

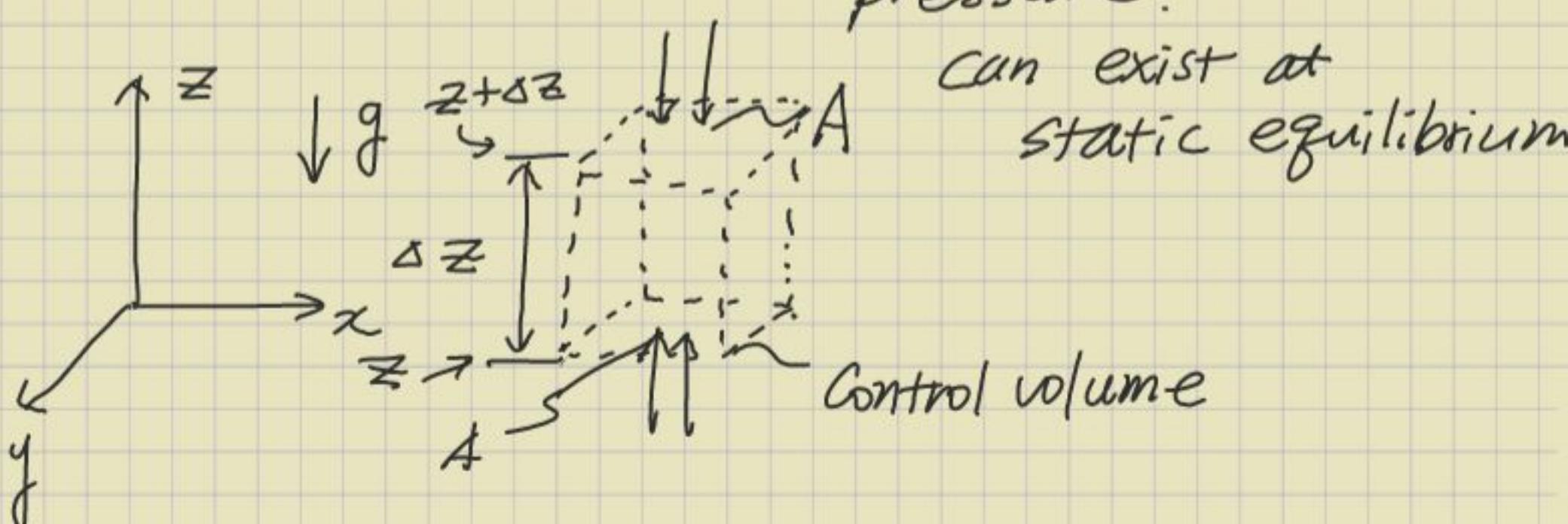
## lecture 2

- Fluid statics
- General equation
- Application
- Buoyancy / Archimedes principle  
(fully or partially immersed)
- Rotating fluid.

### □ Fluid statics

Fluid cannot support shear force  
but can support normal force.

(example)  
pressure.



Consider force balance in  
the z-direction.

$$\sum f_z = 0$$

mass:  $\rho A \Delta z$

$$= -P(z + \Delta z)A + P(z)A - Mg$$

Rearranging

$$\Delta P = P(z + \Delta z) - P(z) = -P\Delta z g$$

Note that  
sign dep. on  
direction.

$$\Rightarrow \frac{\Delta P}{\Delta z} = -Pg$$

$$\Rightarrow \lim_{\Delta z \rightarrow 0} \frac{\Delta P}{\Delta z} = \frac{dP}{dz} = -Pg ?$$

Be careful! cheating

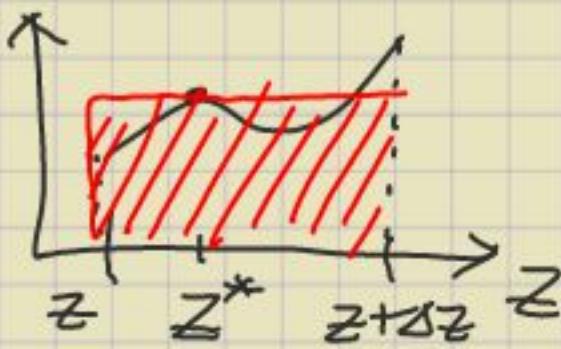
$\because \rho$  does not need to be constant.

