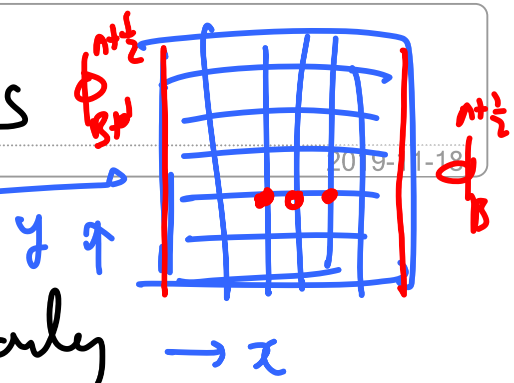
5.9.2 Alternating directional implicit methods



Peaceman & Rachford (1955)

① $$\frac{\phi^{n+\frac{1}{2}} - \phi^{n}}{\Delta t/2} = \alpha\left(\frac{\partial^2 \phi^{n+\frac{1}{2}}}{\partial x^2} + \frac{\partial^2 \phi^{n}}{\partial y^2}\right)$$: implicit only in x-direction

② $$\frac{\phi^{n+1} - \phi^{n+\frac{1}{2}}}{\Delta t/2} = \alpha\left(\frac{\partial^2 \phi^{n+\frac{1}{2}}}{\partial x^2} + \frac{\partial^2 \phi^{n+1}}{\partial y^2}\right)$$: " " in y-direction

CD2 : ① $$\left(I - \frac{\alpha \Delta t}{2} A_x\right)\phi^{n+\frac{1}{2}} = \left(I + \frac{\alpha \Delta t}{2} A_y\right)\phi^{n}$$

② $$\left(I - \frac{\alpha \Delta t}{2} A_y\right)\phi^{n+1} = \left(I + \frac{\alpha \Delta t}{2} A_x\right)\phi^{n+\frac{1}{2}}$$

$$① \to \quad \phi^{n+\frac{1}{2}} = \left(I - \frac{\alpha\Delta t}{2}A_x\right)^{-1}\left(I + \frac{\alpha\Delta t}{2}A_y\right)\phi^n$$

$$② \to \quad \left(I - \frac{\alpha\Delta t}{2}A_y\right)\phi^{n+1} = \left(I + \frac{\alpha\Delta t}{2}A_x\right)\left(I - \frac{\alpha\Delta t}{2}A_x\right)^{-1}\left(I + \frac{\alpha\Delta t}{2}A_y\right)\phi^n$$

$$\underline{\left(I - \frac{\alpha\Delta t}{2}A_x\right)}\left(I - \frac{\alpha\Delta t}{2}A_y\right)\phi^{n+1} = \underline{\left(I - \frac{\alpha\Delta t}{2}A_x\right)}\,\underline{\left(I + \frac{\alpha\Delta t}{2}A_x\right)}\left(I - \frac{\alpha\Delta t}{2}A_x\right)^{-1}\left(I + \frac{\alpha\Delta t}{2}A_y\right)\phi^n$$

commute

$$= \left(I + \frac{\alpha\Delta t}{2}A_x\right)\left(I + \frac{\alpha\Delta t}{2}A_y\right)\phi^n$$

$\Rightarrow$ same as before from approx. factorization

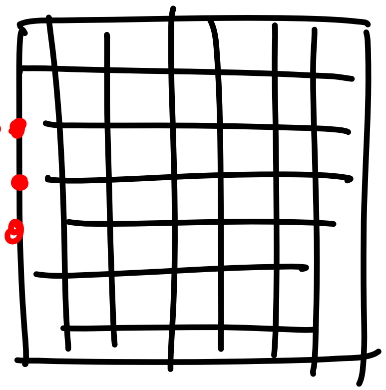
b.c.'s for $\phi^{n+\frac{1}{2}}$: $\left(I - \frac{\alpha\Delta t}{2}A_x\right)\phi_B^{n+\frac{1}{2}} = \left(I + \frac{\alpha\Delta t}{2}A_y\right)\phi_B^n$

$$\left(I - \frac{\alpha\Delta t}{2}A_y\right)\phi_B^{n+1} = \left(I + \frac{\alpha\Delta t}{2}A_x\right)\phi_B^{n+\frac{1}{2}}$$

