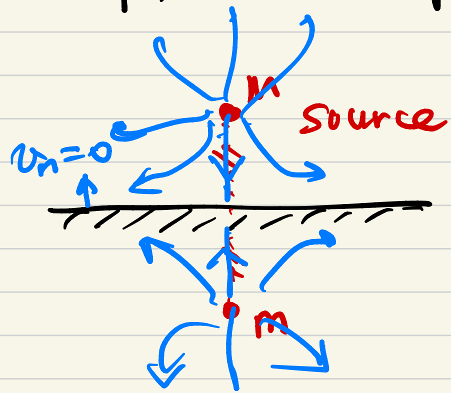
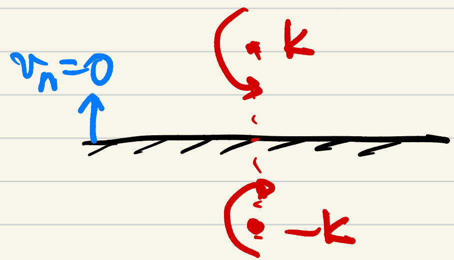
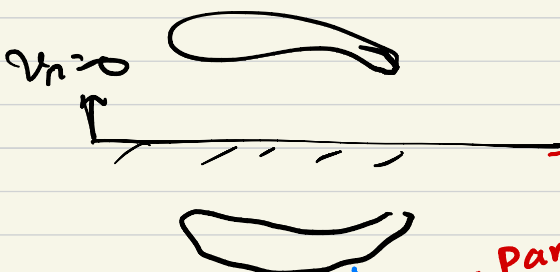
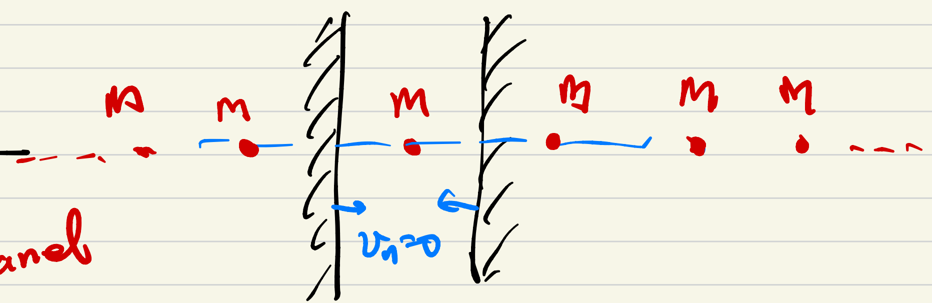
⑧ Images or Image method

