Fusion Plasma Theory I

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3/8/21 Definition of plasma

1. Thermal equilibrium of charged gas particles in near-equilibrium

$$f(\vec{x},\vec{v},t) \propto e^{-U/kT} \qquad U \equiv \frac{1}{2}mv^2 + q\phi$$

particle distribution — Boltzmann — total energy

$$\int f(\vec{x},\vec{v},t)\, d^3v = n \quad (\phi = 0)$$

→ Local (Boltzmann) Maxwellian

$$f(\vec{x},\vec{v},t) = \frac{n(\vec{x},t)}{(2\pi kT(\vec{x},t))^{3/2}} e^{-\left(\frac{1}{2}mv^2 + q\phi(\vec{x},t)\right)/kT(\vec{x},t)}$$

— phase-space average (moment)

$$\langle A \rangle = \frac{\int A f\, d^3v}{\int f\, d^3v} = \frac{1}{n}\int A f\, d^3v$$

$$\langle E \rangle = \langle \frac{1}{2}mv^2 \rangle = \frac{3}{2}kT$$

def. thermal velocity $v_t \equiv (2T/m)^{\frac{1}{2}}$

$$f = \frac{n}{(\sqrt{\pi}\, v_t)^3} e^{-v^2/v_t^2} \cdot e^{-q\phi/kT}$$

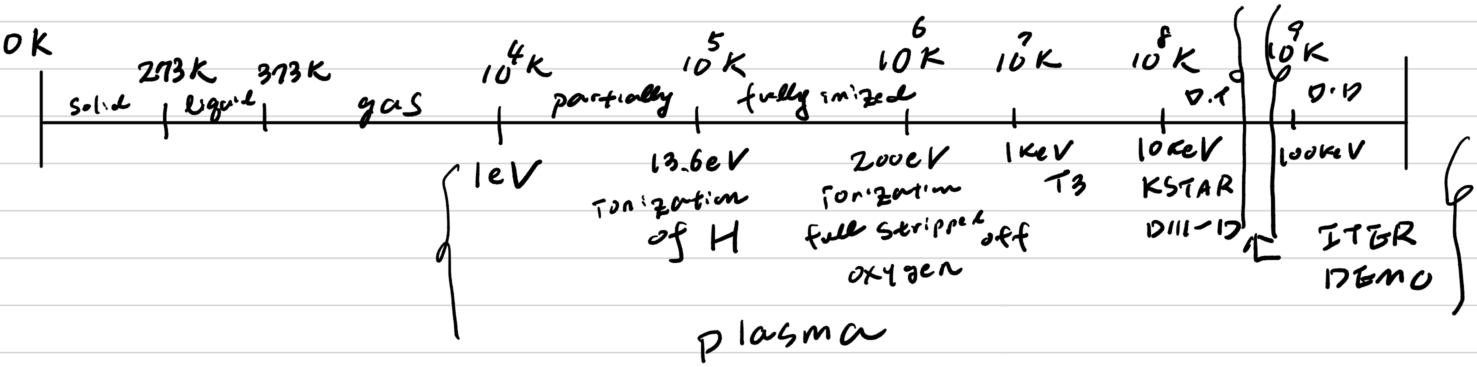
— Units (SI) throughout. except.

$$k\,[1.38\times10^{-23}\ \mathrm{J/K}]\cdot T[\mathrm{K}] = e\,[1.6\times10^{-19}\ \mathrm{C}]\ T[\mathrm{eV}]$$

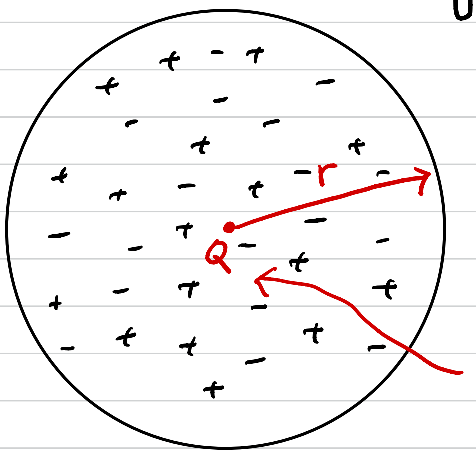
$$11600\ \mathrm{K} \simeq 1\mathrm{eV}$$

convenient since we will often see "T/e"