$$\rightarrow \quad 2\phi_B^{n+\frac{1}{2}} = \left(I + \frac{\alpha\Delta t}{2}A_y\right)\phi_B^n + \left(I - \frac{\alpha\Delta t}{2}A_y\right)\phi_B^{n+1}$$

known

$$\frac{1}{\Delta y^2}\left(\phi_{j+1}^n - 2\phi_j^n + \phi_{j-1}^n\right)_B$$



5.9.3 Mixed time advancement

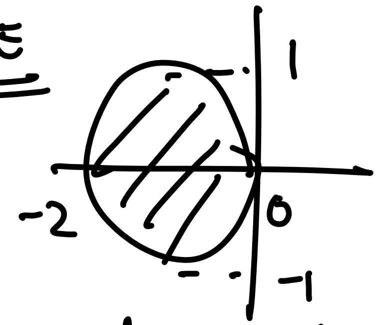
$$\boxed{\frac{\partial\phi}{\partial t} = -c\frac{\partial\phi}{\partial x} + \alpha\frac{\partial^2\phi}{\partial x^2}}$$: convection-diffusion eq.

$$\frac{\partial\phi}{\partial t} = -c\frac{\partial\phi}{\partial x} \rightarrow \frac{d\psi}{dt} = -ick'\psi$$: purely imaginary

$$\frac{\partial\phi}{\partial t} = \alpha\frac{\partial^2\phi}{\partial x^2} \rightarrow \frac{d\psi}{dt} = -\alpha k'^2\psi$$: real & negative

$$\hookrightarrow \frac{d\psi}{dt} = \underbrace{\left(-\alpha k_D'^2 - i c k_c'\right)}_{\text{complex number}} \psi$$

EE



conditionally stable

$\Delta t \propto \Delta x^2$

not good.

- EE on conv. term + CN on diff. term

$$\rightarrow \frac{\phi^{n+1}-\phi^{n}}{\Delta t} = -c\frac{\partial \phi^{n}}{\partial x} + \frac{\alpha}{2}\left(\frac{\partial^2 \phi^{n+1}}{\partial x^2} + \frac{\partial^2 \phi^{n}}{\partial x^2}\right) \rightarrow O(\Delta t)$$

TDMA

$\hookrightarrow$ $\Delta t \propto \Delta x$ not bad

- AB2 on conv. term + CN on diff. term

$$\rightarrow \frac{\phi^{n+1}-\phi^{n}}{\Delta t} = -\frac{c}{2}\left(3\frac{\partial \phi^{n}}{\partial x} - \frac{\partial \phi^{n-1}}{\partial x}\right) + \frac{\alpha}{2}\left(\frac{\partial^2 \phi^{n+1}}{\partial x^2} + \frac{\partial^2 \phi^{n}}{\partial x^2}\right) \rightarrow O(\Delta t^2)$$

TDMA

$\Rightarrow$ conditionally stable : $\frac{\Delta t\, c}{\Delta x} \le 1$

- CN on all terms

$$\rightarrow \frac{\phi^{n+1}-\phi^{n}}{\Delta t} = -\frac{c}{2}\left(\frac{\partial\phi^{n+1}}{\partial x}+\frac{\partial\phi^{n}}{\partial x}\right)+\frac{\alpha}{2}\left(\frac{\partial^2\phi^{n+1}}{\partial x^2}+\frac{\partial^2\phi^{n}}{\partial x^2}\right) \rightarrow O(\Delta t^2)$$

TDMA

absolutely stable

* How about $c=\phi$?

Burgers eq.

$$\frac{\partial\phi}{\partial t} = -\phi\frac{\partial\phi}{\partial x}+\alpha\frac{\partial^2\phi}{\partial x^2}$$: nonlinear conv.-diff. eq.

- CN on all terms

$$\frac{\phi^{n+1}-\phi^{n}}{\Delta t} = -\frac{1}{2}\left(\phi^{n+1}\frac{\partial\phi^{n+1}}{\partial x}+\phi^{n}\frac{\partial\phi^{n}}{\partial x}\right)+\frac{\alpha}{2}\left(\frac{\partial^2\phi^{n+1}}{\partial x^2}+\frac{\partial^2\phi^{n}}{\partial x^2}\right)$$

fully implicit method.