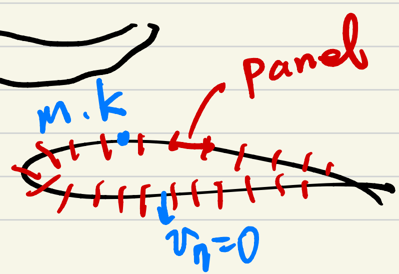


$$\psi = \psi_m\big|_{y=a} + \psi_m\big|_{y=-a}$$



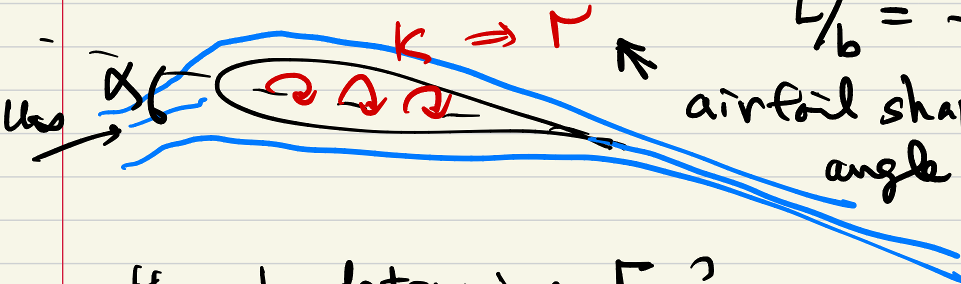






panel method

⊙ Airfoil theory



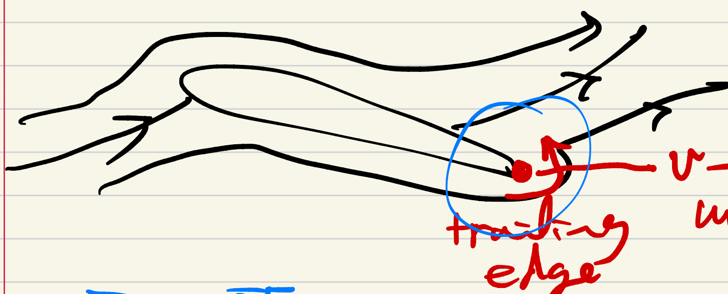
$L/b = -\rho U_\infty \Gamma$

airfoil shape

angle of attack α

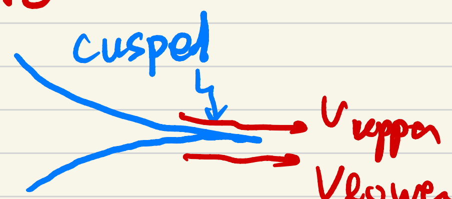
How to determine Γ ?

→ Kutta condition

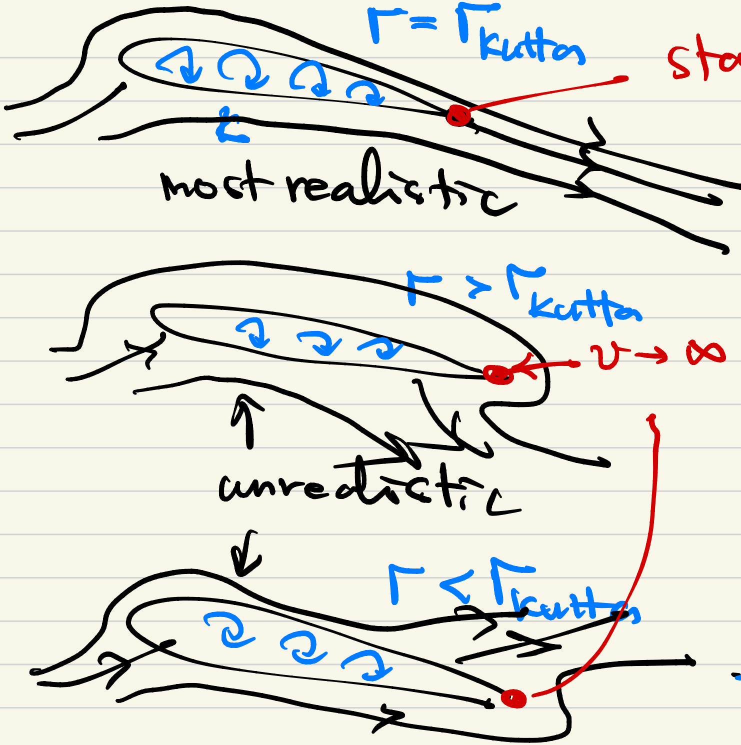


$V_{lower} = V_{upper} = 0$

stag. pt. @ TE

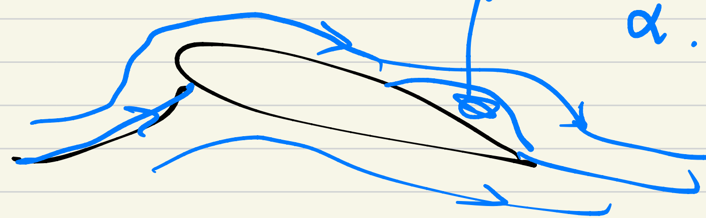


$V_{upper} = V_{lower}$ @ TE

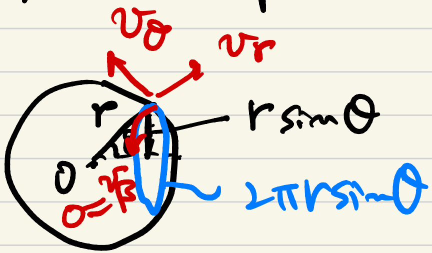


Kutta condition:
velocity difference vanishes at the trailing edge.

<real flow> flow sep. at high α.



