

# Engineering Mathematics 2

Lecture 18

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Previously, we discussed

- Fourier transform

$$\mathcal{F}(f) = \hat{f}(w) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$$

- Inverse Fourier transform

$$\mathcal{F}^{-1}(\hat{f}) = f(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$$

- Discrete Fourier transform:

$$\hat{f} = \underline{F}_N f, \text{ where } F_N = [w^{nk}], w = e^{-\frac{2\pi i}{N}}$$

- Inverse transform:

$$f = F_N^{-1} \hat{f}, \text{ where } F_N^{-1} = \frac{1}{N} \bar{F}$$

# Fast Fourier transform

$$= \sum_{m=0}^{M-1} W_M^{mn} f_{2m} + \sum_{m=0}^{\frac{M-1}{2}} W_N^{n \cdot 2m} W_M^{mn} f_{2m+1}$$

- $\hat{f}_n = \sum_{k=0}^{N-1} w_N^{kn} f_k = \sum_{m=0}^{M-1} W_N^{2mn} f_{2m} + \sum_{m=0}^{M-1} W_N^{(2m+1)n} f_{2m+1}, \quad N = 2M$

- For  $N = 2M$ ,  $w_N^2 = (e^{-\frac{2\pi i}{N}})^2 = e^{-\frac{4\pi i}{2M}} = e^{-\frac{2\pi i}{M}} = w_M$

$$\mathbf{f}_{ev} = [f_0, f_2, \dots, f_{N-2}]^T,$$

$$\hat{\mathbf{f}}_{ev} = \mathbf{F}_M \mathbf{f}_{ev}$$

$$\mathbf{f}_{od} = [f_1, f_3, \dots, f_{N-1}]^T,$$

$$\hat{\mathbf{f}}_{od} = \mathbf{F}_M \mathbf{f}_{od}$$

$$W_N^n$$



- Fast Fourier transform

$$\hat{\mathbf{f}}_n = \hat{\mathbf{f}}_{ev,n} + w_N^n \hat{\mathbf{f}}_{od,n}$$

for  $n = 0, 1, \dots, \underline{M-1}$

$$\hat{\mathbf{f}}_{n+M} = \hat{\mathbf{f}}_{ev,n} - w_N^n \hat{\mathbf{f}}_{od,n}$$

for  $n = 0, 1, \dots, M-1$

$$\Theta(N \log N) \ll \Theta(N^2)$$

- Example 5: FFT for  $N = 4$

$$\text{F}_4$$

$$\begin{array}{c} n \\ \hline 0 & 1 \\ 2 & -1 \\ 3 & i \\ \hline w_N^n & \\ & e^{-\frac{2\pi i}{4}} \end{array}$$

$$M = 2, w = e^{-\frac{2\pi i}{2}} = -1, F_2 = [w^{nk}] = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$$

$$\hat{f}_{ev} = F_2 f_{ev} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_0 \\ f_2 \end{bmatrix} = \begin{bmatrix} f_0 + f_2 \\ f_0 - f_2 \end{bmatrix}$$

$$W_N = e^{-\frac{2\pi i}{4}} = -i$$

$$\hat{f}_{od} = F_2 f_{od} = \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} f_1 \\ f_3 \end{bmatrix} = \begin{bmatrix} f_1 + f_3 \\ f_1 - f_3 \end{bmatrix}$$

$$\hat{f} = \begin{bmatrix} f_0 + f_2 + f_1 + f_3 \\ f_0 - f_2 - i(f_1 - f_3) \\ f_0 + f_2 - f_1 - f_3 \\ f_0 - f_2 + i(f_1 - f_3) \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -i & -1 & i \\ 1 & -1 & 1 & -1 \\ 1 & i & -1 & -i \end{bmatrix} \begin{bmatrix} f_0 \\ f_1 \\ f_2 \\ f_3 \end{bmatrix}$$