→ no stability limit but iteration.

• AB2 on conv. term + CN on diff. term

$$\frac{\phi^{n+1}-\phi^{n}}{\Delta t} = -\frac{1}{2}\left(3\phi^{n}\frac{\partial\phi^{n}}{\partial x} - \phi^{n-1}\frac{\partial\phi^{n-1}}{\partial x}\right) + \frac{\alpha}{2}\left(\frac{\partial^2\phi^{n+1}}{\partial x^2} + \frac{\partial^2\phi^{n}}{\partial y^2}\right)$$

semi-implicit method

→ $\Delta t \propto \Delta x$, no iteration

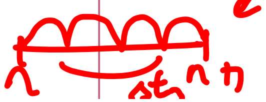
not self-starting, fixed Δt

carry information of ϕ^n & ϕ^{n-1}.

$y' = f(y,t)$

• RK3 on conv. term + CN on diff. term) semi-implicit method

• RK4 on " + CN " .



$\Delta t_3 > \Delta t_2 > \Delta t_1$

→ self-starting, $\Delta t \propto \Delta x$.

AB2 or RK3 ~~RK4~~ exp or imp. CN can change δt

exp. method

CN & Approx. Fact. & ADI

* Navier-Stokes eq.

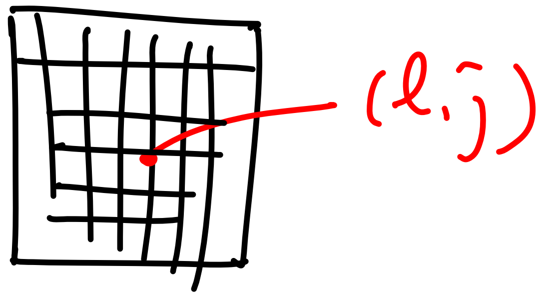
$$\begin{cases} \rho \dfrac{\partial u_i}{\partial t} + \rho u_j \dfrac{\partial u_i}{\partial x_j} = -\dfrac{\partial p}{\partial x_i} + \dfrac{\partial}{\partial x_j}\left(\mu \dfrac{\partial u_i}{\partial x_j}\right) \\ \dfrac{\partial u_i}{\partial x_i} = 0 \end{cases} \qquad i = 1, 2, 3$$

$$\rightarrow \nabla^2 p = \alpha \qquad u_i, p$$

5.10 Elliptic PDEs

$$\begin{cases} \nabla^2 \phi = 0 & \text{Laplace eq.} \\ \nabla^2 \phi = q & \text{Poisson eq.} \\ \nabla^2 \phi + k^2 \phi = q & \text{Helmholtz eq.} \end{cases}$$

$\nabla^2\phi = f$

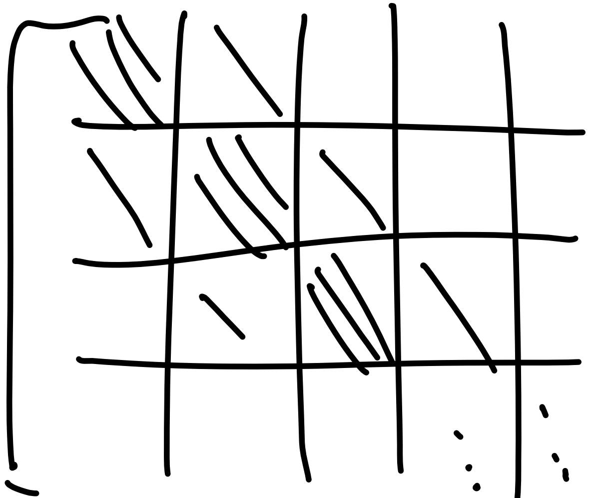


in 2D, $\dfrac{\partial^2\phi}{\partial x^2} + \dfrac{\partial^2\phi}{\partial y^2} = f$

CD2 : $$\boxed{\frac{\phi_{l+1,j} - 2\phi_{l,j} + \phi_{l-1,j}}{\Delta x^2} + \frac{\phi_{l,j+1} - 2\phi_{l,j} + \phi_{l,j-1}}{\Delta y^2} = f_{l,j}}$$

