

- Matrix stability analysis  $\rightarrow$

$$\frac{d\phi}{dt} = A\phi$$

노트 제목

2019-10-30

- Von Neumann Stability analysis

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2} \xrightarrow[\text{CD2}]{S-D} \frac{\partial \phi_j^n}{\partial t} = \alpha \frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta x^2}$$

full discretization (using EE)

$$\frac{\phi_j^{n+1} - \phi_j^n}{\Delta t} = \alpha \frac{\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n}{\Delta x^2} \quad j=1, 2, \dots, N-1$$

Assume sol. of the form

$$\phi_j^n = \theta^n e^{ikx_j}$$

*assume  
spatial  
periodicity*

$$\rightarrow \frac{\sigma^n e^{ikx_j} - \sigma^n e^{ikx_j}}{\delta t} = \alpha \frac{\sigma^n e^{ikx_{j+1}} - 2\sigma^n e^{ikx_j} + \sigma^n e^{ikx_{j-1}}}{\delta x^2} \quad \begin{pmatrix} x_{j+1} = x_j + \delta x \\ x_{j-1} = x_j - \delta x \end{pmatrix}$$

$$\rightarrow \sigma = 1 + \frac{\alpha \delta t}{\delta x^2} (-2 + 2 \cos k \alpha x)$$

for stability,  $|\sigma| \leq 1 \rightarrow -1 \leq 1 + \frac{\alpha \delta t}{\delta x^2} (-2 + 2 \cos k \alpha x) \leq 1$

$$\rightarrow \delta t \leq \frac{2}{\alpha \delta x^2 (2 - 2 \cos k \alpha x)}$$

worst case :  $\cos k \alpha x = -1$

$$\rightarrow \boxed{\delta t \leq \frac{\alpha \delta x^2}{2 \alpha}}$$

\* Von Neumann stability analysis  
works for const. coeff diff eq.  
and assumes periodic boundary conditions.

same as in  
matrix  
stability  
analysis

In many cases, numerical stability comes from full discretization of PDE and NOT from the b.c.s.

### ① Modified wavenumber analysis

$$\frac{\partial \phi}{\partial t} = \alpha \frac{\partial^2 \phi}{\partial x^2}$$

S-D  $\downarrow$  CD2

Assume  $\phi(x, t) = \psi(t) e^{ikx}$

$$\frac{d\psi}{dt} e^{ikx} = \alpha(-k^2) \psi e^{ikx} \Rightarrow \boxed{\frac{d\psi}{dt} = -\alpha k^2 \psi}$$
  

$$\frac{\partial \phi_j}{\partial t} = \alpha \frac{\phi_{j+1} - 2\phi_j + \phi_{j-1}}{\Delta x^2}$$

Assume  $\phi_j = \psi(t) e^{ikx_j}$

$$\rightarrow \frac{d\psi}{dt} e^{ikx_j} = \frac{\alpha}{\Delta x^2} (\psi e^{ikx_{j+1}} - 2\psi e^{ikx_j} + \psi e^{ikx_{j-1}})$$

$$= \frac{\alpha}{\alpha x^2} (-2 + 2 \cos k_0 x) \psi e^{ik_0 x} j$$

→  $\frac{d\psi}{dt} = -\alpha \frac{2}{\alpha x^2} (1 - \cos k_0 x) \psi$   
 $\equiv k'^2$

$k'$ : modified wavenumber

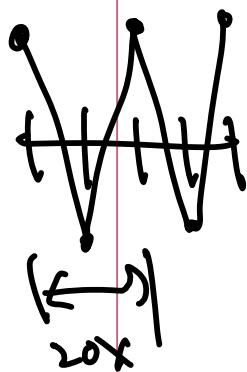
$$(k'_0 x)^2 = 2(1 - \cos k_0 x) = 2 \cdot 2 \sin^2 \frac{k_0 x}{2}$$

$$\rightarrow k'_0 x = 2 \sin \frac{k_0 x}{2} \rightarrow k' = 2 \frac{\sin k_0 x / 2}{\alpha x}$$

$$\Rightarrow \frac{d\psi}{dt} = -\alpha k'^2 \psi$$

Application of any other FD scheme instead of CD2 used here would have also led to the same

