

5.8 Implicit methods in high dimensions

$$\Delta t \leq \frac{\alpha x^2}{4\alpha}$$

노트 제목

$$\frac{\partial \phi}{\partial t} = \alpha \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)$$

Crank - Nicolson method

$$\frac{\phi^{n+1} - \phi^n}{\Delta t} = \frac{\alpha}{2} \left(\frac{\partial^2 \phi^{n+1}}{\partial x^2} + \frac{\partial^2 \phi^n}{\partial x^2} \right) + \frac{\alpha}{2} \left(\frac{\partial^2 \phi^{n+1}}{\partial y^2} + \frac{\partial^2 \phi^n}{\partial y^2} \right)$$

$$\Delta x = \Delta y = h \quad \& \text{ CD2}$$

$$\Rightarrow \phi_{l,j}^{n+1} - \phi_{l,j}^n = \frac{\alpha \Delta t}{2h^2} \left(\phi_{l+1,j}^{n+1} - 2\phi_{l,j}^{n+1} + \phi_{l-1,j}^{n+1} \right) \\ + \beta \left(\phi_{l,j+1}^{n+1} - 2\phi_{l,j}^{n+1} + \phi_{l,j-1}^{n+1} \right)$$

$$+ \text{ " } (\hat{\phi}_{\ell+1,j}^n - 2\hat{\phi}_{\ell,j}^n + \hat{\phi}_{\ell-1,j}^n)$$

$$+ \text{ " } (\hat{\phi}_{\ell,j+1}^n - 2\hat{\phi}_{\ell,j}^n + \hat{\phi}_{\ell,j-1}^n)$$

$$\Rightarrow -\beta \hat{\phi}_{\ell+1,j}^n + (1+4\beta) \hat{\phi}_{\ell,j}^n - \beta \hat{\phi}_{\ell-1,j}^n - \beta \hat{\phi}_{\ell,j+1}^n - \beta \hat{\phi}_{\ell,j-1}^n = \hat{F}_{\ell,j}^n$$

CN + CD2

$\Theta(\alpha t^2)$ $\Theta(\alpha x^2)$

$\ell = 1, 2, \dots, M-1$; $j = 1, 2, \dots, N-1$

Sys. of eqs

$$\begin{bmatrix} B & C \\ A & B & C \\ \vdots & \ddots & \vdots \\ \phi & \ddots & AB \end{bmatrix} \begin{bmatrix} \phi \\ \vdots \\ \phi_j \\ \vdots \\ \phi_{M+N-1} \end{bmatrix} = \begin{bmatrix} F_1 \\ F_2 \\ \vdots \\ F_{\ell-j} \\ \vdots \\ F_{M+N-1} \end{bmatrix}$$

$\underbrace{(M-1)(N-1) \times (M-1)(N-1)}$ Block-tridiagonal matrix.

$M=N=100$: # of elts in the matrix = 10^8

→ too difficult to solve

direct inversion requires $O(M^3 N^3)$ operations

→ may have to introduce an iterative method

→ but actually not \Leftarrow ADI method.

5.9 Alternating directional implicit (ADI) method
and approximate factorization

(no class on this wednesday)

$$\frac{\partial \phi}{\partial t} = \alpha \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right) \quad \text{with} \quad \frac{\phi_{l+1,j}^{n+1} - 2\phi_{l,j}^{n+1} + \phi_{l-1,j}^{n+1}}{\Delta x^2}$$

$$CN : \frac{\phi^{n+1} - \phi^n}{\Delta t} = \frac{\alpha}{2} (A_x \phi^{n+1} + A_y \phi^{n+1}) + \frac{\alpha}{2} (A_y \phi^{n+1} + A_y \phi^n) + O(\Delta x^2) + O(\Delta y^2)$$