$l = 1, 2, \cdots, M-1$

$j = 1, 2, \cdots, N-1$



$(M-1)(N-1) \times (M-1)(N-1)$

$$\begin{bmatrix}\phi_{11}\\ \phi_{21}\\ \vdots\\ \phi_{l,j}\\ \vdots\\ \phi_{M-1,N-1}\end{bmatrix} = \begin{bmatrix}\vdots\\ \vdots\\ \vdots\end{bmatrix}$$

Block-tri-diagonal matrix

$\downarrow$

difficult to solve directly

$\downarrow$

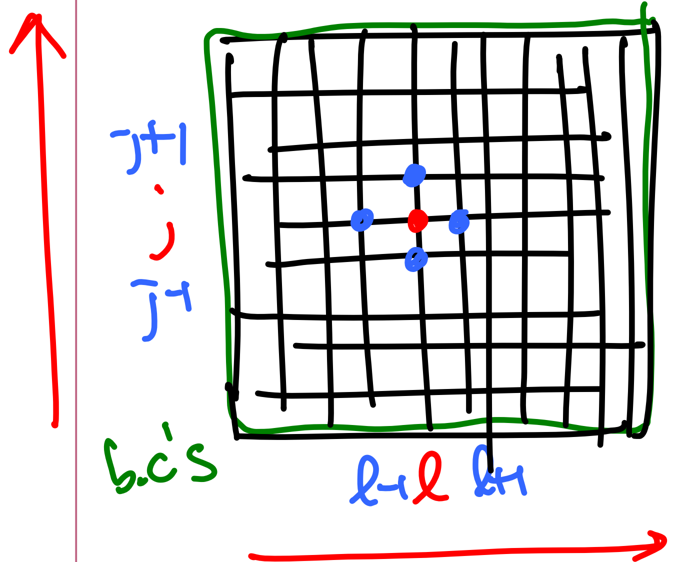
introduce iterative methods

$$\frac{\phi^{k}_{l+1,j} - 2\phi^{k+1}_{l,j} + \phi^{k}_{l-1,j}}{\Delta x^2} + \frac{\phi^{k}_{l,j+1} - 2\phi^{k+1}_{l,j} + \phi^{k}_{l,j-1}}{\Delta y^2} = f_{l,j}$$

k: iteration index

$(\Delta x = \Delta y)$ $\hookrightarrow$ $\boxed{\phi^{k+1}_{l,j} = \frac{1}{4}\left(\phi^{k}_{l+1,j} + \phi^{k}_{l-1,j} + \phi^{k}_{l,j+1} + \phi^{k}_{l,j-1}\right) - \frac{\Delta^2}{4} f_{l,j}}$

Jacobi iteration



$$\frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} = f \Rightarrow \frac{\partial\phi}{\partial t} = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} - f$$

ellip. eq   para. eq.

As $t \to \infty$, steady sol. $\to$ sol. of Poisson eq.

We integrate para. eq. until reaching steady state $\left(\frac{\partial \phi}{\partial t}=0\right)$

EE $(\Delta x=\Delta y)$
+CD2

$$\frac{\phi^{n+1}_{l,j}-\phi^{n}_{l,j}}{\Delta t}=\frac{1}{\Delta x^2}\left(\phi^{n}_{l+1,j}-2\phi^{n}_{l,j}+\phi^{n}_{l-1,j}+\phi^{n}_{l,j+1}-2\phi^{n}_{l,j}+\phi^{n}_{l,j-1}\right)-f^{n}_{l,j}$$

stability: $\Delta t \le \frac{\Delta x^2}{4}$ $\rightarrow$ $\Delta t_{max}=\Delta x^2/4$

$$\Rightarrow \phi^{n+1}_{l,j}=\frac{1}{4}\left(\phi^{n}_{l+1,j}+\phi^{n}_{l-1,j}+\phi^{n}_{l,j+1}+\phi^{n}_{l,j-1}\right)-\frac{\Delta x^2}{4}f^{n}_{l,j}$$

same as Jacobi iteration $\rightarrow$ very slow convergence