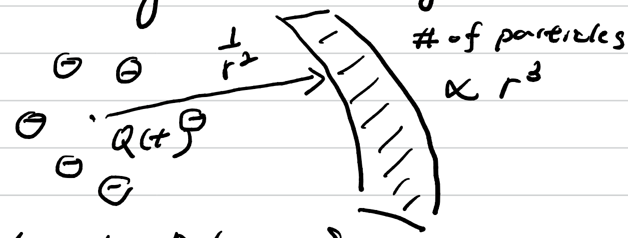
e.g. $\langle \vec{v}_{\nabla B} \rangle = \left(\frac{T}{e}\right)\frac{\vec{B}\times\nabla B}{B^3} \qquad \omega_{Dia} = \frac{1}{en}\frac{dP}{d\psi_p}$



2. charge-neutrality due to Debye shielding



Coulomb force $\propto \frac{1}{r^2}$
is a long-range force
diverging collectively



not due to Debye. ⟩

$$\epsilon_0 \vec{\nabla}\cdot\vec{E} = \rho = Q\,\delta(\vec{r}) + e(n_i - n_e)$$

Assume $\begin{cases} \text{Ion is fixed} \\ \phi, E \to 0 \ \{\text{as } r\to\infty\} \end{cases}$

$n_i = n_e = n_\infty$

if ϕ is finite. $n_i = n_\infty$ $\quad n_e = n_\infty \exp(e\phi/T_e)$

$$\vec{E} = -\vec{\nabla}\phi$$

$$\nabla^2\phi = \frac{e n_\infty}{\epsilon_0}\left(\exp(e\phi/T_e) - 1\right) - \frac{Q}{\epsilon_0}\delta(\vec{r})$$

$(e\phi/T_e \ll 1)$

check the validity "a posteriori"

$$\exp(e\phi/T_e) = 1 + e\phi/T_e$$

slab
[cylinder]
H.W1

$$\nabla^2\phi - \left(\frac{e^2 n_\infty}{\epsilon_0 T}\right)\phi = -\frac{Q}{\epsilon_0}\delta(\vec{r})$$ "Helmholtz" Green fcn.

$$\nabla^2\phi - \lambda_D^{-2}\phi = -\frac{Q}{\epsilon_0}\delta(\vec{r}) \qquad \lambda_D = \left(\frac{\epsilon_0 T}{e^2 n_\infty}\right)^{\frac{1}{2}}$$

Spherical sym.

$$\left\{ \frac{1}{r^2}\frac{d}{dr}\left(r^2\frac{d}{dr}\right)\phi - \lambda_D^{-2}\phi \right\} = -\frac{Q}{\epsilon_0}\delta(\vec{r})$$

$$\phi = \frac{f(r)}{r} = A\frac{e^{r/\lambda_D}}{r} + B\frac{e^{-r/\lambda_D}}{r} \qquad (\text{B.C}) \ \phi(\infty) = 0$$

