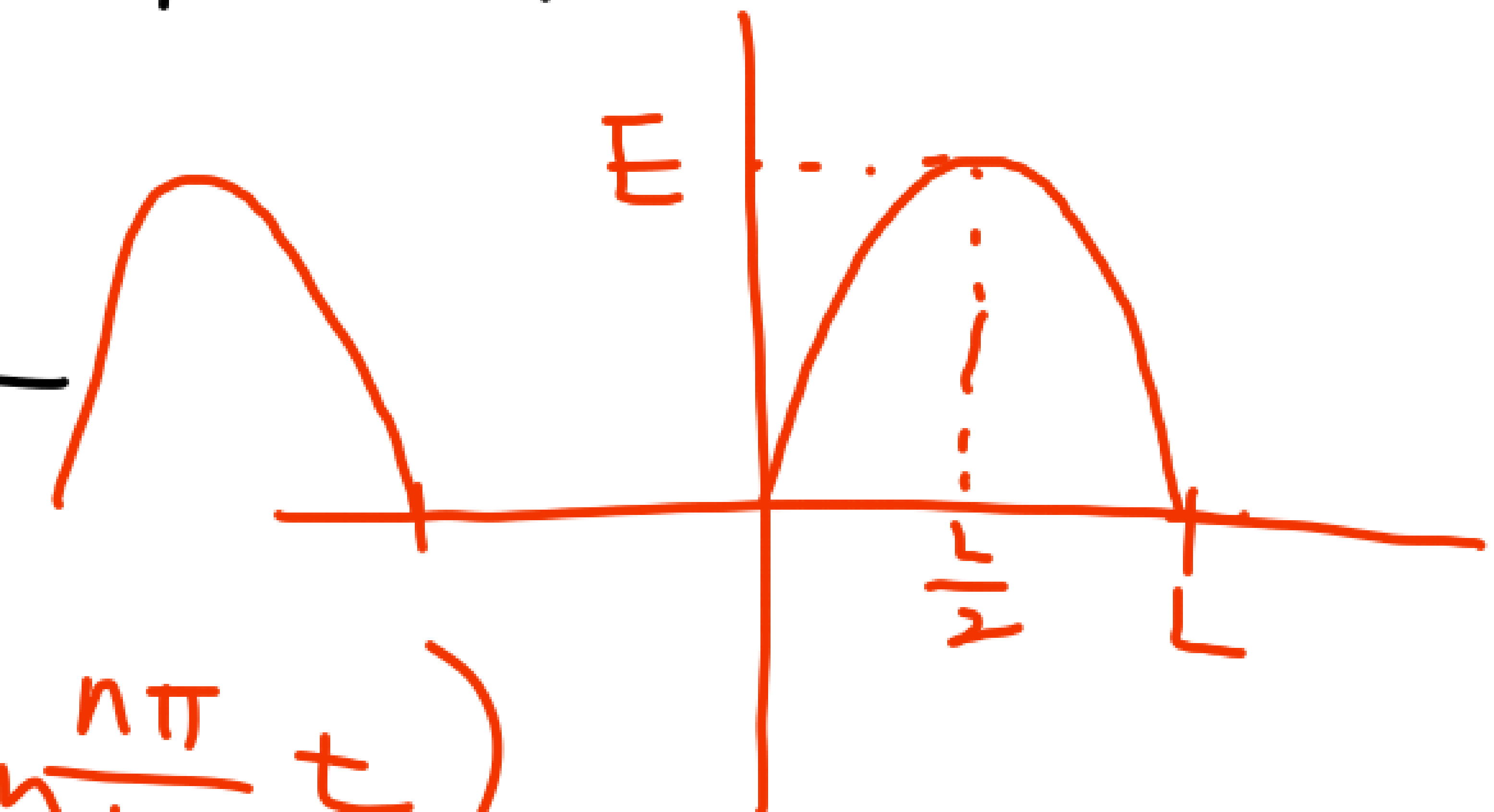
$$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x) dx$$

$$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos \frac{n\pi}{L} x \, dx$$

$$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin \frac{n\pi}{L} x \, dx$$

<u>Example</u> Find the Fourier coefficients of

$$u(t)=\begin{cases} 0 & , -L<t<0 \\ E\sin\omega t & , 0<t<L \end{cases}$$



$$u(t+2L)=u(t), \quad L=\frac{\pi}{\omega}$$

$$u(t)=a_0+\sum_{n=1}^{\infty}\left(a_n\cos\frac{n\pi}{L}t+b_n\sin\frac{n\pi}{L}t\right)$$

$$a_0=\frac{1}{2L}\int_{-L}^{L}u(t)dt=\frac{1}{2L}\int_0^L E\sin\frac{\pi t}{L}dt=-\frac{1}{2L}\frac{EL}{\pi}\cos\frac{\pi t}{L}\Big|_0^L$$

$$=\frac{E}{\pi}$$

$$a_n=\frac{1}{L}\int_0^L E\sin\frac{\pi t}{L}\cos\frac{n\pi t}{L}dt=\frac{E}{2L}\int_0^L\left(\sin\frac{(1+n)\pi t}{L}+\sin\frac{(1-n)\pi t}{L}\right)dt$$

$$a_1=0$$

$$a_n=-\frac{LE}{2\pi L}\left[\frac{\cos\frac{(1+n)\pi t}{L}}{1+n}+\frac{\cos\frac{(1-n)\pi t}{L}}{1-n}\right]_0^L=-\frac{E}{2\pi}\left[\frac{\cos(1+n)\pi-1}{1+n}+\frac{\cos(1-n)\pi-1}{1-n}\right]$$