## 12.1 Basic concepts of PDEs

- Partial differential equations
- Linear vs. nonlinear
- Homogeneous vs. nonhomogeneous
- Boundary conditions and/or initial conditions

$$u^2, uu'$$

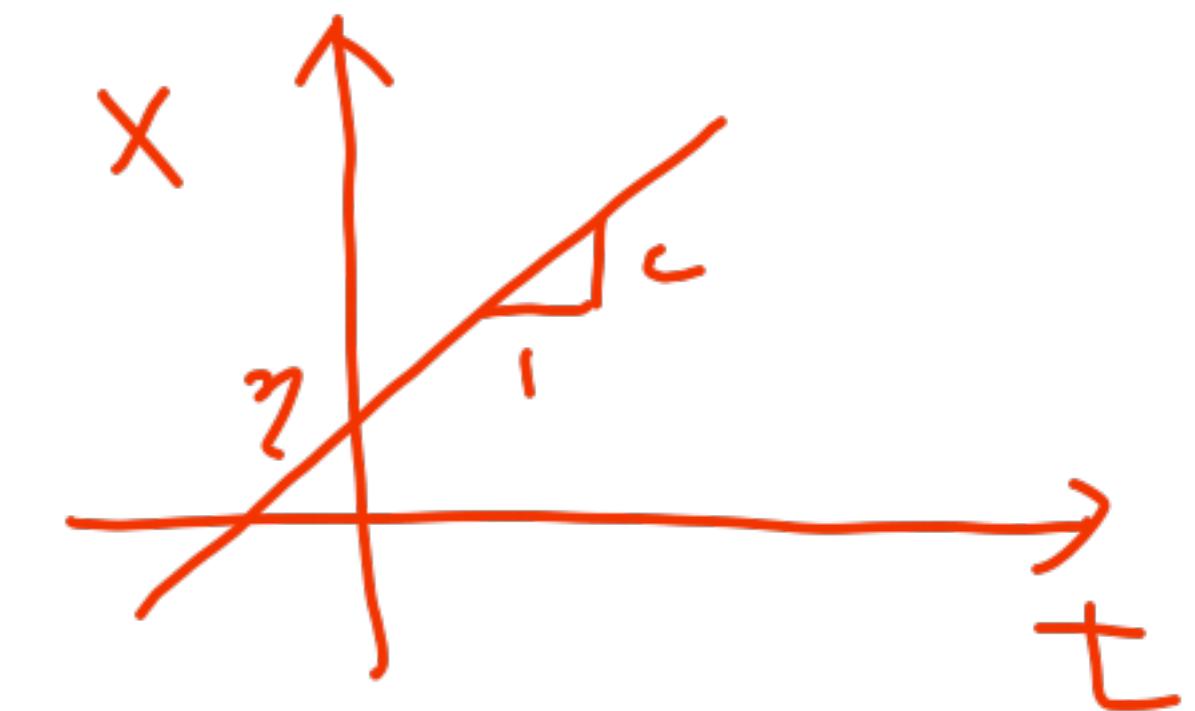
• Example:

Verify that  $u(x, t) = v(x + ct) + w(x - ct)$  with any twice differentiable functions  $v$  and  $w$  satisfies the wave equation:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

$$x + ct = \zeta$$

$$x - ct = \eta, \quad x = ct + \eta$$



$$\frac{\partial u}{\partial x} = \frac{dv}{d\zeta} \frac{\partial \zeta}{\partial x} + \frac{dw}{d\eta} \frac{\partial \eta}{\partial x} = v' + w'$$

↷

$$\frac{\partial^2 u}{\partial x^2} = \frac{d}{d\zeta} (v') \frac{\partial \zeta}{\partial x} + \frac{d}{d\eta} (w') \frac{\partial \eta}{\partial x} = \underline{v'' + w''}$$

$$\frac{\partial^2 u}{\partial t^2} = c^2 (v'' + w'') = c^2 \frac{\partial^2 u}{\partial x^2}$$

## 12.2 Wave equation as a model for vibrating string

- For an elastic string stretched and fastened at  $x = 0$  and  $x = L$ , the deflection  $u(x, t)$  is found as the solution for

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

which is the one-dimensional wave equation.