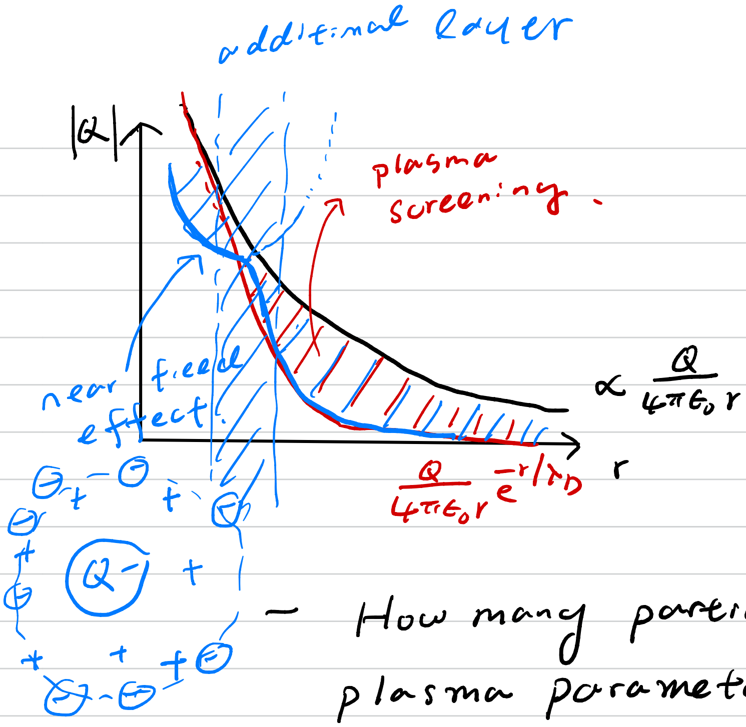
$$\lim_{r\to 0}\int 4\pi r^2 \{\ \}\,dr = -\frac{Q}{\epsilon_0}$$

$$\lim_{r\to 0}\frac{d\phi}{dr} = -\frac{Q}{4\pi r^2\epsilon_0}$$

$$= \left(\frac{Q}{4\pi\epsilon_0 r}\right)e^{-r/\lambda_D}$$

$$\frac{Q}{4\pi\epsilon_0 r}e^{-r/\lambda_D} \ll T_e/e$$

always valid. $r \gg \lambda_D$, $r \gg \frac{eQ}{4\pi\epsilon_0 T_e}$



- system size $L \gg \lambda_D$
  It's quasi-neutral
  e.g. KSTAR $n_e \sim 10^{19} m^{-3}$
  $T_e \sim 1 keV$

$$\lambda_D = \left(\frac{10^{-11} \times 10^{3}}{10^{-19} \cdot 10^{19}} \right)^{\frac{1}{2}} \approx 10^{-4} m$$

- How many particles needed? e.g. KSTAR
  plasma parameter $N_D \equiv \frac{4\pi}{3} \lambda_D^3 \cdot n \sim 5 \cdot 10^{-12} \cdot 10^{19} \sim 5 \times 10^{7}$
  $N_D \gg 1$. collectively behaving.

- How fast is the response?

$$\omega_p^{-1} \equiv \frac{\lambda_D}{v_t} \qquad \omega_p \equiv \left(\frac{e^2 n}{\epsilon_0 m} \right)^{\frac{1}{2}} > 10^{10}/s$$

$\omega_p \tau_n \gg 1$ collision time $\equiv \tau_n$

$\rightarrow$ neutral effect becomes subdominant.

plasma is a quasi-neutral gas of charged particles with collective behaviors.
$\left(\begin{array}{l} L \gg \lambda_D \\ N_D \gg 1 \\ \omega_p \tau_n \gg 1 \end{array} \right.$

- near-field effect?
  strong potential limit.
  (slab model $\rightarrow$ exact sol. of P-B equation)

$$\frac{1}{r^2} \frac{d}{dr} \left(r^2 \frac{d\phi}{dr} \right) = \frac{e n_\infty}{\epsilon_0} \left(\exp(e\phi/T_e) - 1 \right) = -\frac{e n_\infty}{\epsilon_0} - \frac{Q}{\epsilon_0} \delta(\vec{r})$$

if $Q<0$. $\phi<0$, $\phi \rightarrow -\infty$

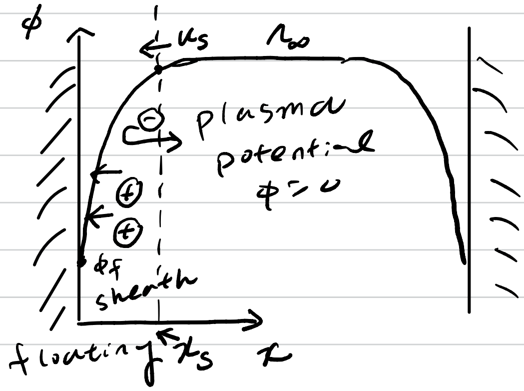
$$r^2 \frac{d\phi}{dr} = -\frac{1}{3} \frac{e n_\infty}{\epsilon_0} r^3 + C_1 \qquad \phi_{in} = -\frac{1}{6} \frac{e n_\infty}{\epsilon_0} r^2 - \frac{C_1}{r} + C_2$$