- Fourier cosine series for even function: $f(-x) = f(x)$

$$a_0 = \frac{1}{L}\int_0^L f(x)dx$$

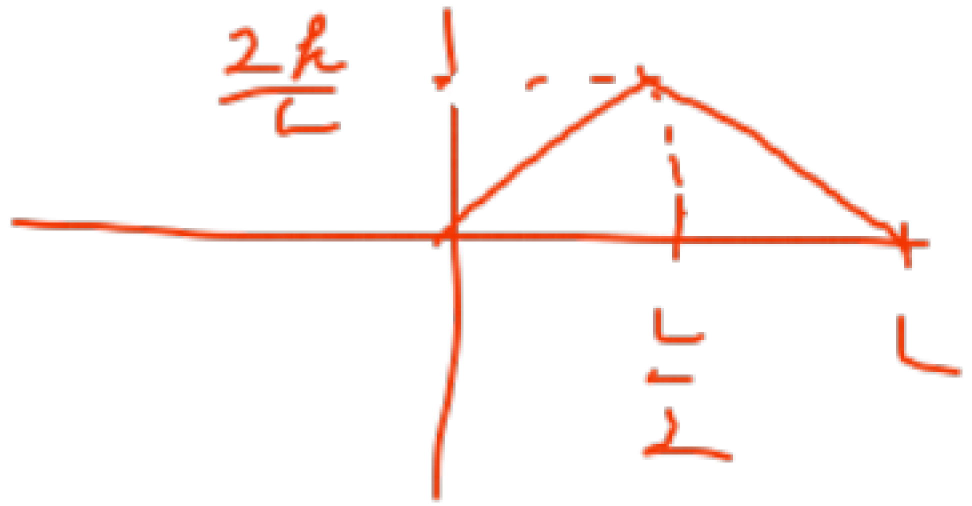
$$a_n = \frac{2}{L}\int_0^L f(x)\cos\frac{n\pi}{L}x\,dx$$