Indeed.

$$M = A \int_z^{z+\Delta z} \rho(z) dz$$

Therefore

$$\Delta P = P(z + \Delta z) - P(z) = - \left( \int_z^{z+\Delta z} \rho(z) dz \right) g$$



by Mean value theorem

$$= \rho(z^*) \Delta z$$

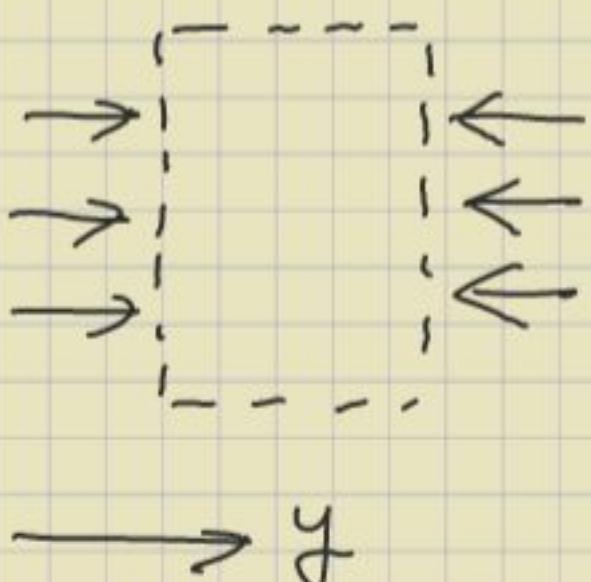
$$\Delta P = -\rho(z^*) g \Delta z$$

$$\Rightarrow \frac{\Delta P}{\Delta z} = -\rho(z^*) g$$

$$\Rightarrow \lim_{\Delta z \rightarrow 0} \frac{\Delta P}{\Delta z} = \frac{dP}{dz} = -\rho(z) g$$

As  $\Delta z \rightarrow 0$ ,  $z^* \rightarrow z$

Consider force acting on  
the side of control volume.



$$\sum f_y = 0$$

$$= P(y)A - P(y + \Delta y)A$$

$$\rightarrow y$$

$$P(y) = P(y + \Delta y)$$

$$\therefore \frac{dP}{dy} = 0$$

$$\text{also } \frac{dP}{dx} = 0$$

In Sum,

$$\frac{dp}{dx} = 0 \quad \frac{dp}{dy} = 0 \quad \frac{dp}{dz} = -\rho g$$

Now we know the change of pressure in all  $x, y$  &  $z$  direction.

→ Concept of gradient

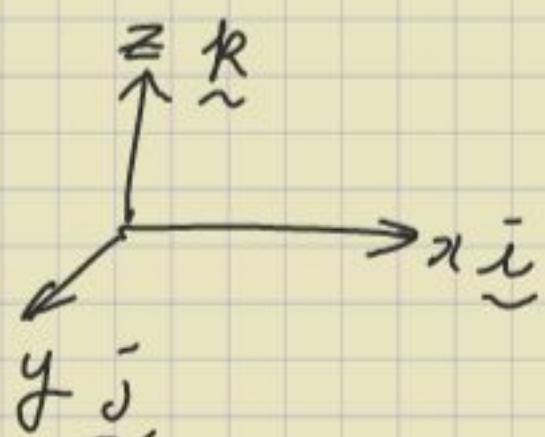
$\nabla P$  (vector)

$$= i \frac{\partial P}{\partial x} + j \frac{\partial P}{\partial y} + k \frac{\partial P}{\partial z}$$

[Now become partial derivative]

$$\frac{\partial P}{\partial x} = \left( \frac{\partial P}{\partial x} \right)_{y,z}$$

fix  $y$  &  $z$



In our example,

$$\nabla P = -k \rho g \Rightarrow \text{pressure change only in } z \text{ direction.}$$

Meanwhile, the gravitational acceleration  
is also vector.

$$\vec{g} = -k\vec{g}$$

Therefore

$$\nabla P = \rho \vec{g}$$

### Application of fluid statics

$\uparrow z$  (elevation)

$\downarrow$   
}  $\rightarrow 0$

(depth)  $\uparrow$  assume that scuba diver  
does not move.