B.C. $\left(\begin{array}{l} \lim_{r\to 0} \frac{d\phi}{dr} \sim -\frac{Q}{4\pi\epsilon_0 r^2} \\ \lim_{r\to\infty} \phi_{in} \sim \lim_{r\to 0} \phi \sim \frac{Q}{4\pi\epsilon_0 r} e^{-r/\lambda_D} \end{array} \right.$

$$\phi_{in} \sim -\frac{1}{6} \frac{e n_\infty}{\epsilon_0} r^2 + \frac{Q}{4\pi\epsilon_0 r} - \frac{Q}{4\pi\epsilon_0 \lambda_D} + \sim$$

3/10/21 Sheath – An edge plasma phenomenon

1-d slab



· sheath essential
  mitigating large electron (heat) flux

· narrow due to Debye screening

· is stretched up to a few λ_D's
  due to ion free flight

1. Bohm-sheath criterion (edge of sheath)

$\nabla^2 \phi = \frac{e}{\epsilon_0}(n_e - n_i)$

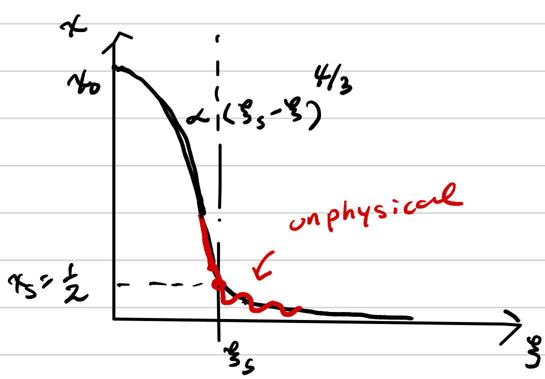
· electron : $n_e = n_\infty \exp(e\phi/T_e)$

· Ion const flux $J_i = e n_i u = \text{const}$

u: drift velocity developed by ϕ

$\frac{1}{2} m u^2 + e\phi \approx 0$ (cold ion)

$u = \left(-\frac{2e\phi}{m}\right)^{\frac{1}{2}}$



Assume at $x = x_s$ n_{is}, u_s can be estimated

where $n_{is} \approx n_\infty$ $\phi_s \approx 0$ $u_s \approx 0$
$\qquad\qquad\qquad\qquad \neq 0 \quad \neq 0$

$n_i = n_{is}\left(\frac{u_i}{u}\right) = n_{is}\left(\frac{\phi}{\phi_s}\right)^{-\frac{1}{2}} = n_{is}\left(\frac{-e\phi}{\frac{1}{2}m u_s^2}\right)^{-\frac{1}{2}}$

[density of plasma as $x \to \infty$ at $\phi = 0$]

$\phi'' = \frac{e n_\infty}{\epsilon_0}\left(\exp(e\phi/T_e) - \left(\frac{-e\phi/T_e}{\frac{1}{2}m u_s^2/T_e}\right)^{-\frac{1}{2}}\right)$

$\chi = -\frac{e\phi}{T_e}$ $\quad \xi = x/\lambda_D$ $\quad \mathcal{M}^2 = \frac{\frac{1}{2}m u_s^2}{T_e} = \frac{1}{2}\left(\frac{u_s}{c_s}\right)^2$

$\chi'' = \mathcal{M}\chi^{-\frac{1}{2}} - e^{-\chi}$

$\frac{1}{2}(\chi')^2 = 2\mathcal{M}\chi^{\frac{1}{2}} + e^{-\chi} - 1$ $\quad (\chi' = \chi = 0$ at $\xi \gg 1)$ $= -$(Sagdeev potential) for shock wave (soliton)