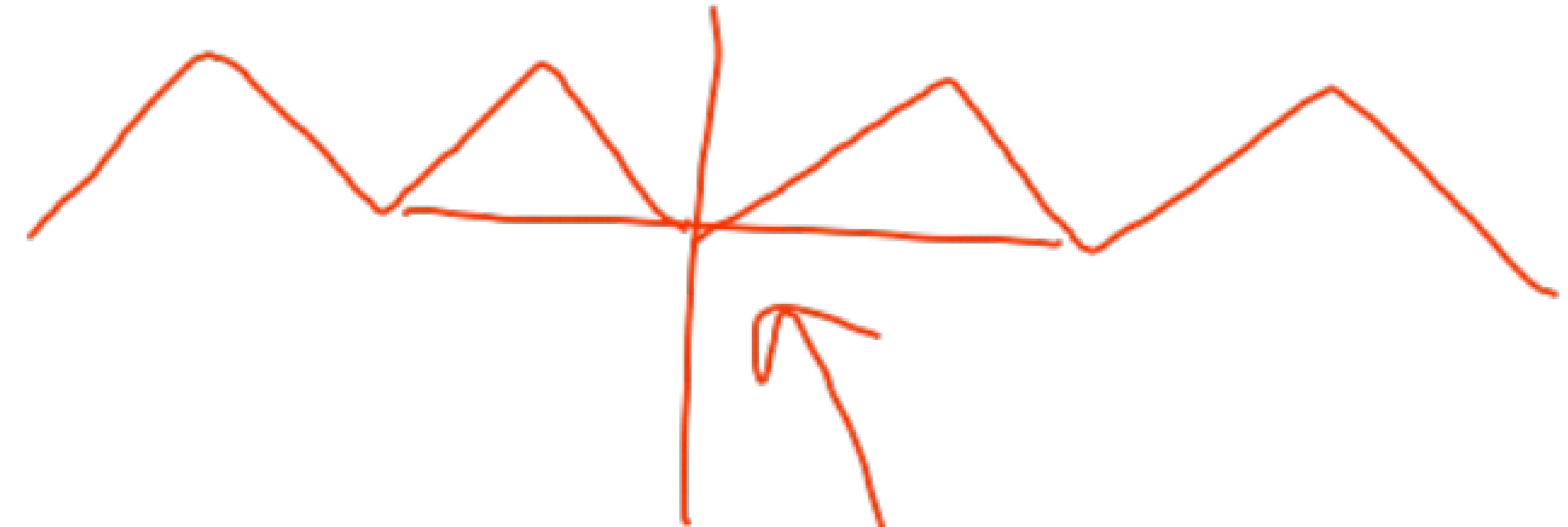
$$b_n = 0$$

- Fourier sine series for odd function: $f(-x) = -f(x)$

$$a_0 = a_n = 0$$

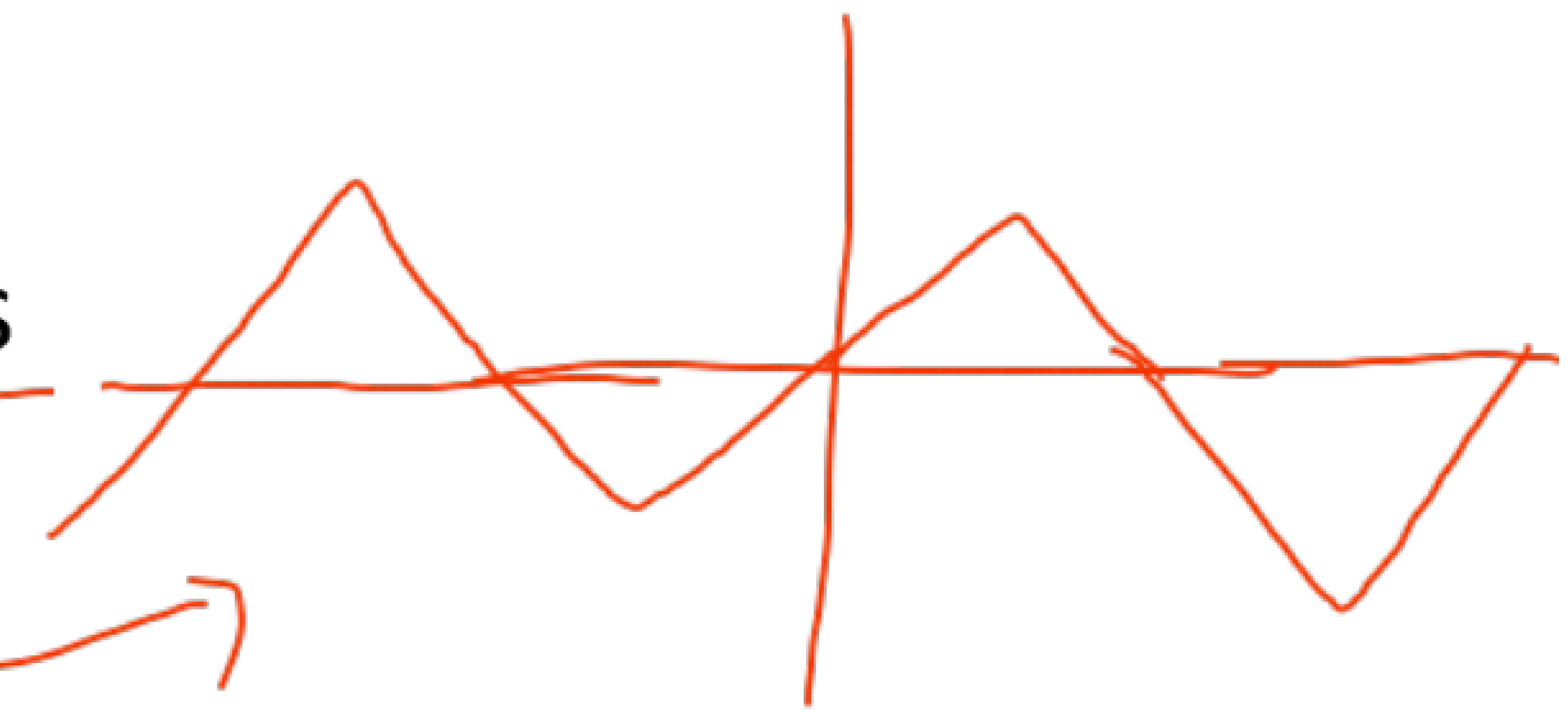
$$b_n = \frac{2}{L}\int_0^L f(x)\sin\frac{n\pi}{L}x\,dx$$





- Half-range extensions

  - Even periodic extension: Fourier cosine series
  - Odd periodic extension: Fourier sine series



- Example 6: Find the two half-range extensions of

$$f(x)=\begin{cases}\dfrac{2k}{L}x, & 0<x<\dfrac{L}{2}\\[2ex] \dfrac{2k}{L}(L-x), & \dfrac{L}{2}<x<L\end{cases}$$

11.3 Forced oscillations

- In the previous semester, we learned to solve

$$my'' + cy' + ky = F_0 \cos \omega t$$

- Now we can solve for more general forcing using Fourier series:

$$my'' + cy' + ky = r(t),$$

where

$$r(t) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos \frac{n\pi}{L} t + b_n \sin \frac{n\pi}{L} t \right)$$