Engineering Mathematics 2

Lecture 17

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Previously, we discussed

- Fourier integral:

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]dw$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv$, $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv$

- Fourier cosine transform for an even function

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_c(w)\cos wx\, dw\,, \qquad \hat{f}_c(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\cos wx\, dx$$

- Fourier sine transform for an odd function

$$f(x) = \sqrt{\frac{2}{\pi}}\int_0^\infty \hat{f}_s(w)\sin wx\, dw\,, \qquad \hat{f}_s(w) = \sqrt{\frac{2}{\pi}}\int_0^\infty f(x)\sin wx\, dx$$

11.9 Fourier transform

- Complex Fourier integral

Starting from Fourier integral

$$f(x) = \int_0^\infty [A(w)\cos wx + B(w)\sin wx]dw$$

where $A(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\cos wv\, dv$, $B(w) = \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)\sin wv\, dv$

arrive at $f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)e^{iw(x-v)}dv\, dw$

$$f(x) = \int_0^\infty \frac{1}{\pi}\int_{-\infty}^{\infty} f(v)(\cos wv \cos wx + \sin wv \sin wx)\, dv\, dw$$

$$= \frac{1}{\pi}\int_0^\infty \underbrace{\int_{-\infty}^{\infty} f(v)\cos w(v-x)\, dv}\, dw$$

$$= \frac{1}{2\pi}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\cos w(x-v)\, dv\, dw + i\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(v)\sin w(x-v)\, dv\, dw\right]$$

- For $f(x)$ absolutely integrable and piecewise continuous on every finite interval

$$f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \left[\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(v) e^{-iwv} dv \right] e^{iwx} dw$$

- Fourier transform

$$\mathcal{F}(f) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

- Inverse Fourier transform

$$\mathcal{F}^{-1}(\hat{f}) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$$

- Example 2: Find the Fourier transform of

$$f(x) = \begin{cases} e^{-ax}, x > 0 \\ 0, \quad\quad x < 0 \end{cases}$$

where $a > 0$.

$$\hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_0^\infty e^{-ax} e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \frac{1}{-(a+iw)} e^{-(a+iw)x} \Big|_0^\infty$$

$$= \frac{1}{\sqrt{2\pi}(a+iw)}$$

- The Fourier transform is linear:

$$\mathcal{F}(af + bg) = a\mathcal{F}(f) + b\mathcal{F}(g)$$

- Fourier transform of derivative (under what condition?):

$$\mathcal{F}(f') = iw\mathcal{F}(f)$$

- $\mathcal{F}(f'') = ?$

$$\mathcal{F}(f') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f'(x) e^{-iwx} dx$$

$$= \frac{1}{\sqrt{2\pi}} \left\{ f(x)e^{-iwx} \Big|_{-\infty}^{\infty} + iw \int_{-\infty}^{\infty} f(x) e^{-iwx} dx \right\}$$

$$= \cancel{\frac{1}{\sqrt{2\pi}} \left[f(x)e^{-iwx} \right]_{-\infty}^{\infty}} + \underline{iw\mathcal{F}(f)}$$

$f \to 0$ as $|x| \to \infty$

$$\mathcal{F}(f'') = iw\mathcal{F}(f')$$

$$= iw\{iw\mathcal{F}(f)\}$$

$$= -w^2\mathcal{F}(f)$$

- Convolution:

$$h(x) = (f * g)(x) = \int_{-\infty}^{\infty} f(p)g(x-p)dp = \int_{-\infty}^{\infty} f(x-p)g(p)dp$$

- If each of f and g has Fourier transform, respectively,

$$\mathcal{F}(f * g) = \sqrt{2\pi}\hat{f}\hat{g} \qquad (f * g)(x) = \int_{-\infty}^{\infty} \hat{f}(w)\hat{g}(w)e^{iwx}dw$$

- Discrete Fourier transform
  (for sampled values rather than functions)

- Suppose $f(x) = f(x + 2\pi)$ which is being sampled N points over the interval $0 \le x \le 2\pi$, namely

$$x_k = \frac{2\pi k}{N}, (k = 0, 1, \cdots, N-1)$$

$f(X_k)$

- We want to find $g(x) = \sum_{n=0}^{N-1} c_n e^{inx}$ s.t.

$$f_k = f(x_k) = g(x_k) = \sum_{n=0}^{N-1} c_n e^{inx_k}$$

$e^{i\theta}$

$e^{i(n-m)x_k}$

$n=m$

$n\neq m$

- Using the orthogonality of the trigonometric functions, we can get

$$c_n = \frac{1}{N}\sum_{k=0}^{N-1} f_k e^{-inx_k}$$

- Discrete Fourier transform

$$\widehat{f_n} = N c_n = \sum_{k=0}^{N-1} f_k e^{-inx_k}$$

- In vector form

$$\hat{\mathbf{f}} = \mathbf{F}_N \mathbf{f}$$

where, $\mathbf{F}_N = [e_{nk}]$, $\quad e_{nk} = e^{-inx_k} = w^{nk}$, $\quad w = e^{-\frac{2\pi i}{N}}$

- Example 4: Find $\mathbf{F}_4$, $w = e^{-\frac{2\pi i}{4}} = e^{-\frac{\pi i}{2}} = -i$

$$F_4 = \begin{bmatrix} w^0 & w^0 & w^0 & w^0 \\ w^0 & w^1 & w^2 & w^3 \\ w^0 & w^2 & w^4 & w^6 \\ w^0 & w^3 & w^6 & w^9 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix}$$