Local expansion near $\chi \sim \chi_s$ for RHS

$\chi'' = \mathcal{M}\chi_s^{-\frac{1}{2}} - \frac{1}{2}\mathcal{M}\chi_s^{-\frac{3}{2}}(\chi - \chi_s) - e^{-\chi_s} + e^{-\chi_s}(\chi - \chi_s) + \mathcal{O}(\chi - \chi_s)^2$

note $\mathcal{M}\chi_s^{-\frac{1}{2}} = 1$

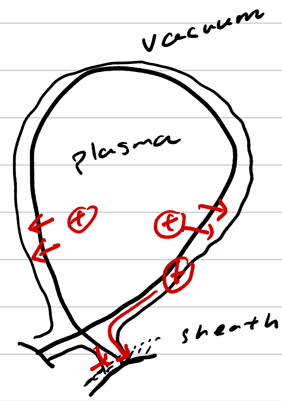
$= \left(e^{-\chi_s} - \frac{1}{2}\mathcal{M}\chi_s^{-\frac{3}{2}}\right)(\chi - \chi_s) + 1 - e^{-\chi_s}$

(1) Take χ_s as small as possible (ξ_s is as far as we can go)

✱ update the note with y instead of χ.

$\chi'' \simeq u(1-\frac{1}{2}\chi_s^{-1})(\chi-\chi_s)$

sol. → smooth when $1-\frac{1}{2}\chi_s^{-1} \geq 0 \qquad \chi_s \geq \frac{1}{2}$



" Bohm sheath criterion "

$\chi_s \geq \frac{1}{2} \qquad -\phi_s \geq T_e/2e \qquad u_s \geq C_s = (T_e/m)^{\frac{1}{2}}$

$\chi_s = \frac{1}{2} \qquad -\phi_s = T_e/2e \qquad u_s = C_s$

→ sheath formation begins $(\chi'' \simeq 0)$

2\. Child-Langmuir Space limited current (SLC) (Inside sheath)

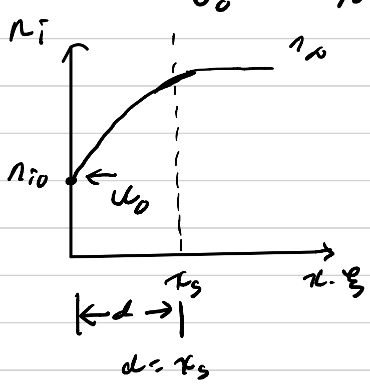
$\chi > \chi_s \qquad 0 < \xi < \xi_s$

$\chi'' = M\chi^{-\frac{1}{2}} - e^{-\chi} \approx M\chi^{-\frac{1}{2}}$ Integrate from ξ_s to ξ

$\frac{1}{2}(\chi')^2 \simeq 2M\chi^{\frac{1}{2}} \qquad \chi' = -2M^{\frac{1}{2}}\chi^{\frac{1}{4}} \qquad \chi^{-\frac{1}{4}}\chi' \simeq -2M^{\frac{1}{2}}$

$\frac{4}{3}\chi^{\frac{3}{4}} \simeq -2M^{\frac{1}{2}}(\xi-\xi_s) \qquad \chi \simeq \left(\frac{3}{2}\right)^{\frac{4}{3}} M^{\frac{2}{3}} (\xi_s-\xi)^{4/3}$

at $\xi = 0$, $\chi_0 = \left(\frac{3}{2}\right)^{\frac{4}{3}} M^{\frac{2}{3}} \xi_s^{4/3} \qquad \chi_0 \to \phi_0 \to u_0 = \left(\frac{-2e\phi_0}{m}\right)^{\frac{1}{2}}$



$J_0 = e n_{i0} u_0 = \sqrt{2}\, e n_\infty M C_s$

$= \sqrt{2}\, e n_\infty \frac{16}{9} \chi_0^{3/2} \xi_s^{-2} C_s$

$= \frac{4}{9}\left(\frac{2e}{m}\right)^{\frac{1}{2}} \frac{\epsilon_0 |\phi_0|^{3/2}}{d^2}$

current is limited due to space "d"