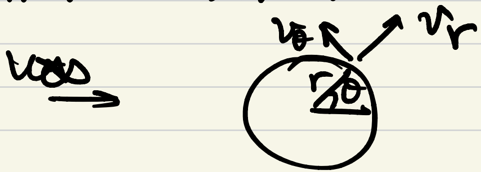
⊚ Axisymmetric potential flow ($v_\phi = 0$) (r, θ, β)



$$\nabla\cdot\underline{V} = 0 \; : \; \frac{\partial}{\partial r}\left(r^2 v_r \sin\theta\right) + \frac{\partial}{\partial \theta}\left(r v_\theta \sin\theta\right) = 0$$

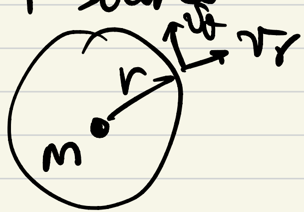
$$\rightarrow \; v_r = -\frac{1}{r^2\sin\theta}\frac{\partial \psi}{\partial \theta} = \frac{\partial \phi}{\partial r}, \quad v_\theta = \frac{1}{r\sin\theta}\frac{\partial \psi}{\partial r} = \frac{1}{r}\frac{\partial \phi}{\partial \theta}$$

- uniform stream



$$\left.\begin{aligned} v_r &= U_\infty \cos\theta \\ v_\theta &= -U_\infty \sin\theta \end{aligned}\right\} \rightarrow \begin{aligned} \psi &= -\frac{1}{2} U_\infty r^2 \sin^2\theta \\ \phi &= U_\infty r \cos\theta \end{aligned}$$

- point source

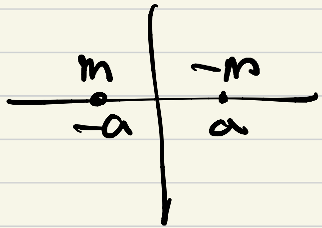


$$Q = 4\pi r^2 v_r$$

$m = \frac{Q}{4\pi}$: source strength

$$\left.\begin{aligned} \rightarrow v_r &= \frac{Q}{4\pi r^2} = \frac{m}{r^2} \\ v_\theta &= 0 \end{aligned}\right\} \rightarrow \begin{aligned} \psi &= m\cos\theta \\ \phi &= -\frac{m}{r} \end{aligned}$$

- point doublet



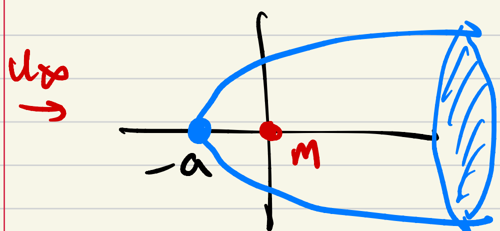
$a \to 0$

$2am = \lambda$

$$\psi = \frac{\lambda \sin^2\theta}{r}$$

$$\phi = \frac{\lambda \cos\theta}{r^2}$$

- Rankine half body of revolution



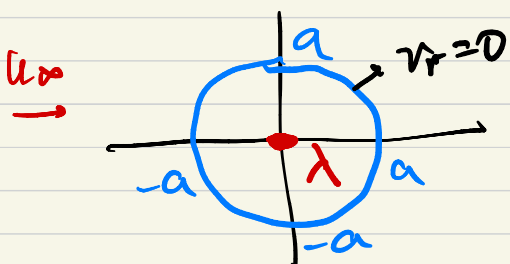
uniform stream + point source

@ $r=a$, $v_r = 0 \Rightarrow \frac{1}{2}U_\infty a^2 - \frac{\lambda}{a} = 0$

$\rightarrow a = \left(\frac{2\lambda}{U_\infty}\right)^{\frac{1}{3}}$

$\lambda = \frac{1}{2} U_\infty a^3$

- sphere : unif stream + point doublet



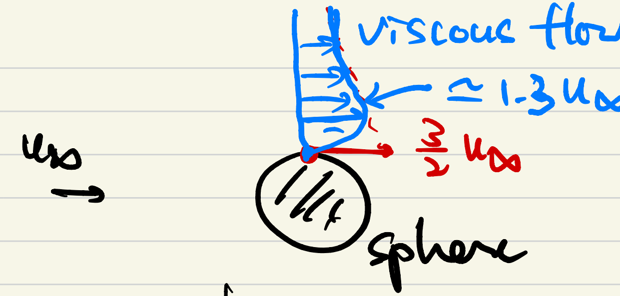
$$\psi = -\frac{1}{2}U_\infty r^2 \sin^2\theta + \frac{\lambda}{r}\sin^2\theta$$