A_x, A_y : difference operators having 2nd-order accuracy representing derivatives in x and y directions

$$\rightarrow \left[I - \frac{\alpha \Delta t}{2} A_x - \frac{\alpha \Delta t}{2} A_y \right] \phi^{n+1} = \left[I + \frac{\alpha \Delta t}{2} A_x + \frac{\alpha \Delta t}{2} A_y \right] \phi^n + \Delta t \begin{bmatrix} O(\Delta t^2) \\ O(\Delta x^2) \\ O(\Delta y^2) \end{bmatrix}$$

$$= (I - \frac{\alpha \Delta t}{2} A_x)(I - \frac{\alpha \Delta t}{2} A_y) - \frac{\alpha^2 \Delta t^2}{4} A_x A_y = (I + \frac{\alpha \Delta t}{2} A_x)(I + \frac{\alpha \Delta t}{2} A_y) - \frac{\alpha^2 \Delta t^2}{4} A_x A_y$$

$$\rightarrow \left(I - \frac{\alpha \delta t}{2} A_x \right) \left(I - \frac{\alpha \delta t}{2} A_y \right) \phi^{n+1} = \left(I + \frac{\alpha \delta t}{2} A_x \right) \left(I + \frac{\alpha \delta t}{2} A_y \right) \phi^n$$

$$+ \boxed{+ \frac{\alpha^2 \delta t^2}{4} A_x A_y (\phi^{n+1} - \phi^n)} + \delta t \left[O(\delta t^2) + O(\alpha_x^2) + O(\alpha_y^2) \right]$$

$\frac{\delta t}{\delta t} \frac{\partial \phi}{\partial t} + \dots$

$O(\delta t^3)$

\therefore neglect □ term
w/o losing accuracy

↑
approximate factorization

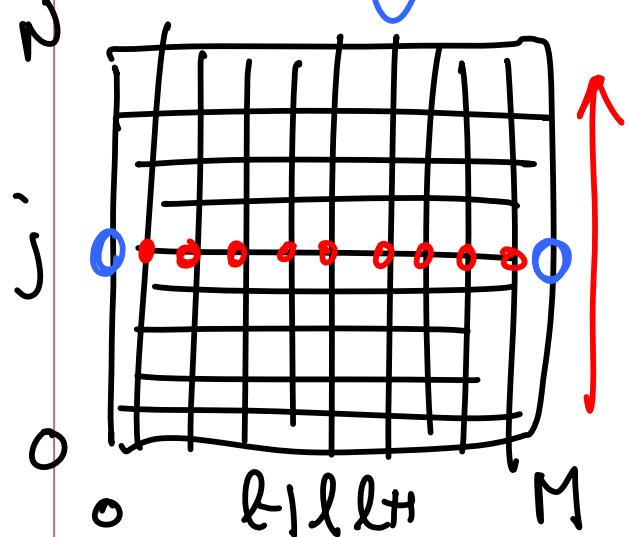
$$\Rightarrow \boxed{\left(I - \frac{\alpha \delta t}{2} A_x \right) \left(I - \frac{\alpha \delta t}{2} A_y \right) \phi^{n+1} = \left(I + \frac{\alpha \delta t}{2} A_x \right) \left(I + \frac{\alpha \delta t}{2} A_y \right) \phi^n}$$

Z F $CN + CD2 + AF$

$$\rightarrow \left(I - \frac{\alpha \Delta t}{2} A_x \right) z = F$$

$$\rightarrow z_{l,j} - \frac{\alpha \Delta t}{2} \cdot \frac{z_{l,j+1} - 2z_{l,j} + z_{l,j-1}}{\Delta x^2} = F_{l,j}$$

$\ell = 1, 2, \dots, M-1$
 $j = 1, 2, \dots, N-1$



tri-diagonal matrix for ℓ

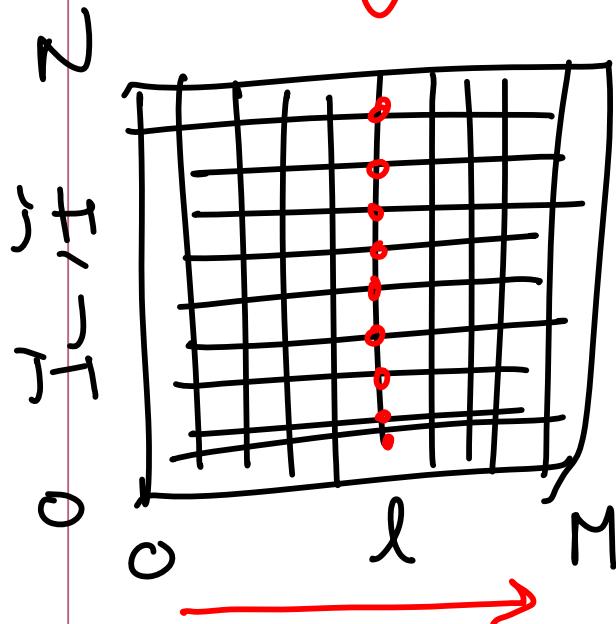
For each j , solve a tri-diagonal matrix for

$z_{l,j}$.