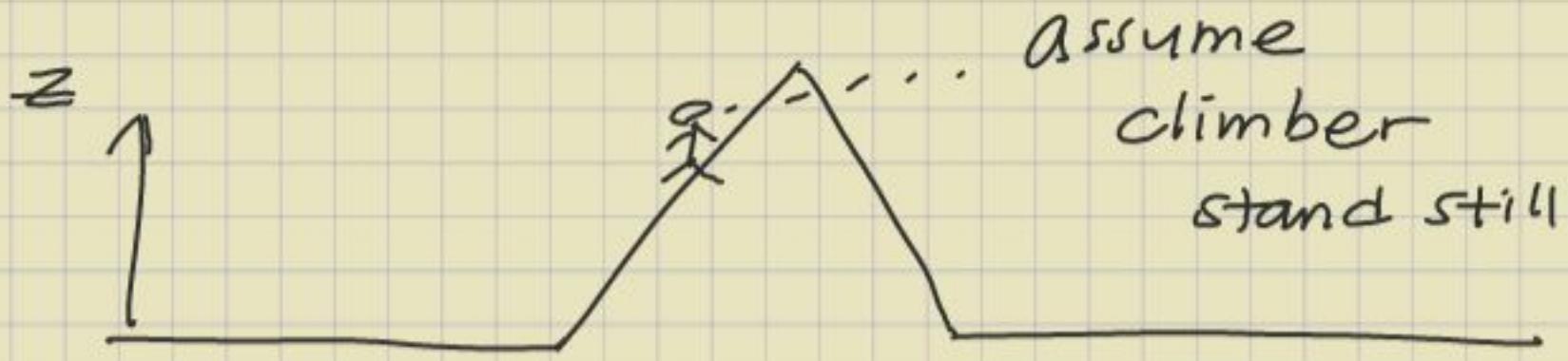
$$\sim 1 \text{ atm} = 101,000 \text{ Pa}$$

$$\frac{dP}{dz} = - \frac{dP}{d\zeta} \Rightarrow \frac{dP}{d\zeta} = \rho g \quad \begin{cases} \text{Water:} \\ \rho \sim 1000 \text{ kg/m}^3 \\ g \sim 10 \text{ m/s}^2 \\ \zeta \sim 10 \text{ m} \end{cases}$$

$$\Rightarrow P(\zeta) = P(0) + \rho g \zeta$$

$$\Rightarrow P(10 \text{ m}) = P(0) + 10^5 \left( \frac{\rho g}{m \cdot s^2} = \text{Pa} \right)$$

# Climb mountain



Consider the air as ideal gas

$$PV = nRT \Rightarrow \frac{n}{V} = \frac{P}{RT}$$

$$\rho = \frac{n M_w}{V} = \frac{PM_w}{RT} = P f$$

recall. Coefficient of thermal expansion

$$\alpha = \frac{1}{V_s} \left( \frac{\partial V_s}{\partial T} \right)_P = \rho \left( \frac{\partial (\rho V)}{\partial T} \right)_P$$

$$= \frac{PM_w}{RT} \left( \frac{\partial}{\partial T} \left( \frac{RT}{PM_w} \right) \right)_P \quad (\rho \sim \frac{1}{V_s})$$

$$= \frac{PM_w}{RT} \left( \frac{R}{PM_w} \right) = \frac{1}{T}$$

$\alpha$  for ideal gas

When we ignore temperature dependency.  
for the simplicity

$$\frac{dP}{dz} = -f Pg \Rightarrow \frac{dP}{P} = -fg dz$$

$$P(z) = P_0 \underbrace{\exp[-fg(z - z_0)]}_{\substack{\uparrow \\ \text{pressure changes}}} \dots \textcircled{*}$$

$P_0 = P(z=0)$  exponentially  
(unlike water  
changes linearly)

while

$$\frac{P_0}{P_0} = \frac{M_w}{RT}$$

\textcircled{\*} becomes (w/  $z_0 = 0$ )

$$\frac{P(z)}{P_0} = \exp\left[-\frac{\rho_0}{P_0} g z\right]$$

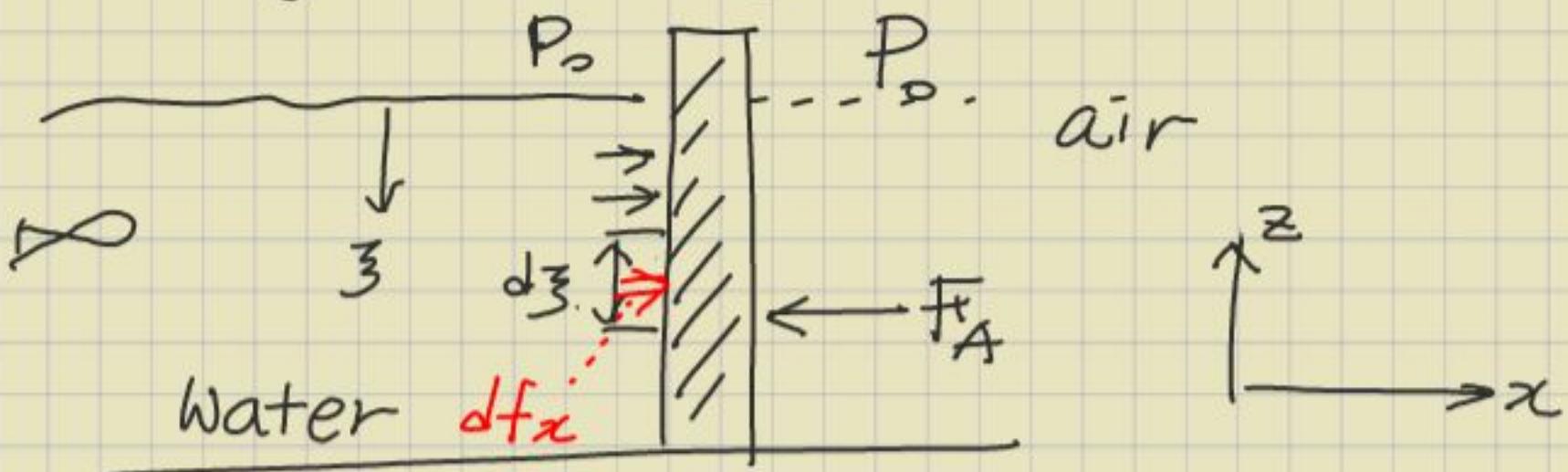
$$= \underbrace{1 - \frac{\rho_0}{P_0} g z}_{\substack{\text{H.O.T}}} + \frac{1}{2} \left(\frac{\rho_0 g}{P_0} z\right)^2 + \text{H.O.T}$$