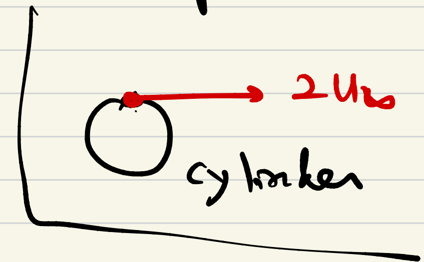
$$v_r = -\frac{1}{r^2\sin\theta}\frac{\partial\psi}{\partial\theta} = \cdots = \frac{2}{r^2}\left(\frac{1}{2}U_\infty r^2 - \frac{\lambda}{r}\right)\cos\theta$$

$$v_\theta = \frac{1}{r\sin\theta}\frac{\partial\psi}{\partial r} = \cdots = -\frac{1}{2}U_\infty \sin\theta\left(2 + \frac{a^3}{r^3}\right)$$

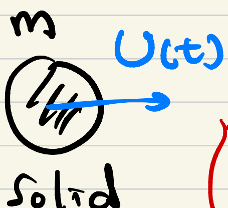
@ $r = a$, $v_\theta = -\frac{3}{2} u_\infty \sin\theta$

$\theta = 90°$: $v_\theta = -\frac{3}{2} u_\infty$





• Concept of hydrodynamic mass m_h



pushes fluid too

→ need force

$$\boxed{\Sigma \underline{F} = (m + m_h)\frac{dU}{dt}}$$

you feel like moving heavier object.

m_h: depend on shape, direction

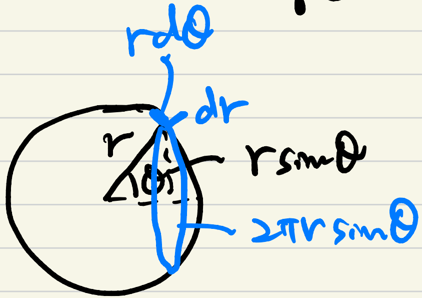
hydrodynamic mass
added mass
virtual mass

$$KE_{fluid} = \int_{\text{flow field}} \frac{1}{2} v^2 \, dm \equiv \frac{1}{2} m_h U^2$$

v = flow over sphere − unif. stream

$$v_r = -\frac{U a^3 \cos\theta}{r^3}, \quad v_\theta = -\frac{U a^3 \sin\theta}{2r^3}$$

$$v^2 = v_r^2 + v_\theta^2$$



$$dm = \rho (2\pi r \sin\theta) \, r \, d\theta \, dr$$

$$KE_{fluid} = \int_{r=a}^{\infty} \int_0^{\pi} \frac{1}{2} v^2 \rho \, 2\pi r \sin\theta \, r \, d\theta \, dr$$

$$= \cdots = \frac{1}{3} \rho \pi a^3 U^2 \equiv \frac{1}{2} m_h U^2$$

$$\therefore \boxed{m_h(\text{sphere}) = \frac{2}{3} \rho \pi a^3}$$

if $\rho_s = \rho$, m_h = half of the sphere mass

For cylinder, $\boxed{m_h(\text{cylinder}) = \rho \pi a^2 L}$ if $\rho_s = \rho$, $m_h = m$ (mass of cylinder)

