Engineering Mathematics 2

Lecture 15

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Previously, we discussed

- Fourier series:

  $f(x)$ a function with the period $p = 2L$:

$$f(x) = a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi}{L}x + b_n \sin\frac{n\pi}{L}x\right)$$

  where Fourier coefficients are given as

$$a_0 = \frac{1}{2L}\int_{-L}^{L} f(x)dx$$

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi}{L}x\,dx$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi}{L}x\,dx$$

- Cases for odd and even functions, respectively, as well as half-range expansions.
- As an application, forced oscillation with the forcing term represented by Fourier series.

11.4 Approximation by trigonometric polynomials

- Suppose $f(x)$ a function on the interval $-\pi \le x \le \pi$ represented by Fourier series:

$$f(x) = a_0 + \sum_{n=1}^{\infty}(a_n \cos nx + b_n \sin nx)$$

- An approximation of the degree N:

$$F(x) = a_0 + \sum_{n=1}^{N}(A_n \cos nx + B_n \sin nx)$$

- The square error: $E = \int_{-\pi}^{\pi}(f - F)^2 dx$

- The square error E has the <u>minimum</u> iff $A_n = a_n$ and $B_n = b_n$:

$$E^* = \int_{-\pi}^{\pi} f^2 dx - \pi[2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2)].$$

- Bessel's inequality

$$2a_0^2 + \sum_{n=1}^{N}(a_n^2 + b_n^2) \leq \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx$$

- Parseval's identity

$$2a_0^2 + \sum_{n=1}^{\infty}(a_n^2 + b_n^2) = \frac{1}{\pi}\int_{-\pi}^{\pi} f^2 dx$$

- Example 1: Find E^* for F with $N = 1, 2, \cdots 10, 20, \cdots, 100, 1000$ relative to

$$f(x) = x + \pi \ \ (-\pi < x < \pi).$$

The Fourier series: $f(x) = \pi + 2\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} \sin nx$, then

$$E^* = \int_{-\pi}^{\pi} (x+\pi)^2 dx - \pi\left[2\pi^2 + 4\sum_{n=1}^{N} \frac{1}{n^2}\right] = \frac{2\pi^3}{3} - 4\pi\sum_{n=1}^{N} \frac{1}{n^2}$$

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% Lecture 15
% Example 1, Section 11.4

N = [1:9, 10:10:100, 1000];
E = zeros(length(N));
for j = 1:length(N)
    e = (2/3)*pi^3;
    for n = 1:N(j)
        e = e - 4*pi/n^2;
    end
    E(j) = e;
end
loglog(N, E, 'ko'), xlabel('N'), ylabel('E^{*}','Rotation',0), grid on
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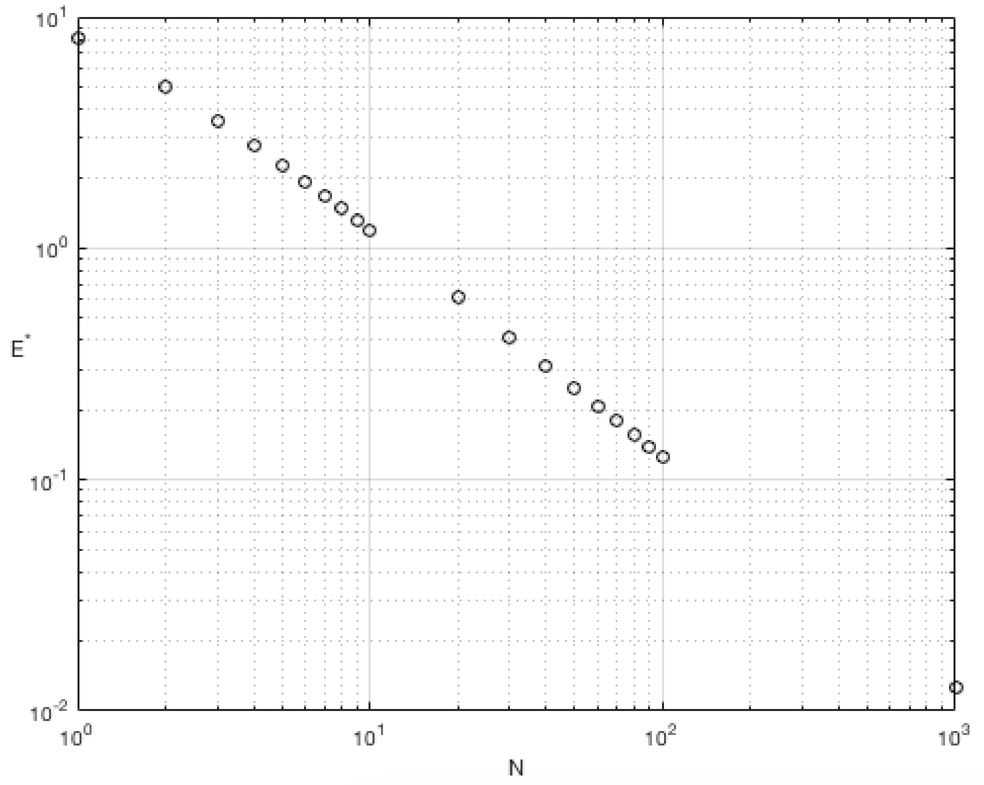

