

Engineering Mathematics 2

Lecture 17

Yong Sung Park

Previously, we discussed

- Fourier integral:

$$f(x) = \int_0^\infty [A(w) \cos wx + B(w) \sin wx] dw$$

where $A(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos wv dv$, $B(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \sin wv dv$

- Fourier cosine transform for an even function

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_c(w) \cos wx dw, \quad \hat{f}_c(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \cos wx dx$$

- Fourier sine transform for an odd function

$$f(x) = \sqrt{\frac{2}{\pi}} \int_0^\infty \hat{f}_s(w) \sin wx dw, \quad \hat{f}_s(w) = \sqrt{\frac{2}{\pi}} \int_0^\infty f(x) \sin wx dx$$

11.9 Fourier transform

- Complex Fourier integral

Starting from Fourier integral

$$f(x) = \int_0^\infty [A(w) \cos wx + B(w) \sin wx] dw$$

where $A(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \cos wv dv$, $B(w) = \frac{1}{\pi} \int_{-\infty}^\infty f(v) \sin wv dv$

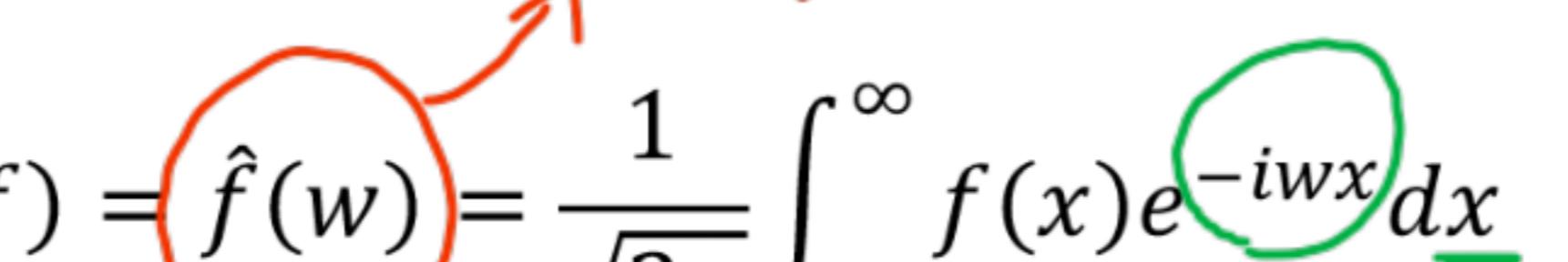
arrive at $f(x) = \frac{1}{2\pi} \int_{-\infty}^\infty \int_{-\infty}^\infty f(v) e^{iw(x-v)} dv dw$

$$\begin{aligned} f(x) &= \int_0^\infty \frac{1}{\pi} \int_{-\infty}^\infty f(v) (\cos wv \cos wx \\ &\quad + \sin wv \sin wx) dv dw \\ &= \frac{1}{\pi} \int_0^\infty \int_{-\infty}^\infty f(v) \cos w(v-x) dv dw \\ &= \frac{1}{2\pi} \left[\int_{-\infty}^\infty \int_{-\infty}^\infty f(v) \cos w(x-v) dv dw \right. \\ &\quad \left. + i \int_{-\infty}^\infty \int_{-\infty}^\infty f(v) \underline{\sin w(x-v)} dv dw \right] \end{aligned}$$

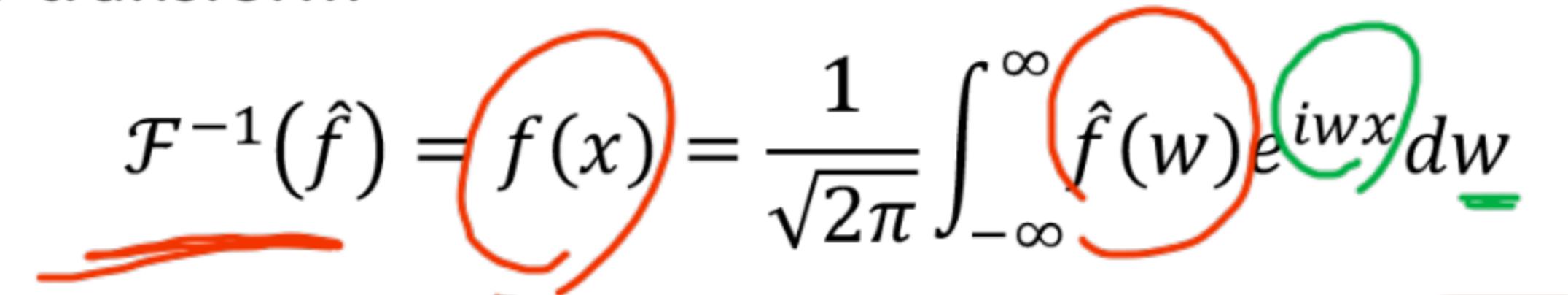
- For $f(x)$ absolutely integrable and piecewise continuous on every finite interval