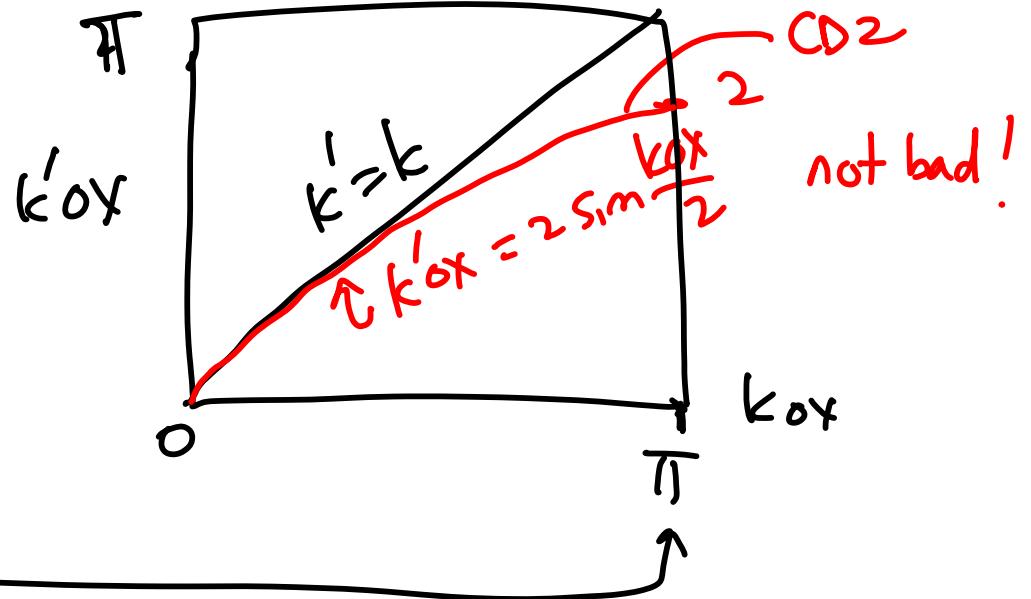
form as  $\textcircled{A}$  but with a different modified wave number.



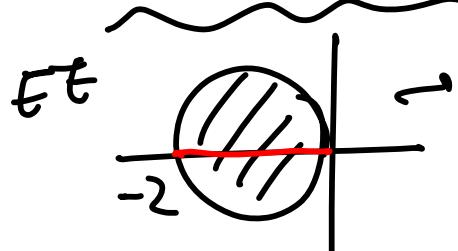
$$k'_{ox} = 2 \sin \frac{k_{ox}}{2}$$

$$K(2ox) = 2\pi$$

$$\rightarrow k = \pi/ox$$

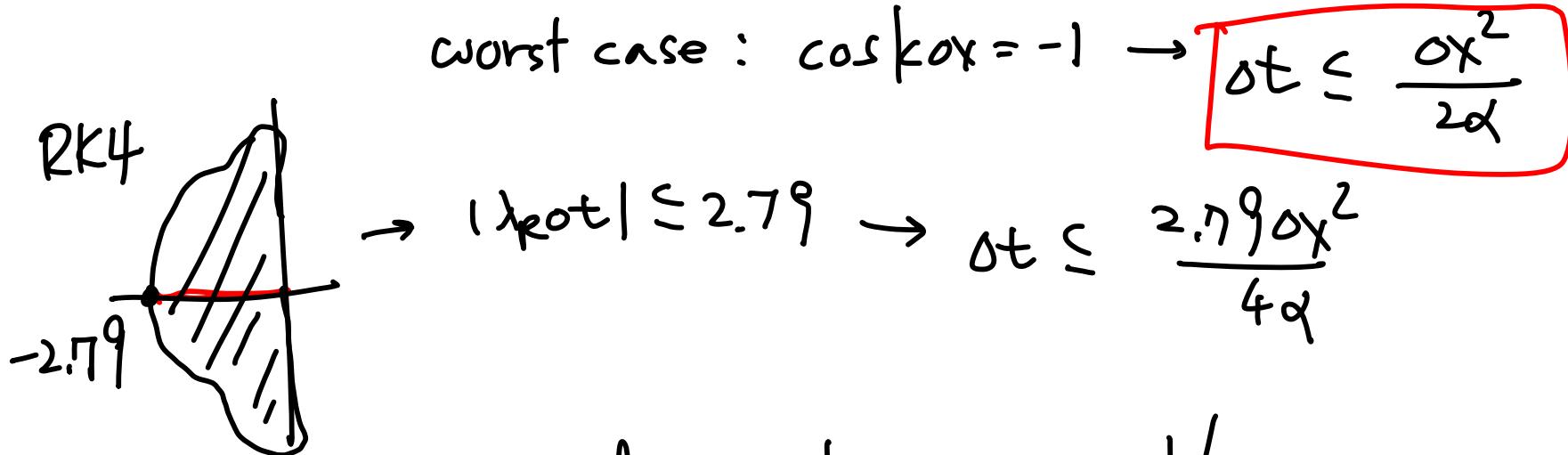


$$\frac{d\psi}{dt} = -\alpha k'^2 \psi = \lambda \psi$$



$$\lambda = -\alpha |k'|^2 : \text{real & negative}$$

$$\rightarrow \omega t \leq \frac{2}{|\lambda_R|} = \frac{2}{\alpha |k'|^2} = \frac{2}{\alpha \frac{2}{ox^2}(1-\cos k_{ox})}$$



Modified wavenumber analysis

- ① calculate the modified wavenumber for spatial derivative
- ② use results from ODE with  $\lambda$  replaced with the worst case for  $k'$ .