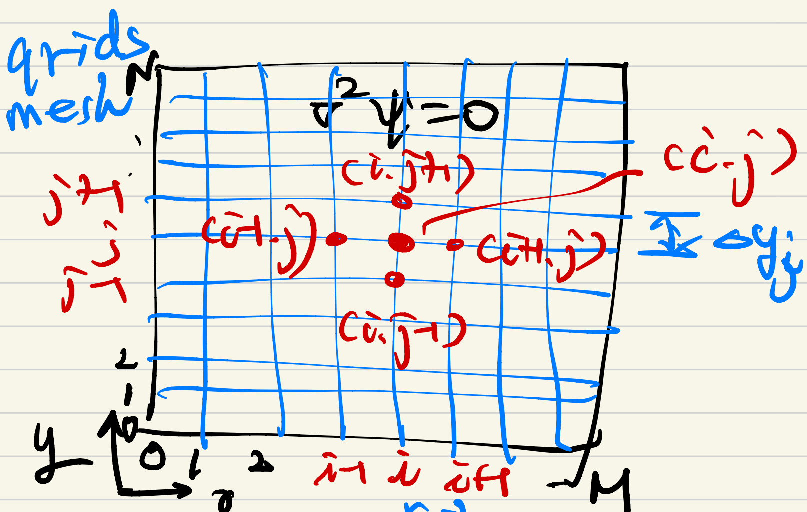
⑤ Numerical analysis

$$\begin{cases} \nabla^2 \psi = 0 \\ \nabla^2 \phi = 0 \end{cases} \rightarrow \text{separation of variable} \quad \psi = X(x)\,Y(y)$$

① finite difference method (FDM) 유한차분법 $\boxed{\nabla^2 \psi = 0}$ numerical method

② " volume " (FVM) " 체적 " ← 유체

③ " element " (FEM) " 요소 " ← 고체



$\Delta y_j = y_{j+1} - y_j$

$\Delta x_i = x_{i+1} - x_i$: grid spacing

FDM ← Taylor series expansion

$$\psi_{i+1,j} = \psi_{i,j} + \Delta x_i \frac{\partial \psi}{\partial x}\Big|_{i,j} + \frac{1}{2}\Delta x_i^2 \frac{\partial^2 \psi}{\partial x^2}\Big|_{i,j} + \frac{1}{6}\Delta x_i^3 \frac{\partial^3 \psi}{\partial x^3}\Big|_{i,j} + \cdots$$

$$\psi_{i-1,j} = \psi_{i,j} - \quad " \quad + \quad " \quad - \quad " \quad + \cdots$$

$+$

$$\psi_{i+1,j} + \psi_{i-1,j} = 2\psi_{i,j} + \Delta x_i^2 \frac{\partial^2 \psi}{\partial x^2}\Big|_{i,j} + \frac{1}{12}\Delta x_i^4 \frac{\partial^4 \psi}{\partial x^4}\Big|_{i,j} + \cdots$$

$$\Rightarrow \boxed{\frac{\partial^2 \psi}{\partial x^2}\Big|_{i,j} = \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta x_i^2}} - \frac{1}{12}\Delta x_i^2 \frac{\partial^4 \psi}{\partial x^4}\Big|_{i,j} + \cdots$$

central difference method

$$\boxed{\frac{\partial^2 \psi}{\partial y^2}\Big|_{i,j} = \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{\Delta y_j^2}} - \frac{1}{12}\Delta y_j^2 \frac{\partial^4 \psi}{\partial y^4}\Big|_{i,j} + \cdots$$

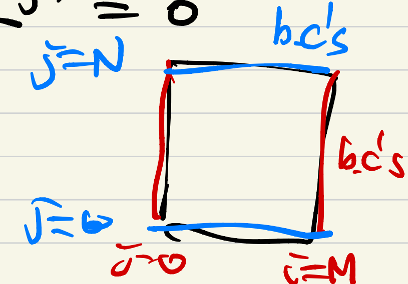
truncation error

leding "

second-order accuracy

$$\nabla^2\psi = 0 \rightarrow \frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2} = 0$$

$$\xrightarrow{\text{FDM}} \frac{\psi_{i+1,j} - 2\psi_{i,j} + \psi_{i-1,j}}{\Delta x_i^2} + \frac{\psi_{i,j+1} - 2\psi_{i,j} + \psi_{i,j-1}}{\Delta y_j^2} = 0$$