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-ivw} dv \right] e^{iwx} dw$$


- Fourier transform

$$\mathcal{F}(f) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-ixw} dx$$


- Inverse Fourier transform

$$\mathcal{F}^{-1}(\hat{f}) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{ixw} dw$$


- Example 2: Find the Fourier transform of

$$f(x) = \begin{cases} e^{-ax}, & x > 0 \\ 0, & x < 0 \end{cases}$$

where $a > 0$.

$$\begin{aligned}\hat{f}(w) &= \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ax} e^{-iwx} dx \\ &= \frac{1}{\sqrt{2\pi}} \left[\frac{1}{-(a+iw)} e^{-(a+iw)x} \right]_0^\infty\end{aligned}$$

$$= \frac{1}{\sqrt{2\pi} (a+iw)}$$

- The Fourier transform is linear:

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

- Fourier transform of derivative (under what condition?):

$$\underline{\mathcal{F}(f') = iw\mathcal{F}(f)}$$

- $\mathcal{F}(f'') = ?$

$$\begin{aligned}\mathcal{F}(f'') &= iw\mathcal{F}(f') \\ &= iw\{iw\mathcal{F}(f)\} \\ &= -w^2 \mathcal{F}(f)\end{aligned}$$

$$\begin{aligned}\mathcal{F}(f') &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-ixw} dx \\ &= \frac{1}{\sqrt{2\pi}} \left\{ f(x) e^{-ixw} \Big|_{-\infty}^{\infty} + iw \int_{-\infty}^{\infty} f(x) e^{-ixw} dx \right\}\end{aligned}$$

$$\begin{aligned}&= \frac{1}{\sqrt{2\pi}} \left[f(x) e^{-ixw} \right]_{-\infty}^{\infty} + iw \mathcal{F}(f) \\ &\quad f \rightarrow 0 \quad \text{as } |x| \rightarrow \infty\end{aligned}$$

- Convolution:

$$h(x) = (f * g)(x) = \underbrace{\int_{-\infty}^{\infty} f(p)g(x-p)dp}_{\text{Convolution}} = \int_{-\infty}^{\infty} f(x-p)g(p)dp$$

- If each of f and g has Fourier transform, respectively,

$$\underbrace{\mathcal{F}(f * g)}_{\text{Fourier Transform}} = \sqrt{2\pi} \hat{f} \hat{g} \quad (f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(w) \hat{g}(w) e^{iwx} dw$$

- Discrete Fourier transform
(for sampled values rather than functions)
- Suppose $f(x) = f(x + 2\pi)$ which is being sampled N points over the interval $0 \leq x \leq 2\pi$, namely

$$x_k = \frac{2\pi k}{N}, (k = 0, 1, \dots, N - 1)$$

$$f(x_k)$$

- We want to find $g(x) = \sum_{n=0}^{N-1} c_n e^{inx}$ s.t.

$$f_k$$

$$f_k = f(x_k) = g(x_k) = \sum_{n=0}^{N-1} c_n e^{inx_k}$$



$e^{j\theta}$

$e^{j(n-m)x_k}$

$n=m$

$n \neq m$

- Using the orthogonality of the trigonometric functions, we can get

$$c_n = \frac{1}{N} \sum_{k=0}^{N-1} f_k e^{-inx_k}$$

- Discrete Fourier transform

$$\widehat{f}_n = N c_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}$$

- In vector form

$$\hat{\mathbf{f}} = \underline{\mathbf{F}_N} \underline{\mathbf{f}}$$

where, $\underline{\mathbf{F}_N} = [\underline{e_{nk}}]$, $e_{nk} = e^{-inx_k} = w^{nk}$, $w = e^{-\frac{2\pi i}{N}}$

• Example 4: Find (\mathbf{F}_4) , $w = e^{-\frac{2\pi i}{4}} = e^{-\frac{\pi i}{2}} = -i$

$$\mathbf{F}_4 = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$