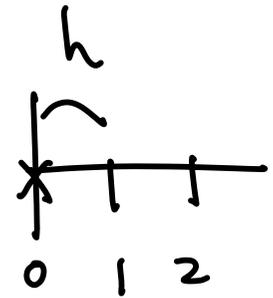


• Types of boundary conditions

Dirichlet b.c. : $y(0) = y_0, y(L) = y_L$

Neumann b.c. : $\frac{dy}{dx}(0) = 0$

mixed b.c. : $\beta y(0) + \alpha \frac{dy}{dx}(0) = g$



$\hookrightarrow \frac{y_1 - y_0}{h} + O(h)$ (blue circle around y_0 , arrow from "unknown" points to it, blue circle around the whole expression)

 $\hookrightarrow \frac{-3y_0 + 4y_1 - y_2}{2h} + O(h^2)$ (red circle around the whole expression)

$$\begin{bmatrix}
 * & * & * & & & \\
 * & * & 0 & 0 & \dots & 0 \\
 & * & * & & & \\
 & & * & * & & \\
 & & & \dots & \dots & \\
 & & & & \dots & \dots
 \end{bmatrix}
 \begin{bmatrix}
 y_0 \\
 y_1 \\
 y_2 \\
 \vdots
 \end{bmatrix}
 = \begin{bmatrix}
 \\
 \\
 \\
 \\
 \end{bmatrix}$$

Difficulty near boundaries when higher-order FD is used at (or near) the boundary.

Ch. 5 Numerical solution of partial diff'l eq. (PDE)

* Physical classification

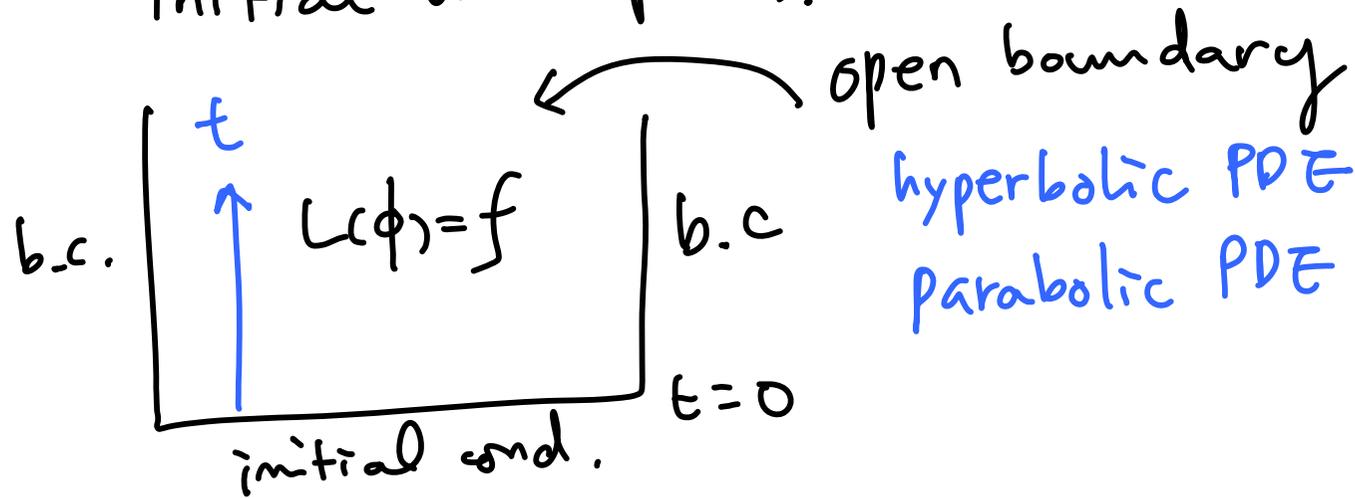
① Equilibrium problems — steady-state problems

domain $L(\phi) = f$ L : diff'l operator
 closed boundary: specify something

Elliptic PDE

about ϕ on the boundary.

- ② Propagation problems - transient nature
initial value probs.



- * Mathematical classification
Quasi-linear second-order PDE-

$$a u_{xx} + b u_{xy} + c u_{yy} = f$$

$$u_{xx} = \partial^2 u / \partial x^2$$

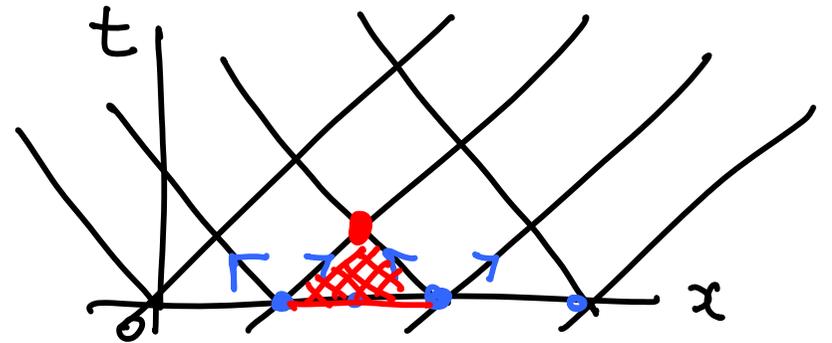
$$a, b, c \sim f(x, y, u, u_x, u_y)$$

- hyperbolic PDE if $b^2 - 4ac > 0$, two real characteristics
- parabolic PDE if $b^2 - 4ac = 0$, one " "
- elliptic PDE if $b^2 - 4ac < 0$, no " "

ex) $\frac{\partial^2 u}{\partial t^2} - a^2 \frac{\partial^2 u}{\partial x^2} = f$