$$i = 1, 2, \cdots, M-1 \; ; \quad j = 1, 2, \cdots, N-1$$

$$\underbrace{\begin{bmatrix} -4 & 1 & 0 & \cdots & 0 & 1 & & 0 \\ 1 & -4 & 1 & \cdots & \cdots & 1 & 0 & \\ & \ddots & \ddots & \ddots & & \ddots & & 0 \\ 1 & & \ddots & \ddots & \ddots & & \ddots & \\ & 1 & 0 & \ddots & \ddots & \ddots & & \\ & & 1 & & \ddots & \ddots & \ddots & \\ & & & \ddots & & \ddots & \ddots & \end{bmatrix}}_{A} \underbrace{\begin{bmatrix} \psi_{1,1} \\ \psi_{2,1} \\ \vdots \\ \psi_{i,j} \\ \vdots \\ \psi_{M-1,N-1} \end{bmatrix}}_{\psi} = \underbrace{\begin{bmatrix} \square \\ 0 \\ 0 \\ 0 \\ \square \\ 0 \\ \square \\ 0 \\ \square \end{bmatrix}}_{r}$$

$(M-1)(N-1) \times (M-1)(N-1)$

$$A\,\psi = r$$

$A\psi = r \quad \rightarrow \quad \psi = A^{-1} r$ ✗

iterative method   k : iteration index (k=1, 2, ...)

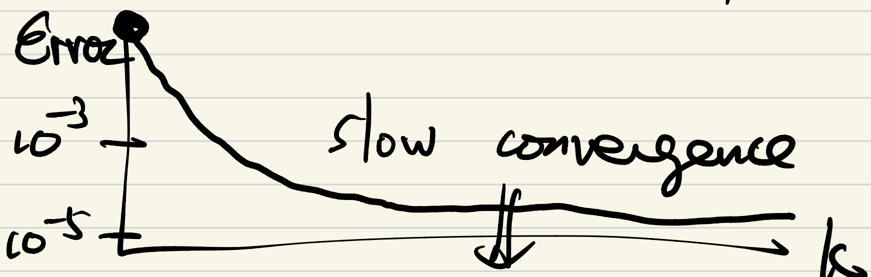
$$\frac{\psi^{k}_{i+1,j} - 2\psi^{k+1}_{i,j} + \psi^{k}_{i-1,j}}{\Delta x_i^2} + \frac{\psi^{k}_{i,j+1} - 2\psi^{k+1}_{i,j} + \psi^{k}_{i,j-1}}{\Delta y_j^2} = 0$$

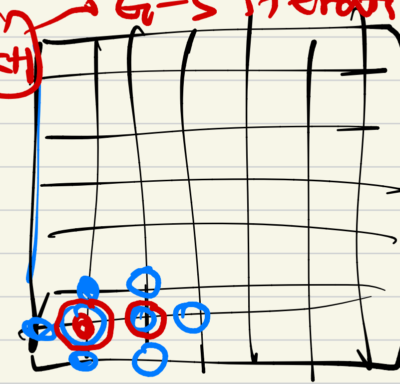
Jacobi iteration

G-S iteration

$$\psi^{k} \quad \psi^{k+1} \quad k+1 \qquad + \qquad k \quad k+1 \quad k+1$$

〃 〃





speed-up ① SOR, CGS

② multi-grid method

- Navier-Stokes eq. ($\rho = \text{const}$)

$$\rho u \frac{\partial u}{\partial x} + \rho v \frac{\partial u}{\partial y} = -\frac{\partial p}{\partial x} + \mu \nabla^2 u$$

$\xrightarrow{\text{FDM}}$

$$\rho u_{i,j} \frac{u_{i+1,j} - u_{i-1,j}}{2\Delta x} + \rho v_{i,j} \frac{u_{i,j+1} - u_{i,j-1}}{2\Delta y} = -\frac{p_{i+1} - p_{i-1}}{2\Delta x}$$
$$+ \mu \frac{u_{i+1,j} - 2u_{i,j} + u_{i-1,j}}{\Delta x^2}$$

$$\rho u \frac{\partial v}{\partial x} + \rho v \frac{\partial v}{\partial y} = -\frac{\partial p}{\partial y} + \mu \nabla^2 v$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

coupled (u, v, p)

$\Rightarrow$ CFD (Computational Fluid Dynamics)

commercial package