when  $z$  is small,

valid up to

$$P(z) = P_0 - \rho_0 g z \nearrow 100m \quad (\text{for air } \rho_0 \text{ is small})$$

# Dyke (height H)

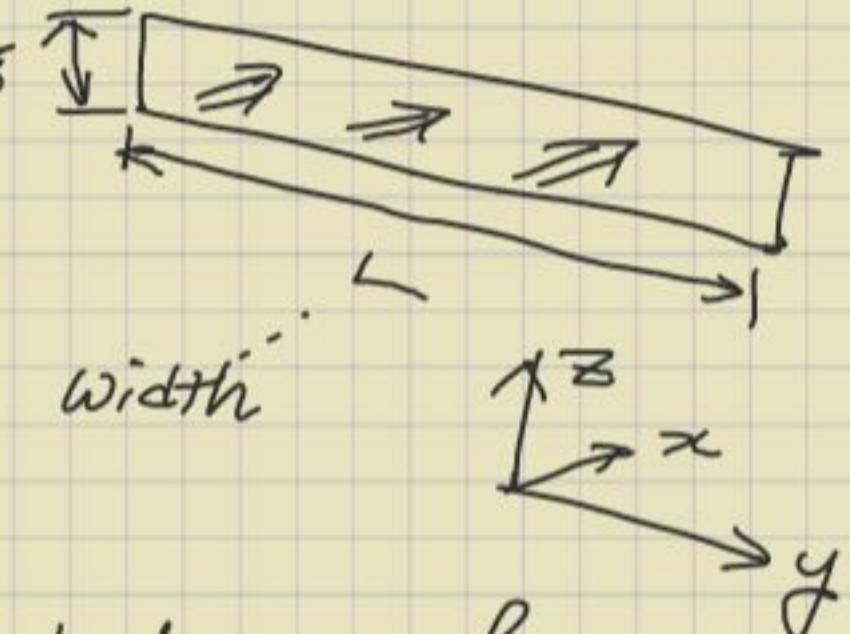


Note  $\rho_{\text{air}} \ll \rho_{\text{water}}$

$$df_x = P(\zeta) \underbrace{d\zeta L}_{\text{area}}$$

force acting on

a thin strip of dyke surface,



$F_x$  (force acting on the surface of dyke from water side)

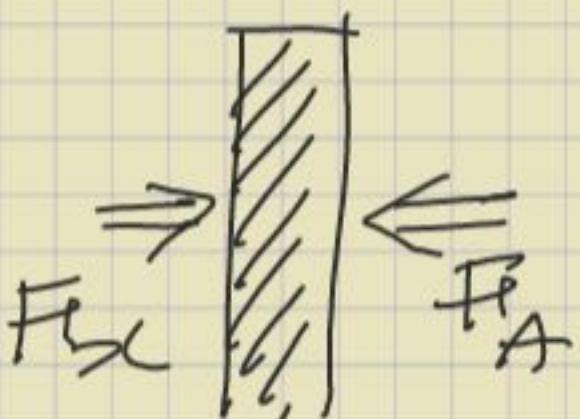
$$= \int_0^H df_x = L \int_0^H P(\zeta) d\zeta$$

$$= L \int_0^H (P_0 + \rho_{\text{water}} g \zeta) d\zeta = P_0 L H + \underbrace{\frac{1}{2} \rho_{\text{water}} g \frac{H^2}{2}}_{\text{waterside}}$$