12.3 Solution by separating variables and Fourier series

- The governing equation for a vibrating string:

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2}$$

- The boundary conditions:

$$u(0, t) = u(L, t) = 0$$

- The initial conditions:

$$u(x, 0) = f(x), \quad u_t(x, 0) = g(x)$$

$$u(x, t) = F(x) G(t)$$

$$\boxed{\begin{array}{l} F(0) = 0 \\ F(L) = 0 \end{array}}$$

$$u(x, t) = F(x) G(t)$$

$$\frac{\partial^2 u}{\partial x^2} = F'' G$$

$$\frac{\partial^2 u}{\partial t^2} = \ddot{F} G$$

$$\ddot{F} G = c^2 F'' G$$

$$\frac{\ddot{G}}{c^2 G} = \frac{F''}{F} = k$$

$$F'' - kF = 0$$

$$\ddot{G} - kc^2 G = 0$$

$$k = -p^2$$

$$F'' + p^2 F = 0$$

$$F = A \cos px + B \sin px$$

$$F(0) = A = 0$$

~~$$F(L) = B \sin pl = 0,$$~~

$$F(x) = \sin \frac{n\pi x}{L}$$

$$G + p^2 c^2 G = 0 \rightarrow \lambda_n$$

$$G(t) = C_n \cos \frac{C_n \pi}{L} t + D_n \sin \frac{C_n \pi}{L} t$$

$$u_n(x, t) = (C_n \cos \lambda_n t + D_n \sin \lambda_n t) \sin \frac{n\pi x}{L}$$

$$u = \sum_{n=1}^{\infty} u_n$$

$$u(x, 0) = \sum_{n=1}^{\infty} C_n \sin \frac{n\pi x}{L} = f(x)$$

$$C_n = \frac{2}{L} \int_0^L f(x) \sin \frac{n\pi x}{L} dx$$

$$\left. \frac{\partial u}{\partial t} \right|_{t=0} = \sum_{n=1}^{\infty} \lambda_n D_n \sin \frac{n\pi x}{L} = g(x)$$

$$\lambda_n D_n = \frac{2}{L} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

$$D_n = \frac{2}{C_n \pi} \int_0^L g(x) \sin \frac{n\pi x}{L} dx$$

### 12.4 D'Alembert's solution

- $u(x, t) = \phi(x + ct) + \psi(x - ct)$

- With initial conditions:  $u(x, 0) = f(x), u_t(x, 0) = g(x)$

- $u(x, t) = \frac{1}{2}[f(x + ct) + f(x - ct)] + \frac{1}{2c} \int_{x-ct}^{x+ct} g(s) ds$

$$u(x, t) = \phi(x + ct) + \psi(x - ct)$$

$$u(x, 0) = \underbrace{\phi(x)}_{\psi(x)} + \underbrace{\psi(x)}_{\phi(x)} = f(x)$$

$$u_t(x, 0) = c\phi_t - c\psi_t = g(x)$$

$$\phi(x) = \frac{1}{2} f(x) + \frac{1}{2c} \int_{x_0}^x g(s) ds + a$$

$$\psi(x) = \frac{1}{2} f(x) - \frac{1}{2c} \int_{x_0}^x g(s) ds - a$$

$$c[\phi(x) - \phi(x_0)]$$

$$-c[\psi(x) - \psi(x_0)] = \int_{x_0}^x g(s) ds$$

$$\underbrace{\phi(x) - \psi(x)}_{a} = \frac{1}{c} \int_{x_0}^x g(s) ds + \underbrace{c(\phi(x_0) - \psi(x_0))}_{a}$$

- Classification of 2<sup>nd</sup>-order quasilinear PDEs

$$Au_{xx} + 2Bu_{xy} + Cu_{yy} = F(x, y, u, u_x, u_y)$$

$AC - B^2 < 0$	Hyperbolic
$AC - B^2 = 0$	Parabolic
$AC - B^2 > 0$	Elliptic