$\frac{\partial u}{\partial t} + c \frac{\partial u}{\partial x} = 0$  : convection eq.  
 $u(x, t) = \psi(t) e^{ikx}$  →  $\frac{d\psi}{dt} e^{ikx} + c \psi (ik) e^{ikx} = 0$   
 $\frac{d\psi}{dt} = -ikc\psi$

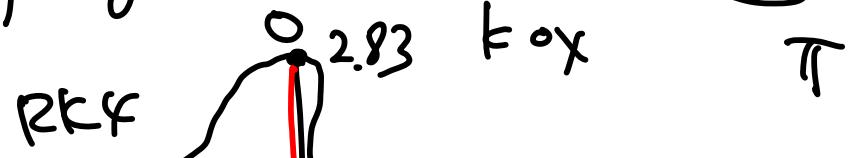
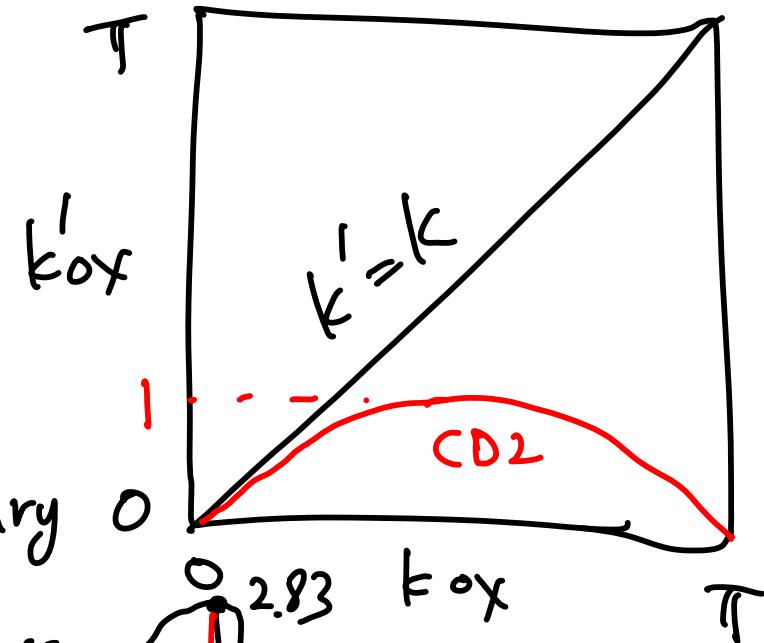
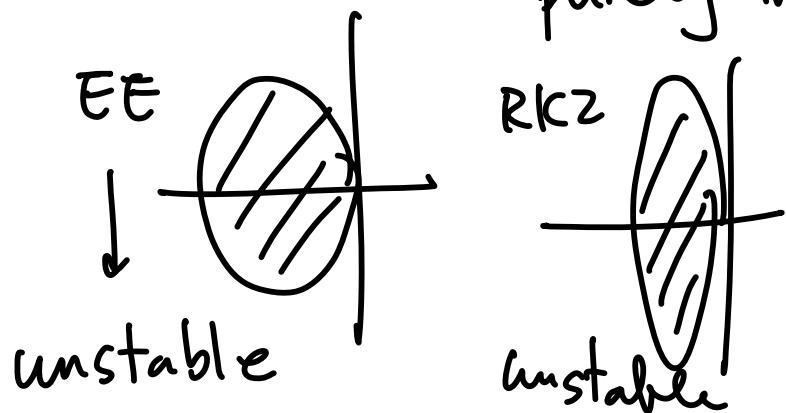
S-D → CD2  
 $\frac{\partial u_j}{\partial t} + c \frac{u_{j+1} - u_{j-1}}{2\Delta x} = 0$   
 Assume  $u_j = \psi(t) e^{ikx_j}$   
 $\rightarrow \frac{d\psi}{dt} e^{ikx_j} + c \frac{\psi e^{ikx_{j+1}} - \psi e^{ikx_{j-1}}}{2\Delta x} = 0$   
 $\frac{d\psi}{dt} = -i \frac{\sin k \Delta x}{\Delta x} c \psi$ 
 $k'$ : modified wave number

$$k'_0x = \sin k_0x$$

$$K(2\alpha x) = 2\pi$$

$$\Rightarrow \frac{d^2k}{dt^2} = \omega^2 k : \omega = -i \frac{\sin k_0 x}{\alpha x}$$