$F_A$  (force acting on the dyke surface from air side)

$$F_A = P_0 L H + \frac{1}{2} \rho_{air} g \frac{H^2}{2}$$

$\circ \quad \because \rho_{air} \ll \rho_{water}$



Therefore net force is

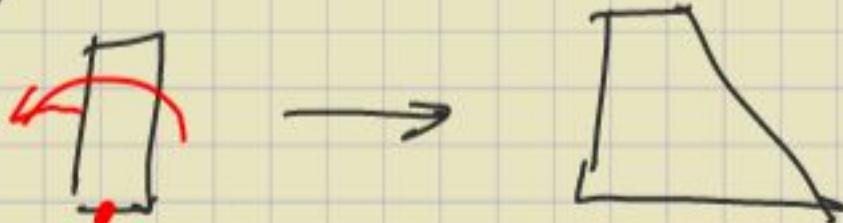
$$F_x - F_A = \rho g \frac{L H^2}{2}$$

What is missing?

Torque analysis.

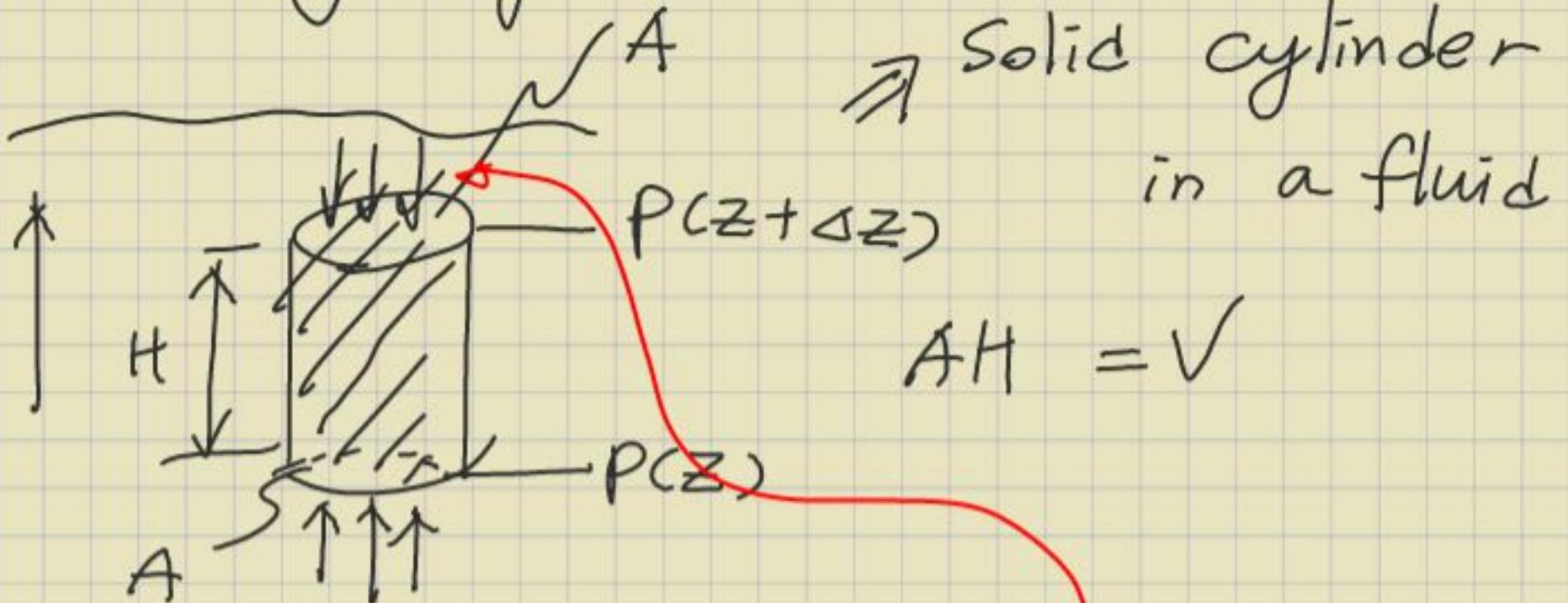
Net torque acting on dyke

so design need to be changed from



to support torque.

## Bouyancy



$\Rightarrow$  Solid cylinder  
in a fluid

$$AH = V$$

$F_B$  (net upward force  
due to the difference between  
opposing pressure)

$$= P(z)A - P(z + \Delta z)A$$

$$= A(P(z) - P(z + \Delta z))$$

but  $P = P_0 - \rho g z$  (for const. density)

$$\Rightarrow P(z) - P(z + H) = \rho g H$$

$\therefore F = \rho g H A = \rho g V > 0$   $\uparrow$   
↑ liquid density  $\uparrow$  solid volume  
Net force pointing upward.

This is called Archimedes principle.