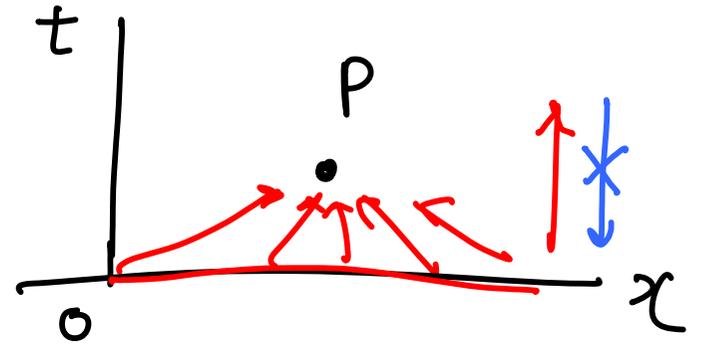
→ hyperbolic eq. $-\frac{1}{a}$ $\frac{1}{a}$

two char. lines : $\frac{dt}{dx} = \pm \frac{1}{a}$



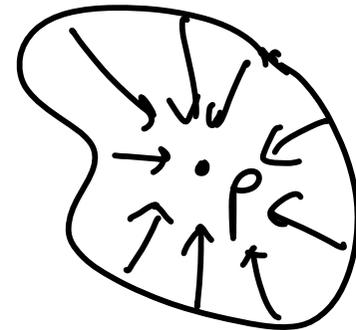
$$\frac{\partial u}{\partial t} = \alpha \frac{\partial^2 u}{\partial x^2} \rightarrow \text{parabolic eq.}$$

P knows what has happened previously along the entire x-axis.



$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = 0 \rightarrow \text{elliptic eq.}$$

No char. line



At any P, the sol. is influenced by all other pts.

5.1 Semi-discretization (S-D) : PDE \rightarrow system of ODEs.

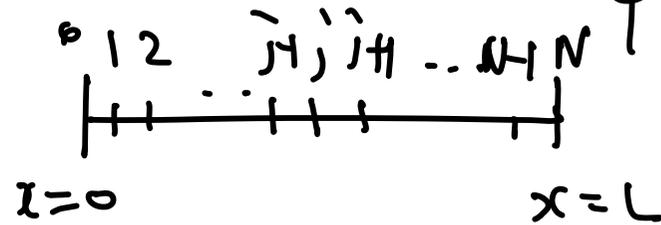
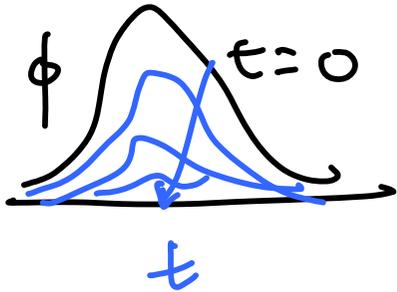
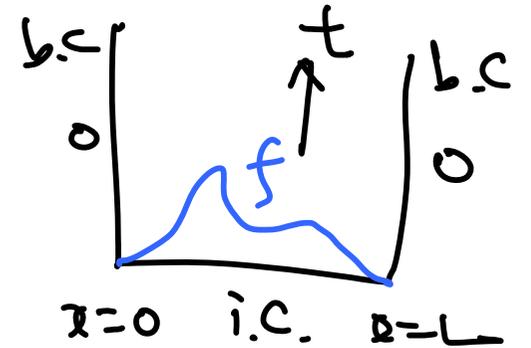
$$\boxed{\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2}}$$

diffusion eq.

$$\phi(0,t) = 0$$

$$\phi(L,t) = 0$$

$$\phi(x,0) = f$$



$$\text{@ } \tilde{j}, \text{ CD2} \rightarrow \frac{\partial \phi_j}{\partial t} = \alpha \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\Delta x^2}, \quad \tilde{j} = 1, 2, \dots, N-1$$

Semi-discretization

$$\phi = \begin{pmatrix} \phi_1 \\ \phi_2 \\ \vdots \\ \phi_{N-1} \end{pmatrix}$$

$$\rightarrow \frac{\partial \phi}{\partial t} = A \phi$$

$$A = \frac{\alpha}{\Delta x^2} \begin{pmatrix} -2 & 1 & & 0 \\ 1 & -2 & 1 & \\ & \ddots & \ddots & \\ 0 & & 1 & -2 & 1 \\ & & & & -2 \end{pmatrix}$$