3\. Ion saturation current $\qquad \phi_s = -2T_e/e$

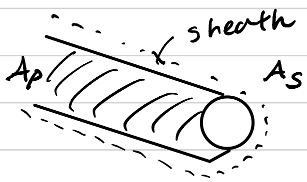
$J_0 = e n_0 u_0 = e n_{is} u_s = e n_\infty \exp(e\phi_s/T_e)(T_e/m)^{\frac{1}{2}}$

$= \exp(-\frac{1}{2})\, e n_\infty (T_e/m)^{\frac{1}{2}}$

$\approx 0.61\, e n_\infty (T_e/m)^{\frac{1}{2}} = J_{is}$

$d \simeq$ sheath thickness (H.W.)

4. V − I characteristics of Langmuir probe



Let $V = \phi_0$, $I = \underset{\uparrow\ \text{probe area}}{A_p} J_e - \underset{\nwarrow\ \text{effective area}}{A_s} J_i$

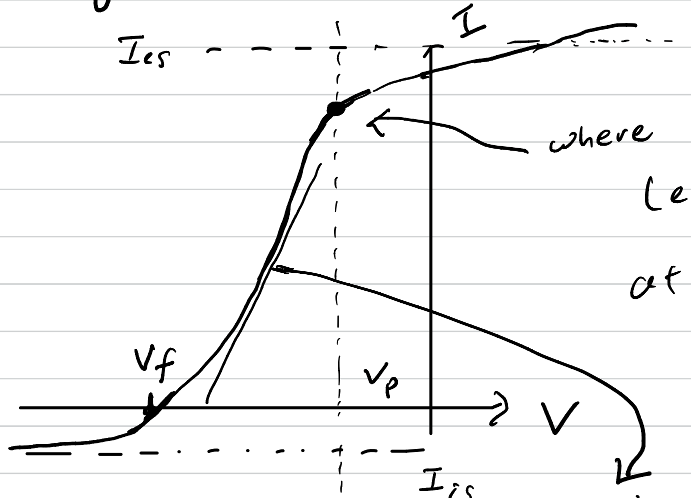
J_e due to kinetic motions of electrons

$$J_e = e n_\infty \int_0^\infty v_x \, dv_x \int_{-\infty}^{\infty} dv_y \int_{-\infty}^{\infty} dv_z \, f_M \cdot \exp(e\phi/T_e)$$

$$= \frac{1}{2\sqrt{\pi}} e n_\infty v_{te} \exp(e\phi/T_e)$$

$$I = A_p e n_\infty (T_e/m_i)^{\frac{1}{2}} \left(\frac{1}{2\sqrt{\pi}} \left(\frac{2m_i}{m_e}\right)^{\frac{1}{2}} \exp(eV/T_e) - \frac{A_s}{A_p} \exp(-\tfrac{1}{2}) \right)$$

* Floating potential $I = 0$ $\quad V_f \simeq -\frac{T_e}{2e} \ln\left(\frac{m_i}{2\pi m_e}\right)$

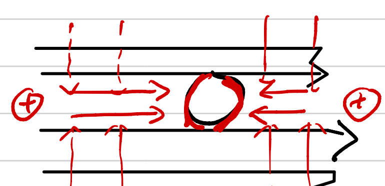


where perturbation is smallest (electron is saturated current) at plasma potential < 0 unknown.

Temperture : $T_e = e(I - I_{is}) / dI/dV$

density : $I_{is} \rightarrow n_\infty$

* effect of magnetic field



* $\nabla^2 y + k^2 y = f(x)$

$\underline{\nabla^2 G + k^2 G = \delta(x)}$ $\quad G = \begin{cases} y_h^+ & (x > 0) \\ y_h^- & (x < 0) \end{cases}$ $\quad y = \int_0^\infty f(x') G(x, x') dx' + \int_{-\infty}^0 f(x') G(x, x') dx'$