If the presence of body neither affects appreciably the body force field in the region it occupies, then the pressure distribution outside it is the same though the body were replaced by additional fluid.

$$F_B = - \oint_S \vec{n} \cdot \vec{P} dS$$

$$= \int_V \nabla P dV$$

density of fluid.

$$= \int_V \rho g dV$$



Volume  
 $V$

Surface area  
 $S$

Buoyancy force:

Total pressure force on the surface of any body totally submerged in fluid.

Back to the example



Find the net force on the cylinder

Net force = Grav. force on cylinder  
( $F_w$ )

+ Buoyancy force  
( $F_B$ )

$$F_w = -\rho_s V g$$

A diagram of a cylinder partially submerged in a fluid. An arrow points downwards from the center of the cylinder, labeled "density of cylinder  $\rho_s$ ".

$$F_B = \rho V g$$

A diagram of a cylinder completely submerged in a fluid. An arrow points upwards from the center of the cylinder, labeled "density of fluid".

$$\therefore F_{net} = (\rho - \rho_s) V$$

If  $\rho > \rho_s$  float  $F_{net} > 0$

$\rho < \rho_s$  sink  $F_{net} < 0$

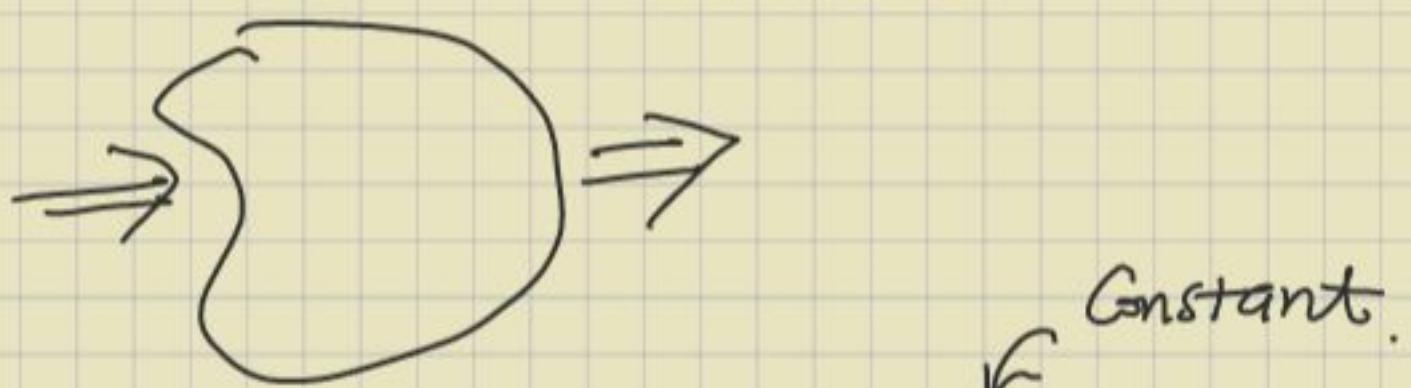
cf) Maths Corollary from divergence theorem

$$\int_V \nabla \cdot \underline{F} dV = - \oint_S \underline{n} \cdot \underline{F} dS$$

Generation  
or = In - Out

Consumption

Flux of  $\underline{F}$  through  
surface  $S$



If  $\underline{F} = f \underline{C} = f(x, y, z) \underline{C}$ ,

$$\nabla \cdot \underline{F} = \nabla \cdot (f \underline{C}) = \nabla f \cdot \underline{C} + f \nabla \cdot \underline{C}$$

$$\text{so } \int_V \nabla \cdot \underline{F} dV = \int_V \nabla \cdot (f \underline{C}) dV$$

$$= \int_V [\nabla f \cdot \underline{C} + f \nabla \cdot \underline{C}] dV$$

$$= \underline{C} \cdot \int_V \nabla f dV \quad \begin{matrix} 0 \\ \dots \\ \emptyset \end{matrix} \because \underline{C} \text{ is constant.}$$