$\underbrace{\quad}_{\mathcal{O}(MN)}$ $\underbrace{\mathcal{O}(M)}_{\mathcal{O}(M)}$

Having solved for $\bar{z}_{l,j}$, $(I - \frac{\alpha_{\text{tot}}}{2} A_y) \phi^{n+1} = z$

$$\rightarrow \phi_{l,j}^{n+1} - \frac{\alpha_{\text{tot}}}{2} \underbrace{\phi_{2,j+1}^{n+1} - 2\phi_{l,j}^{n+1} + \phi_{l,j-1}^{n+1}}_{\Delta y^2} = \bar{z}_{l,j} - \cancel{\dots}$$



for each l , solve a tri-diagonal matrix for

$\phi_{l,j}^{n+1}$.

M

$O(N)$

$O(MN)$ operations

\therefore total $O(2MN)$ operations are required.

ADI method.

④ requires b.c's, $z_{0,j}$ & $z_{M,j}$ for $j=1, 2, \dots, N-1$.

* * ⑤ $l=0$: $z_{0,j} = \phi_{0,j}^{n+1} - \frac{\alpha t}{2} \frac{\phi_{0,j+1}^{n+1} - 2\phi_{0,j}^{n+1} + \phi_{0,j-1}^{n+1}}{\sigma q^2}$

$l=M$: $z_{M,j} = \dots - - - - -$

\Rightarrow ADI method w/ approximate factorization

no iteration is required
implicit method \Rightarrow great!

$$\text{ADI} : (I - \frac{\alpha t}{2} A_x)(I - \frac{\alpha t}{2} A_y) \phi^{n+1} = (I + \frac{\alpha t}{2} A_x)(I + \frac{\alpha t}{2} A_y) \phi^n$$

stability? $\phi^{n+1} = \sigma \phi^n$

$$\rightarrow \delta = \frac{(1 + \frac{\alpha \Delta t}{2} (-k_1'^2))(1 + \frac{\alpha \Delta t}{2} (-k_2'^2))}{(1 - \frac{\alpha \Delta t}{2} (-k_1'^2))(1 - \frac{\alpha \Delta t}{2} (-k_2'^2))}$$

$$= \frac{(1 - \frac{\alpha \Delta t}{2}(1 - \cos k_1 \alpha x))(1 - \frac{\alpha \Delta t}{2}(1 - \cos k_2 \alpha y))}{(1 + \quad \quad \quad)(1 + \quad \quad \quad)} \leq 1$$

$k_1'^2 = \frac{2(1 - \cos k_1 \alpha x)}{\alpha x^2}$
 $k_2'^2 = \frac{2(1 - \cos k_2 \alpha y)}{\alpha y^2}$
 \uparrow
 CD2

\therefore unconditionally stable

⑥ delta-form ($\delta \phi = \phi^{n+1} - \phi^n$)

$$\frac{\partial \phi}{\partial t} = \alpha \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} \right)$$

$$CN + CD2 : \frac{\phi^{n+1} - \phi^n}{\Delta t} = \frac{\alpha}{2} (A_x \phi^{n+1} + A_x \phi^n) + \frac{\alpha}{2} (A_y \phi^{n+1} + A_y \phi^n) \\ + O(\Delta t^2) + O(\alpha x^2) + O(\alpha y^2)$$

$$\delta\phi = \phi^{n+1} - \phi^n$$

$$\Rightarrow \frac{\delta\phi}{\Delta t} = \frac{\alpha}{2} (A_x (\delta\phi + \phi^n) + A_x \phi^n) + \frac{\alpha}{2} (A_y (\delta\phi + \phi^n) + A_y \phi^n)$$

$$\rightarrow \underbrace{(I - \frac{\alpha \Delta t}{2} A_x - \frac{\alpha \Delta t}{2} A_y)}_{\parallel} \delta\phi = \alpha \Delta t (A_x \phi^n + A_y \phi^n) \\ + \Delta t [O(\Delta t^2) + O(\alpha x^2) + O(\alpha y^2)]$$

$$(I - \frac{\alpha \Delta t}{2} A_x) (I - \frac{\alpha \Delta t}{2} A_y) \delta\phi - \frac{\alpha \Delta t^2}{4} A_x A_y \delta\phi \\ \sim \phi^{n+1} - \phi^n = \frac{\alpha \Delta t}{2} \frac{\partial \phi}{\partial t} + \dots$$