11.5 Sturm-Liouville problems. Orthogonal functions

- Functions $y_1, y_2, \cdots$ on the interval $a \leq x \leq b$ are called orthogonal with respect to the weight function $r(x) > 0$, if for all m and all n $(m \neq n)$,

$$(y_m, y_n) = \int_a^b r(x) y_m(x) y_n(x) dx = 0.$$

- Trigonometric system is orthogonal, e.g.

$$\int_{-\pi}^{\pi} \sin mx \sin nx \, dx = \begin{cases} \pi, & m = n \\ 0, & m \neq n \end{cases}$$

- The norm of y_n is

$$\|y_n\| = \sqrt{(y_n, y_n)} = \left[\int_a^b r(x)\, y_n^2\, dx\right]^{1/2}.$$

- Orthonormal if $y_1, y_2, \cdots$ have norm 1. Then,

$$(y_m, y_n) = \delta_{mn} = \begin{cases} 1, & m = n \\ 0, & m \neq n \end{cases}$$

- Example 3: Find the norm of

$$y_n = \sin nx$$

on the interval $-\pi \leq x \leq \pi$ with respect to $r(x) = 1$ and the corresponding orthonormal set of functions.

- Sturm-Liouville problem

$$[p(x)y']' + [q(x) + \lambda r(x)]y = 0, \ (a \leq x \leq b)$$

$$\begin{aligned} k_1 y + k_2 y' &= 0 \quad \text{at } x = a \\ l_1 y + l_2 y' &= 0 \quad \text{at } x = b \end{aligned}$$

- The <u>eigenfunction</u> of Sturm-Liouville problem for each of <u>eigenvalue</u>s λ are orthogonal.

- Example 1:

$$y'' + \lambda y = 0, y(0) = 0, y(\pi) = 0$$

This is a Sturm-Liouville problem with

$$p = 1, q = 0, r = 1, k_1 = 1, k_2 = 0, l_1 = 1, l_2 = 0.$$

The solution $y(x) = \sin\sqrt{\lambda}x$ is the eigenfunction for the eigenvalue $\lambda = 1, 4, 9, \cdots$.

- Example 4: Legendre equation

$$[(1 - x^2)y']' + n(n + 1)y = 0$$

is a Sturm-Liouville equation with the eigenvalue $\lambda = n(n + 1)$.

Therefore Legendre polynomials are the eigenfunctions of the equation and are orthogonal.

11.6 Orthogonal series. Generalised Fourier series

- Generalised Fourier series:

$$f(x) = \sum_{m=0}^{\infty} a_m y_m(x)$$

where $y_1, y_2, \cdots$ orthogonal on the interval $a \leq x \leq b$ w.r.t. $r(x) > 0$ and

$$a_m = \frac{1}{\|y_m\|^2} \int_a^b r(x) f(x) y_m(x) dx.$$

- Example 1: Fourier-Legendre series on the interval $-1 \leq x \leq 1$

$$f(x) = \sum_{m=0}^{\infty} a_m P_m(x) = a_0 + a_1 x + a_2 \left(\frac{3}{2}x^2 - \frac{1}{2}\right) + \cdots$$

where

$$a_m = \frac{2m+1}{2} \int_{-1}^{1} f(x) P_m(x) dx.$$

- Example 2: Fourier-Bessel series on the interval $0 \leq x \leq R$

$$f(x) = \sum_{m=0}^{\infty} a_m J_n\left(\frac{\alpha_{n,m} x}{R}\right)$$

where

$$a_m = \frac{2}{R^2 J_{n+1}^2(\alpha_{n,m})} \int_0^R x f(x) J_n\left(\frac{\alpha_{n,m} x}{R}\right) dx.$$

- Convergence in norm or mean-square convergence:

$$\lim_{k\to\infty} \|f_k(x) - f(x)\| = \lim_{k\to\infty} \int_a^b r(x)\,[f_k(x) - f(x)]^2 dx = 0$$

- An orthonormal set $y_1, y_2, \cdots$ on the interval $a \le x \le b$ is <u>complete</u> in a set of functions S defined on the same interval, if for every $\epsilon > 0$ we can find constants $a_0, a_1, \cdots, a_k$ s.t.

$$\|f - (a_0 y_0 + \cdots + a_k y_k)\| < \epsilon$$

for all $f \in S$.

- Bessel's inequality

$$\sum_{m=0}^{k} a_m^2 \leq \|f\|^2 = \int_a^b r(x) f(x)^2 dx$$

- Parseval equality

$$\sum_{m=0}^{\infty} a_m^2 = \|f\|^2 = \int_a^b r(x) f(x)^2 dx$$