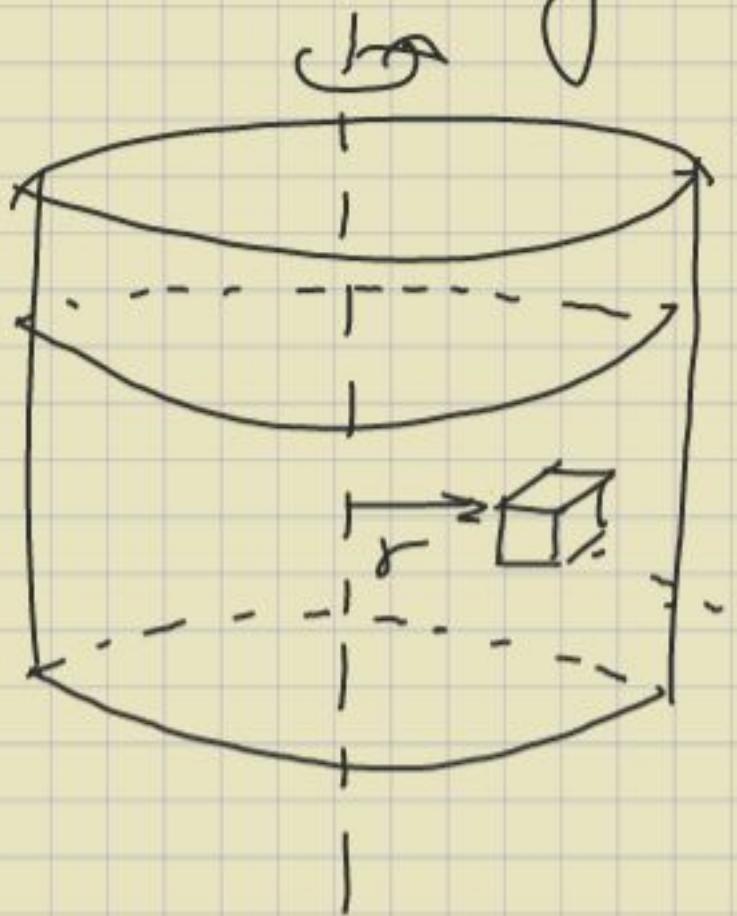
while

$$\begin{aligned} -\int_S \tilde{n} \cdot \tilde{F} dS &= -\int_S \tilde{n} \cdot \tilde{f} \tilde{c} dS \\ &= \tilde{c} \cdot \left( -\int_S \tilde{n} f dS \right) \dots \textcircled{2} \end{aligned}$$

from \textcircled{1} & \textcircled{2}

$$\underline{\int_V \nabla f dV = - \int_S \tilde{n} f dS}$$

## ⑩ Rotating fluid



→ Solid body rotation  
(fluid statistics)

$z$  (axial direction)

$r$  (radial direction)

- Consider small fluid element

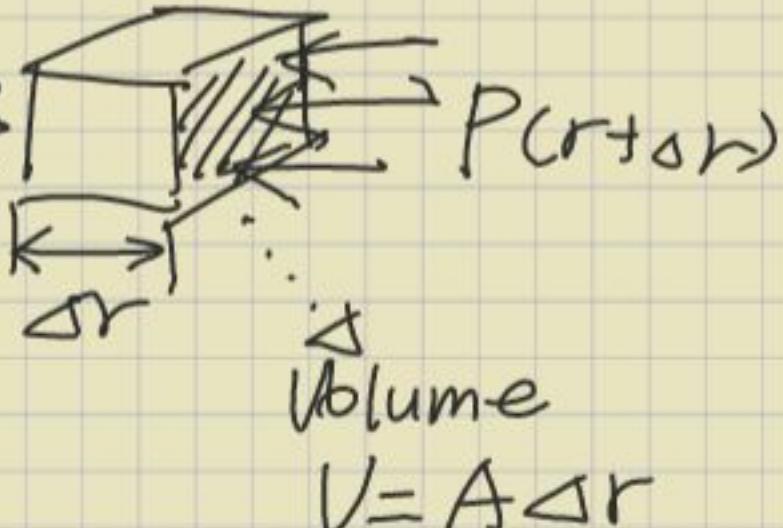


- axial balance

$$\frac{dP}{dz} = -\rho g$$

- radial balance  $P(r) \rightarrow P(r+\Delta r)$

$$\left\{ \begin{array}{l} V_\theta = r\omega \text{ (azimuthal velocity)} \\ \Delta r = -r\omega^2 \text{ (inward)} \end{array} \right.$$



Cause to rotate  
(centripetal force)