$O(\Delta t^3) \therefore \text{neglect}$

$$\rightarrow \left(I - \frac{\alpha \otimes A_x}{2} \right) \left(I - \frac{\alpha \otimes A_y}{2} \right) \delta \phi = \alpha \otimes (A_x \phi' + A_y \phi'')$$

ADI
in δ -form

$$\text{let } \left(I - \frac{\alpha \otimes A_y}{2} \right) \delta \phi = z$$

$$\text{Then, } \left(I - \frac{\alpha \otimes A_x}{2} \right) z = \alpha \otimes (A_x \phi' + A_y \phi'')$$

$$\rightarrow z_{l,j} - \frac{\alpha \otimes}{2} \frac{z_{l+1,j} - 2z_{l,j} + z_{l-1,j}}{\alpha \otimes^2} = RHS_{l,j}, \quad \begin{matrix} l=1, 2, \dots, M-1 \\ j=1, 2, \dots, N-1 \end{matrix}$$

for each j , solve tri-diagonal matrix to get $z_{l,j}$.

(we need $z_{0,j}$ and $z_{M,j}$ as boundary conditions).

$$\text{Then, } \left(I - \frac{\alpha \dot{t}}{2} A_y \right) \delta \phi = z$$

$$\rightarrow \delta \phi_{l,j} - \frac{\alpha \dot{t}}{2} \underbrace{\frac{\delta \phi_{l,j+1} - 2\delta \phi_{l,j} + \delta \phi_{l,j-1}}{\sigma y^2}}_{\text{(*)}} = z_{l,j}$$

$$\delta \phi_{l,0} = \phi_{l,0}^{n+1} - \phi_{l,0}^n$$

$$\delta \phi_{l,M} = \phi_{l,M}^{n+1} - \phi_{l,M}^n$$

known

for each l , solve tri-diagonal matrix to get $\delta \phi_{l,j}$.

$$\text{Once } \delta \phi \text{ is obtained, } \phi^{n+1} = \phi^n + \delta \phi$$

How about $z_{0,j}$ and $z_{M,j}$?

$$\text{(*)} \rightarrow @ l=0: z_{0,j} = \delta \phi_{0,j} - \frac{\alpha \dot{t}}{2} \underbrace{\frac{\delta \phi_{0,j+1} - 2\delta \phi_{0,j} + \delta \phi_{0,j-1}}{\sigma y^2}}_{\phi_{0,j+1}^{n+1} - \phi_{0,j-1}^n}$$

$$l=M: z_{M,j} = \delta \phi_{M,j} - \dots$$

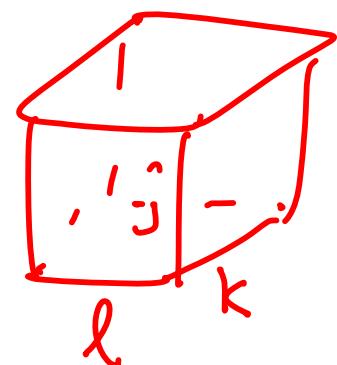
if b.c's do not change in time, $\delta \phi_{0,j} = \phi_{0,j}^{n+1} - \phi_{0,j}^n = 0 \rightarrow z_{0,j} = 0$

$$\delta \phi_{Mij} = \dots \rightarrow \bar{\phi}_{Mij} = 0$$

How about 3D?

$$\frac{\partial \phi}{\partial t} = \alpha \left(\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2} \right)$$

$$\Rightarrow \text{CDT+CN} \quad (I - \alpha A_x - \alpha A_y - \alpha A_z) \phi^{n+1} = \text{RHS}$$



$$(I - \alpha A_x) (I - \alpha A_y) (I - \alpha A_z) \phi^{n+1} = \text{RHS}'$$

Z

tri-diagonal matrix for each j and k

$$(I - \alpha A_1) (I - \alpha A_2) \phi^{\text{NH}} = Z$$

Y

" for each l and k

$$(I - \alpha A_2) \phi^{\text{NH}} = Y$$

" for each l and j