purely imaginary

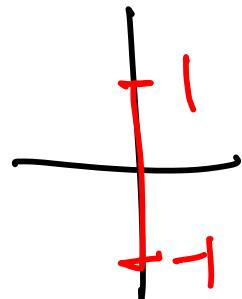


$$|\frac{\sin k_0 x}{\alpha x} \cdot \cot t| \leq 2.83$$

$$|\frac{\sin k_0 x}{\alpha x}| \leq 2.83$$

$$\hookrightarrow \frac{c\omega t}{\Delta x} \leq \frac{2.83}{|\sin k\Delta x|}$$

leapfrog method



$$|\omega \Delta t| \leq 1$$

$$\Delta t \leq \frac{1}{|\omega|} = \frac{\Delta x}{C|\sin k\Delta x|}$$

$$\frac{c\omega t}{\Delta x}$$

: non-dimensional variable

CFL (Courant, Friedrich & Lewy) number

RK4 : CFL  $\leq 2.83$

Leapfrog : CFL  $\leq 1$

worst case :  $|\sin k\Delta x| = 1$

much better than diff. eq.

$$\hookrightarrow \boxed{\frac{c\omega t}{\Delta x} \leq 2.83}$$

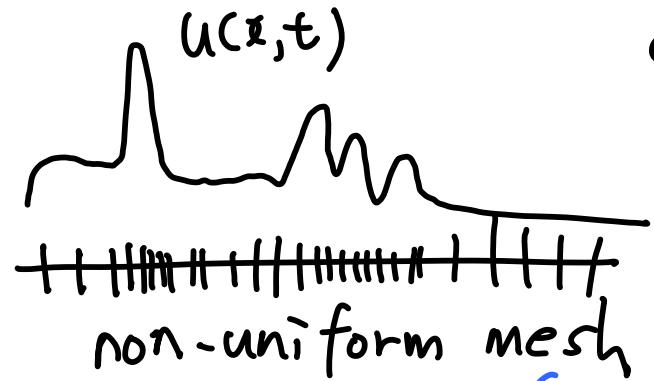
RK4

$\Delta x \rightarrow \Delta x/2 \Rightarrow \Delta t \rightarrow \Delta t/2 \rightarrow$  CPU 4 times!

$$\Delta t \leq \frac{\Delta x}{C}$$

$$\boxed{\frac{c\omega t}{\Delta x} \leq 1}$$

leapfrog

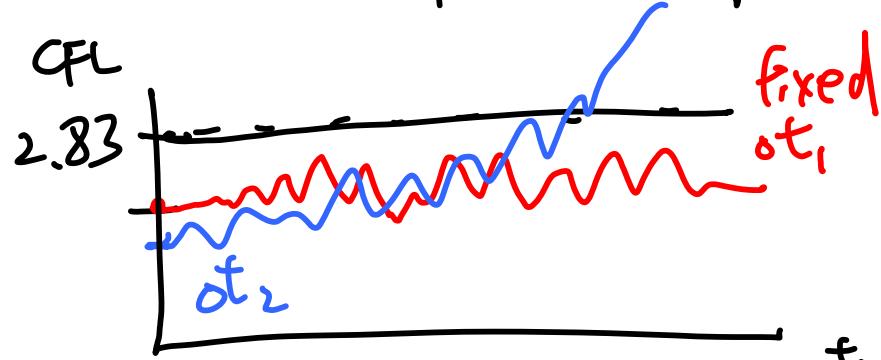


$$CFL = \left| \frac{u \delta t}{\delta x} \right| = \left| \frac{u(x,t) \delta t}{\delta x(x)} \right| \leq 2.83 \text{ for RKC}$$

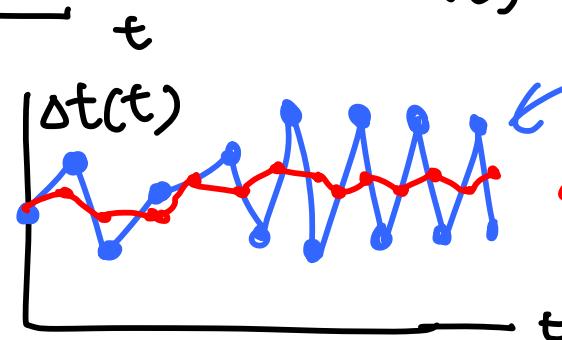
$$\delta t \leq \frac{2.83 \delta x(x)}{|u(x,t)|}$$

worst case:  $\delta x/y$  minimum

$$\delta t_{\max} = 2.83 \left| \frac{\delta x}{|u|} \right|_{\min}$$



Fix CFL # in time



not good

$$\delta t = \alpha \delta t_{\text{new}} + (1-\alpha) \delta t_{\text{old}} \quad (0 \leq \alpha \leq 1)$$

For FFT, const  $\omega$  is required.