$$F_c = -m r\omega^2$$

Therefore forcebalance is

$$( \text{sth happen on Surface} ) = ( \text{sth. happen in Control vol.} )$$

$$( \text{In} - \text{out} ) = ( \text{Gen} )$$

$$p(r)A - p(r+\Delta r)A = -(\rho A \Delta r) r \omega^2$$

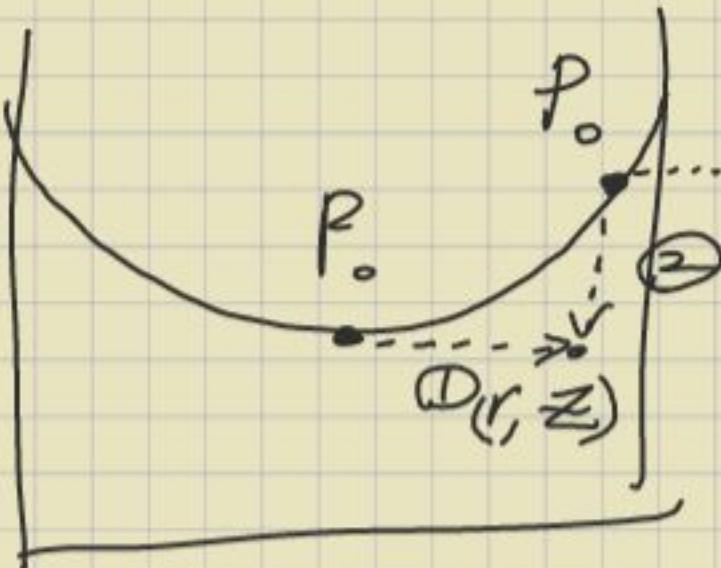
mass of  
fluid element

$$\Rightarrow \rho \omega^2 r \Delta r = p(r+\Delta r) - p(r)$$

$$\Rightarrow \frac{\partial P}{\partial r} = \rho r \omega^2 \quad \rightarrow \text{integrate}$$

$$\therefore P(r) = P_0 + \frac{1}{2} \rho r \omega^2$$

In reality, the interface will be curved



Two ways to find pressure at  $(r, z)$ ,

$$\begin{aligned} \textcircled{1} \quad P(r, z) &= P_0 + \int_0^r \frac{\partial P}{\partial r} dr \\ &= P_0 + \int_0^r \rho \omega^2 \zeta d\zeta \\ &= P_0 + \rho \omega^2 \frac{r^3}{2} \quad \cdots \square \end{aligned}$$

$$\begin{aligned} \textcircled{2} \quad P(r, z) &= P_0 + \int_{z+\Delta z}^z \frac{dp}{dz} dz \\ &= P_0 + \int_{z+\Delta z}^z -\rho g \zeta d\zeta \\ &= P_0 + \rho g \Delta z \quad \cdots \square \end{aligned}$$

$\square$  must be the same as  $\square$

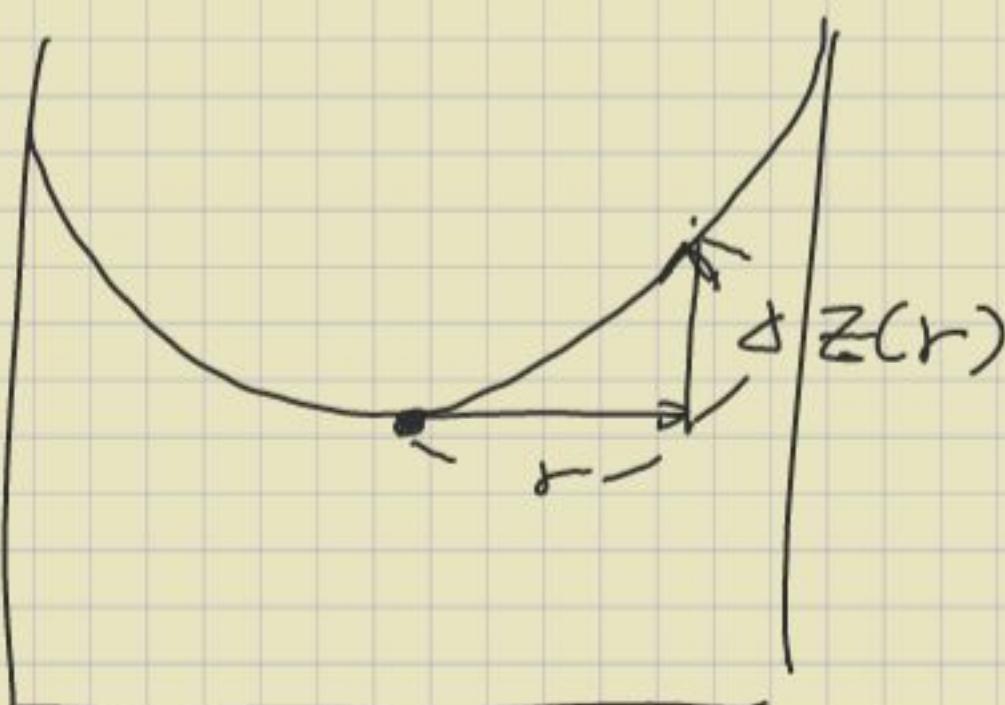
Therefore

$$P_0 + \rho g \Delta z = P_0 + \rho \omega^2 \frac{r^2}{2}$$

$$\Delta z = \frac{\omega^2}{g} \frac{r^2}{2}$$

$$\Rightarrow \Delta z(r) = \frac{\omega^2}{g} \frac{r^2}{2}$$

the shape of interface is parabola.



What is missing in this analysis?  
Surface tension,