PDE $\xrightarrow{\text{semi-discretization}}$

System of ODEs $(N-1) \times (N-1)$

$$\left(\begin{array}{l} A = B[a, b, c] \quad m \times m \\ \lambda_{\bar{j}} = b + 2\sqrt{ac} \cos \alpha_{\bar{j}} \\ \alpha_{\bar{j}} = \hat{j} \pi / (m+1) \\ \bar{j} = 1, 2, \dots, m \end{array} \right)$$

$$A = \frac{\alpha}{\Delta x^2} B[1, -2, 1] \quad (N-1) \times (N-1)$$

$$\lambda_{\bar{j}} = \frac{\alpha}{\Delta x^2} \left(-2 + 2 \cos \frac{\hat{j} \pi}{N} \right), \quad \hat{j} = 1, 2, \dots, N-1$$

$$\lambda_1 = \frac{\alpha}{\alpha x^2} (-2 + 2 \cos \frac{\pi}{N}) = -\frac{\alpha}{\alpha x^2} \left(\frac{\pi}{N}\right)^2 + \dots$$

$$\left(\text{for large } N, \cos \frac{\pi}{N} = 1 - \frac{1}{2!} \left(\frac{\pi}{N}\right)^2 + \dots \right)$$

$$\lambda_{N-1} = \frac{\alpha}{\alpha x^2} \left(-2 + 2 \cos \frac{(N-1)\pi}{N}\right) \approx \frac{-4\alpha}{\alpha x^2}$$

$$\left| \frac{\lambda_{N-1}}{\lambda_1} \right| \sim 4 \frac{N^2}{\pi^2} \rightarrow \text{large for large } N \rightarrow \text{system is stiff.}$$

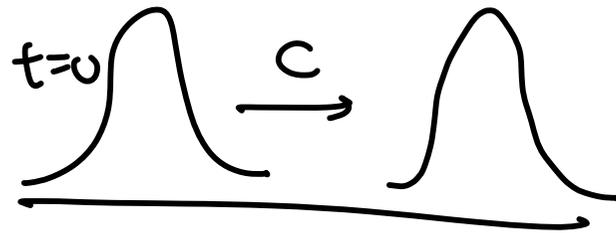
eigenvalues are real & negative. $\left(\frac{dy}{dt} = \lambda y\right)$

↳ sol. decays in time.

$$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$$

convection eq.

c : velocity



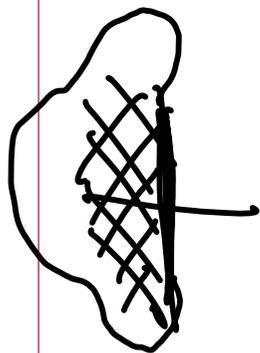
$$S-D (CD2) : \frac{\partial u_j}{\partial t} + c \frac{u_{j+1} - u_{j-1}}{2\Delta x} = 0, \quad j=1, 2, \dots, N-1$$

$$\rightarrow \frac{du}{dt} = -\frac{c}{2\Delta x} B[-1, 0, 1] u$$

$$\lambda_j = -\frac{c}{2\Delta x} \cdot 2 \cdot i \cos \frac{j\pi}{N} = -i \frac{c}{\Delta x} \cos \frac{j\pi}{N}$$

purely imaginary

← wave-like behavior



⑥ Matrix stability analysis

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} \xrightarrow[\text{CD2}]{\text{S-D}} \frac{d\phi}{dt} = \frac{\alpha}{\Delta x^2} B [1, -2, 1] \phi$$

$$\text{EE: } \frac{\phi^{n+1} - \phi^n}{\Delta t} = \frac{\alpha}{\Delta x^2} B [1, -2, 1] \phi^n$$

$$\rightarrow \phi^{n+1} = \left(I + \Delta t \frac{\alpha}{\Delta x^2} B \right) \phi^n \rightarrow \phi^n = \left(I + \Delta t \frac{\alpha}{\Delta x^2} B \right)^n \phi^0$$

For stability, $|1 + \Delta t \frac{\alpha}{\Delta x^2} \lambda_i| \leq 1$ $\left(\lambda_i = -2 + 2 \cos \frac{j\pi}{N} \right)$
 real & negative

$$\rightarrow -1 \leq 1 + \Delta t \frac{\alpha}{\Delta x^2} \lambda_i \leq 1$$

$$\rightarrow \Delta t \leq \frac{2}{\frac{\alpha}{\Delta x^2} |\lambda_i|}$$

worst case $|\lambda_{\max}| = 4 \rightarrow \boxed{\Delta t \leq \frac{\Delta x^2}{2\alpha}}$ $\Delta t_{\max} = \frac{\Delta x^2}{2\alpha}$
 \leftarrow very restrictive

more accuracy in $x \rightarrow \Delta x \downarrow \rightarrow \Delta t \sim \Delta x^2 \downarrow$

$N \rightarrow 2N \Rightarrow \Delta t \rightarrow \frac{1}{4} \Delta t$ $\xrightarrow[t=0]{t=T}$

CPU time \rightarrow 8 times!